18.702 March 5, 2008

Solutions to Quiz 1

1. (25 points) Let G be the group generated by elements x, y, with relations $x^3 = 1$, $y^3 = 1$ and $xyxyx = 1$. Use Todd-Coxeter to determine the order of the group.

Using Todd-Coxeter on the subgroup $< x >$ shows that $< x >$ has index 1, which means that $G =< x >$. Since $x^3 = 1$ is a given relation, the order of G is either 3 or 1.

Todd-Coxeter also shows that that x permutes the cosets of the subgroup $< y >$ cyclically. Therefore $x \neq 1$, and G ahs order 3.

Another way to show that $x \neq 1$ is to add the relation $y = 1$ to the list, and to notice that this gives the cyclic group of order 3.

2. (30 points) A nonabelian group G has order 21. Determine:
(a) the number of Sylow 7 subgroups and the number of Sylow 3 subgroups.
(b) the number of elements of each order.
(c) the Class Equation.
(d) the dimensions of the irreducible representations.

(a) The number of 7-subgroups is 1, so it is a normal subgroup. The number of 3-subgroups divides 7, so it is either 1 or 7. If both were normal, G would be isomorphic to $C_7 \times C_3$, and would be an abelian group. So there are 7 3-subgroups.

(b) Since there are seven 3-subgroups, each of which contains two elements of order 3, G contains 14 elements of order three. Since there is one 7-subgroups, G contains 6 elements of order seven. There is 1 element of order one.

(c) The class equation is $21 = 1 + 3 + 3 + 7 + 7$.

Let x and y be elements of orders 7 and 3 respectively. The centralizer of x contains $< x >$ and it is not the whole group, so it is equal to $< x >$, and the conjugacy class $C(x)$ has order 3. Since $< x >$ is a normal subgroup, it contains $C(x)$. There are six elements of $< x >$ different from the identity, each one of which generates that cyclic group. So these six elements form two classes of order 3.

The centralizer of y must be the cyclic group $< y >$, so $C(y)$ has order 7. Since there are seven conjugate 3-subgroups, there each one must contain one conjugate of y. Then the class of y^2 also has order 7 and is not the same as the class of y.

(d) since there are five conjugacy classes, there are five irreducible characters. The sum of the squares of their dimensions is equal to 21, and each dimension divides 21. The only possibility is $21 = 1^2 + 1^2 + 1^2 + 3^2 + 3^2$. The dimensions are $1, 1, 1, 3, 3$.

3. (20 points) Let ρ, ρ' be representations of a finite group G on complex vector spaces V, V', and let $T : V \to V'$ be an invariant linear transformation. Prove that the kernel of T is an invariant subspace of V.

The statement that T is invariant means that for all $g \in G$, $\rho'_g \circ T = T \circ \rho_g$, or $g(Tv) = T(gv)$. To show that the kernel is invariant, we must show that if v is in the kernel, so is gv, for all $g \in G$. To say v is in the kernel means that $Tv = 0$. Then $gTv = g0 = 0$ and also $gTv = Tgv$. So $Tgv = 0$. This shows that gv is in the kernel.

4. (25 points) Below is the character table of the group of 24 rotational symmetries of a cube (the octahedral group), where e is an edge rotation with angle π, f is a face rotation with angle $\pi/2$, g is a face rotation with angle π, and v is a vertex rotation with angle $2\pi/3$.

size	(1)	(6)	(6)	(3)	(8)
elt	1	e	f	g	v
χ_1	1	1	1	1	1
χ_2	1	-1	-1	1	1
χ_3	2	0	0	2	-1
χ_4	3	-1	1	-1	0
χ_5	3	1	-1	-1	0

The group operates on the six faces of the cube. Let χ be the character of the corresponding representation. Compute χ, and decompose χ into irreducible characters.

The character is

size	(1)	(6)	(6)	(3)	(8)
elt	1	e	f	g	v
χ	6	0	2	2	0

The projection formula shows that $\chi = \chi_1 + \chi_2 + \chi_4$.