**1**8.085 Quiz 1 October 20, 1998 Professor Strang

# **S**olutions

1) (1a) The determinant is positive if $q^2 < 6$ or $-\sqrt{6} < q < \sqrt{6}$.

(1b) $f(x_1, x_2) = \left\| Kx - \left[\begin{smallmatrix}4\\5\end{smallmatrix}\right] \right\|^2$ and also $f(x_1, x_2) = \frac{1}{2}x^T Kx - 4x_1 - 5x_2$ are minimized at the point which solves the linear equation.

(1c) The only combination of columns to solve $Ax = 0$ will be $x = 0$, so the nullspace contains only the zero vector.

2) (2a)

$$\begin{aligned} m_1 g &= y_1 - y_2 \\ m_2 g &= y_2 - y_3 + y_5 - y_6 \\ m_3 g &= y_3 - y_4 \end{aligned}$$

(2b) The matrix in part (a) is

$$A^T = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{bmatrix}$$

$$K = A^T C A = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 & & & & & \\ & c_2 & & & & \bigcirc \\ & & c_3 & & & \\ & & & c_4 & & \\ \bigcirc & & & & c_5 & \\ & & & & & c_6 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} c_1 & -c_2 & 0 & 0 & 0 & 0 \\ 0 & c_2 & -c_3 & 0 & c_5 & -c_6 \\ 0 & 0 & c_3 & -c_4 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} c_1 + c_2 & -c_2 & 0 \\ -c_2 & c_2 + c_3 + c_5 + c_6 & -c_3 \\ 0 & -c_3 & c_3 + c_4 \end{bmatrix}$$

(2c) With all $c = 5$

$$K = \begin{bmatrix} 10 & -5 & 0 \\ -5 & 20 & -5 \\ 0 & -5 & 10 \end{bmatrix}$$

Solving:

$$\begin{bmatrix} 10 & -5 & 0 \\ -5 & 20 & -5 \\ 0 & -5 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6g \\ 6g \\ 6g \end{bmatrix}.$$

via gaussian elimination, we get $\begin{cases} \ell_{22} & = & -\frac{1}{2} \\ \ell_{32} & = & -\frac{2}{7} \end{cases}$

$$\begin{bmatrix} 10 & -5 & 0 \\ 0 & 17.5 & -5 \\ 0 & 0 & \frac{60}{7} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6g \\ 6g \\ \frac{60g}{7} \end{bmatrix}.$$

Back substitution gives $x_3 = g$, then

$$x_2 = \tfrac{4}{5}g\,, \quad x_1 = g\,, \quad \text{so} \quad x = \begin{bmatrix} g \\ \frac{4}{5}g \\ g \end{bmatrix}.$$

(2d) Observe that $K_{ij} = -C_\ell$ if $i$ and $j$ are nodes connected by spring $\ell$. Therefore, $K_{13}$ and $K_{31}$ decrease by $c_7$, when the seventh spring is added. Also $K_{11}$ is the sum of $g$'s such that spring $j$ is attached to mass $m_i$. So $K_{11}$ and $K_{33}$ increase by $c_7$. The other entries of $K$ remain unchanged.

3) (3a)

$$A_0 = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}$$

Let $v = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$, then $A_0 v = 0$ and $A_0^T A_0 v = 0$.

Also, $\operatorname{rank}(A_0 A_0^T) \le \operatorname{rank} A_0 < 6 < 8$. So $\det(A_0 A_0^T) = 0$.

(3b) $A^T C A x = -f$,

$$f = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

(3c) It is easy to see that $y_1 = y_8$, $y_2 = y_7$, $y_4 = y_5$. The tricky part is to observe that $y_3 = -y_6$!

(3d) The loops yield

$$y_1 = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad y_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad y_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

All solutions to $A^T y = 0$ are combinations of $y_1$, $y_2$, $y_3$.

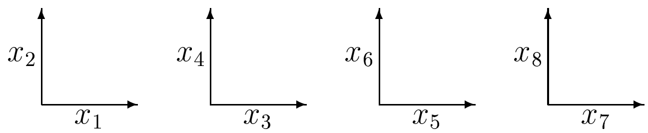
(3e) Take for example $y_1$ above. Then $y_1^T e = 0$ is the same as $-e_1 + e_2 - e_3 = 0$. It is Kirchhoff's Voltage Law.

(3f)

$$Y = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \text{ is a particular solution to } A^T y = f.$$

The general solution has the form $y = d_1 y_1 + d_2 y_2 + d_3 y_3 + Y$, where $y_1$, $y_2$, $y_3$ were found in (3d).

4) (4a)




$$
\begin{aligned}
e_1 &= x_3 - x_1 \\
e_2 &= x_2 - x_6 \\
e_3 &= x_4 - x_8 \qquad\qquad e = Ax \\
e_4 &= x_7 - x_5 \\
e_5 &= \tfrac{1}{\sqrt{2}}x_5 + \tfrac{1}{\sqrt{2}}x_6 \\
e_6 &= -\tfrac{1}{\sqrt{2}}x_7 + \tfrac{1}{\sqrt{2}}x_8
\end{aligned}
$$

$$
\text{where } A = \begin{bmatrix}
-1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1/\sqrt{2} & 1/\sqrt{2}
\end{bmatrix}
$$

Movement with no elongations ($Ax = 0$) is a mechanism. If $Ax = 0$, then $x_1 = x_3$, $x_2 = x_6 = -x_5 = -x_7 = -x_8 = -x_4$. Here are 2 independent mechanisms.

$$
x = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ (top bar slides right) and } x = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \\ -1 \\ 1 \\ -1 \\ -1 \end{pmatrix} \text{ (the square rotates).}
$$

(4b) $m = 6$, $n = 2N - r = 12 - 4 = 8$, so we need two new bars. The crossbars within the box would not work, since the box would still be able to rotate. So let us add boxes connecting joints 3 and 2 and support $(0, 0)$ to joint 4. The length of bar 8 is $\sqrt{5}$. So, $\cos\theta = \frac{2}{\sqrt{5}}$ and $\sin\theta = \frac{1}{\sqrt{5}}$.

Now write out $A_{\text{new}}$ and solve

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 & -1 & 1/\sqrt{2} & 0 & -1/\sqrt{2} & 0 \\ 0 & -1 & 0 & 0 & 1/\sqrt{2} & 0 & -1/\sqrt{2} & 0 \\ 0 & 0 & 0 & 1 & 0 & -1/\sqrt{2} & 0 & 2/\sqrt{5} \\ 0 & 0 & -1 & 0 & 0 & 1/\sqrt{2} & 0 & 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \\ y_8 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ -1 \\ 0 \\ -1 \\ 0 \\ -1 \end{bmatrix}$$

$$\begin{array}{l} (1) \\ (2) \\ (3) \\ (4) \\ (5) \\ (6) \\ (7) \\ (8) \end{array} \begin{cases} -y_1 = 0 \\ y_2 = -1 \\ y_1 + \frac{1}{\sqrt{2}} y_7 = 0 \\ y_3 + \frac{1}{\sqrt{2}} y_7 = -1 \\ -y_4 + \frac{1}{\sqrt{2}} y_5 - \frac{1}{\sqrt{2}} y_7 = 0 \\ -y_2 + \frac{1}{\sqrt{2}} y_5 - \frac{1}{\sqrt{2}} y_7 = -1 \\ y_4 - \frac{1}{\sqrt{2}} y_6 + \frac{2}{\sqrt{5}} y_8 = 0 \\ -y_3 + \frac{1}{\sqrt{2}} y_6 + \frac{1}{\sqrt{5}} y_8 = -1 \end{cases}$$

From (1), $y_1 = 0$. Then from (3) $y_7 = 0$. From (4) $y_3 = -1$. From (6) and (2) $(-1) + \frac{1}{\sqrt{2}} y_5 = -1$, so $y_5 = -2\sqrt{2}$. Now $-y_4 + \frac{1}{\sqrt{2}}(-2\sqrt{2}) - \frac{1}{\sqrt{2}} \cdot 0 = 0$ from (5), so $y_4 = -2$. Now we know that $y_1 = 0$, $y_2 = y_3 = -1$, $y_4 = -2$, $y_5 = -2\sqrt{2}$, $y_7 = 0$. Rewrite (7) and (8):

$$\begin{cases} -\frac{1}{\sqrt{2}} y_6 + \frac{2}{\sqrt{5}} y_8 = 2 & \Rightarrow & \frac{3}{\sqrt{5}} y_8 = 0 \Rightarrow y_8 = 0 \Rightarrow \\ \frac{1}{\sqrt{2}} y_6 + \frac{1}{\sqrt{5}} y_8 = -2 & \Rightarrow & y_6 = -2\sqrt{2} \end{cases}$$

Finally

$$y = \begin{bmatrix} 0 \\ -1 \\ -1 \\ -2 \\ -2\sqrt{2} \\ -2\sqrt{2} \\ 0 \\ 0 \end{bmatrix}$$

(4c) The force balance at 0,

$$\begin{cases} R_H^{(0,0)} &= -\frac{1}{\sqrt{2}}y_5 - \frac{2}{\sqrt{5}}y_8 \\ R_V^{(0,0)} &= -\frac{1}{\sqrt{2}}y_5 - \frac{1}{\sqrt{5}}y_8 \end{cases}$$

At the other support

$$\begin{cases} R_H^{(3,0)} &= \frac{1}{\sqrt{2}}y_6 \\ R_V^{(3,0)} &= -\frac{1}{\sqrt{2}}y_6 \end{cases}$$

Use $y_5 = y_6 = -2\sqrt{2}$ from (4b) and $y_8 = 0$. Then $R_H^{(0,0)} = R_V^{(0,0)} = -2$, $R_H^{(3,0)} = 2$, $R_V^{(3,0)} = -2$.