18.085 Quiz 2 November 17, 1998 Professor Strang

Your name is: ______________________________

Grading	1
	2
	3
	4
Total	

Open book exam, calculators but not laptops, PC's or Crays.

1) **(28 pts.)** (a) Solve the equation $-u'' = \delta(x-a)$, with a unit point load at $x = a$ and with boundary conditions $u'(0) = 0$ and $u(1) = 0$. The solution $u(x, a)$ depends on a!

(b) Graph the solution when $a = \frac{1}{4}$.

(c) What will be the solution $U(x)$ with the same boundary conditions when the right side of the equation changes to

  (a) $C\delta(x-a) + D\delta(x-b)$? I am expecting a three-part answer.

  (b) $\int_0^1 \delta(x-a)\,da$? Give the answer as an integral involving $u(x,a)$ to be done by somebody else.

(d) Show why it is *impossible* to solve $-u'' = \delta(x-\frac{1}{2})$ with only derivative boundary conditions $u'(0) = u'(1) = 0$.

2) **(28 pts.)** The normalized beam equation is $u'''' = \delta(x-a)$, where the right side is a unit load acting at the point $x = a$. The boundary conditions are $u(0) = u'(0) = 0$ and $u''(1) = u'''(1) = 0$.

(a) Write down the form of the solution $u(x)$ in the two intervals $0 < x < a$ and $a < x < 1$ after taking account of the boundary conditions (but not yet the conditions at $x = a$).

(b) *Write down* and *solve* the equations at $x = a$ that determine the unknown constants that were still left in part (a).

(c) For $a = \frac{1}{2}$ graph the functions $w(x) = u''$ and $u(x)$.

(d) Suppose I wanted equations to determine a cubic spline $s(x)$ that has values $s(\frac{1}{4}) = 2$ and $s(\frac{1}{2}) = 6$ and $s(\frac{3}{4}) = 12$ with the same boundary conditions $s(0) = s'(0) = s''(1) = s'''(1) = 0$. If you had the functions $u(x)$ from earlier in this problem, what equations would you solve to find $s(x)$?

3) **(28 pts.)** (a) Suppose you have two real functions $u(x,y)$ and $s(x,y)$ that satisfy the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial s}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial s}{\partial x}\,.$$

Show that $u(x,y)$ solves Laplace's equation and *also* use u to construct a vector $w(x,y)$ that has zero divergence.

(b) Explain clearly why the curve $u(x,y) = 2$ crosses the curve $s(x,y) = 3$ at *right angles*. Suppose the crossing point is (x_0, y_0).

(c) Find the real part $u(x,y)$ and the imaginary part $s(x,y)$ of the function $f(x+iy) = \frac{1}{x+iy}$. (You might multiply by $\frac{x-iy}{x-iy}$.)

(d) For the function in (c), compute $w = \operatorname{grad} u$ and $w \cdot n$ and also $\int w \cdot n \, ds$ around the unit circle $r = 1$. You might expect this integral to be zero (because of $\operatorname{div} w$) but you might also worry about one particular bad point which is $(x, y) = \_\_\_\_$.

4) **(16 pts.)** (a) A flow has velocity vector $v(x,y) = (2x+y, x-2y)$. Test whether v is the gradient of a potential and if it is, find the potential $u(x,y)$.

(b) If a leaf flows past the point $x = y = 1$, find an equation for its path (streamline) with this velocity vector.

(c) Under what conditions on $v_3(x,y,z)$ is velocity vector $v = (2x+y, x-2y, v_3(x,y,z))$ the gradient of a potential $u(x,y,z)$?