LINEAR ALGEBRA IN A NUTSHELL

One question always comes on the first day of class. "*Do I have to know linear algebra*?" My reply gets shorter every year: "*You soon will.*" This section brings together many important points in the theory. It serves as a quick primer, not an official part of the applied mathematics course (like Chapter 1 and 2).

This summary begins with two lists that use most of the key words of linear algebra. The first list applies to invertible matrices. That property is described in 14 different ways. The second list shows the contrast, when A is singular (not invertible). There are more ways to test invertibility of an n by n matrix than I expected.

Nonsingular	**Singular**
A is invertible	A is not invertible
The columns are independent	The columns are dependent
The rows are independent	The rows are dependent
The determinant is not zero	The determinant is zero
$Ax = 0$ has one solution $x = 0$	$Ax = 0$ has infinitely many solutions
$Ax = 0$ has one solution $x = A^{-1}b$	$Ax = b$ has no solution or infinitely many
A has n (nonzero) pivots	A has $r < n$ pivots
A has full rank	A has rank $r < n$
The reduced row echelon form is $R = I$	R has at least one zero row
The column space is all of $\mathbf{R}^n$	The column space has dimension $r < n$
The row space is all of $\mathbf{R}^n$	The row space has dimension $r < n$
All eigenvalues are nonzero	Zero is an eigenvalue of A
$A^{\mathrm{T}}A$ is symmetric positive definite	$A^{\mathrm{T}}A$ is only semidefinite
A has n (positive) singular values	A has $r < n$ singular values

Now we take a deeper look at linear equations, without proving every statement we make. The goal is to discover what $Ax = b$ really means. One reference is my textbook *Introduction to Linear Algebra*, published by Wellesley-Cambridge Press. That book has a much more careful development with many examples (you could look at the course page, with videos of the lectures, on ocw.mit.edu or web.mit.edu/18.06).

The key is to think of every multiplication Ax, a matrix A times a vector x, as a *combination of the columns of* A:

Matrix Multiplication by Columns

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} C \\ D \end{bmatrix} = C\begin{bmatrix} 1 \\ 3 \end{bmatrix} + D\begin{bmatrix} 2 \\ 6 \end{bmatrix} = \textbf{combination of columns}\,.$$

Multiplying by rows, the first component $C + 2D$ comes from 1 and 2 in the first row of A. But I strongly recommend to think of Ax **a column at a time**. Notice how

$x = (1, 0)$ and $x = (0, 1)$ will pick out single columns of A:

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \text{first column} \qquad \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \text{last column}\,.$$

Suppose A is an m by n matrix. Then $Ax = 0$ has at least one solution, the all-zeros vector $x = 0$. There are certainly other solutions in case $n > m$ (more unknowns than equations). Even if $m = n$, there might be nonzero solutions to $Ax = 0$; then A is square but not invertible. It is the number r of *independent* rows and columns that counts. That number r is the **rank** of A ($r \le m$ and $r \le n$).

> The **nullspace** of A is the set of all solutions x to $Ax = 0$. This nullspace $\boldsymbol{N}(A)$ contains only $x = 0$ when the columns of A are **independent**. In that case the matrix A has full column rank $r = n$: independent columns.

For our 2 by 2 example, the combination with $C = 2$ and $D = -1$ produces the zero vector. Thus $x = (2, -1)$ is in the nullspace, with $Ax = 0$. The columns $(1, 3)$ and $(2, 6)$ are "linearly dependent." One column is a multiple of the other column. *The rank is* $r = 1$. The matrix A has a whole line of vectors $cx = c(2, -1)$ in its nullspace:

Nullspace is a line
$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and also} \quad \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} \boldsymbol{2c} \\ \boldsymbol{-c} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

If $Ax = 0$ and $Ay = 0$, then every combination $cx + dy$ is in the nullspace. Always $Ax = 0$ asks for a combination of the columns of A that produces the zero vector:

$\boldsymbol{x}$ in nullspace $\qquad x_1\,(\text{column } 1) + \cdots + x_n\,(\text{column } n) = \text{zero vector}$

When those columns are independent, the only way to produce $Ax = 0$ is with $x_1 = 0$, $x_2 = 0, \ldots, x_n = 0$. Then $x = (0, \ldots, 0)$ is the only vector in the nullspace of A. Often this will be our requirement (independent columns) for a good matrix A. In that case, $A^{\mathrm{T}}A$ also has independent columns. The square n by n matrix $A^{\mathrm{T}}A$ is then invertible and symmetric and positive definite. If A is good then $A^{\mathrm{T}}A$ is even better.

I will extend this review (*still optional*) to the geometry of $Ax = b$.

Column Space and Solutions to Linear Equations

$Ax = b$ **asks for a linear combination of the columns that equals** b. In our 2 by 2 example, the columns go in the same direction! Then b does too:

Column space
$$Ax = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} \text{ is always on the line through } \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

We can only solve $Ax = b$ when the vector b is on that line. For $b = (1, 4)$ there is no solution, it is off the line. For $b = (5, 15)$ there are many solutions (5 times column 1 gives b, and this b is on the line). The big step is to look at a space of vectors:

Definition: The *column space* contains all combinations of the columns.

In other words, $\boldsymbol{C}(A)$ contains all possible products A times x. Therefore $Ax = b$ is **solvable** exactly when the vector b is in the column space $\boldsymbol{C}(A)$.

For an m by n matrix, the columns have m components. The column space of A is in m-dimensional space. The word **"space"** indicates that the key operation of linear algebra is allowed: *Any combination of vectors in the space stays in the space.* The zero combination is allowed, so the vector $x = 0$ is in every space.

How do we write down all solutions, when b belongs to the column space of A? Any one solution to $Ax = b$ is a **particular solution** $\boldsymbol{x_p}$. Any vector $\boldsymbol{x_n}$ in the nullspace solves $Ax = 0$. Adding $Ax_p = b$ to $Ax_n = 0$ gives $A(x_p + x_n) = b$. **The complete solution to** $Ax = b$ **has this form** $x = x_p + x_n$:

Complete solution $\quad x = x_{\text{particular}} + x_{\text{nullspace}} = (\text{one } x_p) + (\text{all } x_n)$.

In the example, $b = (5, 15)$ is 5 times the first column, so one particular solution is $x_p = (5, 0)$. To find all other solutions, add to x_p any vector x_n in the nullspace—which is the line through $(2, -1)$. Here is $x_p + (\text{all } x_n)$:

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 5 \\ 15 \end{bmatrix} \quad \text{gives} \quad x_{\text{complete}} = \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix} + \begin{bmatrix} 2c \\ -c \end{bmatrix}.$$

This line of solutions is drawn in Figure A1. *It is not a subspace.* It does not contain $(0, 0)$, because it is shifted over by the particular solution $(5, 0)$. We only have a "space" of solutions when b is zero (then the solutions fill the nullspace).

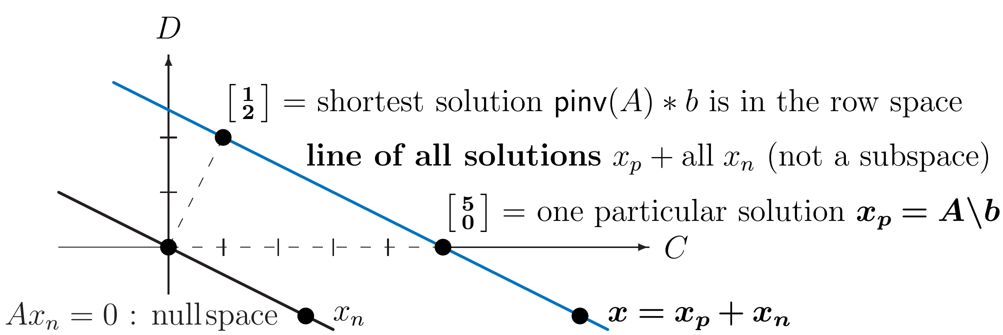

Figure A1: Parallel lines of solutions to $Ax_n = 0$ and $\left[\begin{smallmatrix} 1 & 2 \\ 3 & 6 \end{smallmatrix}\right](\boldsymbol{x_p + x_n}) = \left[\begin{smallmatrix} 5 \\ 15 \end{smallmatrix}\right]$

May I collect three important comments on linear equations $Ax = b$.

1. Suppose A is a square *invertible* matrix (the most common case in practice). Then the nullspace only contains $x_n = 0$. The particular solution $x_p = A^{-1}b$ is the only solution. The complete solution $x_p + x_n$ is $A^{-1}b + 0$. Thus $x = A^{-1}b$.

2. $Ax = b$ has infinitely many solutions in Figure A1. The shortest x always lies in the "row space" of A. That particular solution $(1, 2)$ is found by the *pseudo-inverse* pinv (A). The backslash $A\backslash b$ finds an x with at most m nonzeros.

3. Suppose A is tall and thin ($m > n$). The n columns are likely to be independent. But if b is not in the column space, $Ax = b$ has no solution. The least squares method minimizes $\|b - Ax\|^2$ by solving $A^{\mathrm{T}}A\widehat{x} = A^{\mathrm{T}}b$.

The Four Fundamental Subspaces

The nullspace $\boldsymbol{N}(A)$ contains all solutions to $Ax = 0$. The column space $\boldsymbol{C}(A)$ contains all combinations of the columns. When A is m by n, $\boldsymbol{N}(A)$ is a subspace of $\mathbf{R}^n$ and $\boldsymbol{C}(A)$ is a subspace of $\mathbf{R}^m$.

The other two fundamental spaces come from the transpose matrix A^{T}. They are $\boldsymbol{N}(A^{\mathrm{T}})$ and $\boldsymbol{C}(A^{\mathrm{T}})$. We call $\boldsymbol{C}(A^{\mathrm{T}})$ the "row space of A" because the rows of A are the columns of A^{T}. What are those spaces for our 2 by 2 example?

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \quad \text{transposes to} \quad A^{\mathrm{T}} = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}.$$

Both columns of A^{T} are in the direction of $(1, 2)$. The line of all vectors $(c, 2c)$ is $\boldsymbol{C}(A^{\mathrm{T}})$ = row space of A. The nullspace of A^{T} is in the direction of $(3, -1)$:

$$\textbf{Nullspace of } \boldsymbol{A}^{\mathbf{T}} \qquad A^{\mathrm{T}}y = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} E \\ F \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{gives} \quad \begin{bmatrix} E \\ F \end{bmatrix} = \begin{bmatrix} 3c \\ -c \end{bmatrix}.$$

The four subspaces $\boldsymbol{N}(A)$, $\boldsymbol{C}(A)$, $\boldsymbol{N}(A^{\mathrm{T}})$, $\boldsymbol{C}(A^{\mathrm{T}})$ combine beautifully into the big picture of linear algebra. Figure A2 shows how the nullspace $\boldsymbol{N}(A)$ is perpendicular to the row space $\boldsymbol{C}(A^{\mathrm{T}})$. Every input vector x splits into a row space part x_r and a nullspace part x_n. Multiplying by A always(!) produces a vector in the column space. Multiplication goes from left to right in the picture, from x to $Ax = b$.

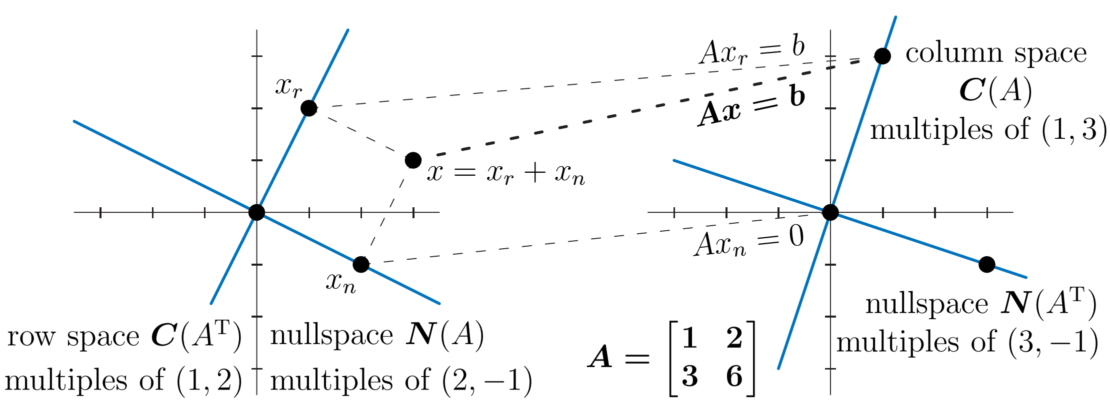

Figure A2: The four fundamental subspaces (lines) for the singular matrix A.