The Orthogonality Relations

1. Statement of the relations

Let $\rho : G \to GL(V)$ and $\sigma : G \to GL(W)$ be representations of a finite group G on complex vector spaces V and W, and let χ, χ' be the characters of ρ, σ, respectively. The orthogonality relations that we prove here assert that if ρ and σ are irreducible and not isomorphic, then

$$\langle \chi', \chi \rangle = 0, \tag{1.1a}$$

and that if ρ is irreducible, then

$$\langle \chi, \chi \rangle = 1. \tag{1.1b}$$

2. Projection Operators

A linear operator f on a vector space V is called a *projection operator* if $f^2 = f$. Another way to say this is that f is a projection operator if it acts as the identity on its image. (We are not speaking of the more restrictive concept of orthogonal projection.)

Lemma 2.1. *(a) Let K and U be the kernel and image of a projection operator operator f on a vector space V. Then V is the direct sum $U \oplus K$.*
(b) The trace of a projection operator f is equal to the dimension of the image U.

Proof. (a) Suppose that f is a projection operator. We must show that $U + K = V$ and that $U \cap K = 0$. Let $v \in V$, let $u = f(v)$, and let $x = v - u$. Then

$$f(x) = f(v) - f^2(v) = 0,$$

so $x \in K$. Since $u + x = v$ and v was arbitrary, $\mathcal{U} + K = V$. Next, if $z \in U \cap K$, then $f(z) = z$ because $z \in U$ and $f(z) = 0$ because $z \in K$. So $z = 0$. This shows that $K \cap U = 0$.

(c) To determine the trace of a projection operator f, we choose a basis for V by appending bases for U and for K. The matrix of f with respect to this basis will have a block form

$$\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix},$$

where the number of rows in I is equal to dim U. □

3. G-invariant transformations

Let $\rho : G \to GL(V)$ and $\sigma : G \to GL(W)$ be representations, as before. A linear transformation $T : V \to W$ is *G-invariant* if

$$f(gv) = gT(v) \tag{3.1}$$

for all $g \in G$ and all $v \in W$. Note that on the left side, g operates via ρ, while on the right g operates via σ.

G-invariance is a very strong restriction on a linear transformation.

Schur's Lemma 3.2. *(a) Let $\rho : G \to GL(V)$ and $\sigma : G \to GL(W)$ be* irreducible *representations, and let $T : V \to W$ be a G-invariant linear transformation. Then either T is an isomorphism, or else $T = 0$.*
(b) Let $\rho : G \to GL(V)$ be an irreducible representation, and let $T : V \to V$ be a G-invariant linear operator. Then T is multiplication by a scalar: $T = cI$.

Sublemma 3.3. *Let T be a G-invariant linear transformation. The kernel of T is a G-invariant subspace of V and the image of T is a G-invariant subspace of W.*

Proof. We must show that if v is in the kernel of T and g is in G, then gv is also in the kernel. This is true because $T(gv) = gT(v) = g\,0 = 0$. Next, we must show that if w is in the image of T, then gw is also in the image. To say w is in the image means that $w = T(v)$ for some v in V. Then $gw = gT(v) = T(gv)$. so gw is in the image too. $\square$

Proof of Schur's Lemma. (a) Assume that $T \neq 0$. Since ρ is irreducible and the kernel of T is an invariant subspace, the kernel is either $\{0\}$ or V. It is not V because $T \neq 0$, so the kernel is $\{0\}$. Next, since σ is irreducible and the image is invariant, the image is either $\{0\}$ or W. It is not $\{0\}$ because $T \neq 0$. So the image is W. Then T is an isomorphism.

(b) The linear operator T has an eigenvalue λ. This means that the kernel of $T - \lambda I$ is not trivial. But the operator $T - \lambda I$ is also G-invariant, so its kernel is equal to V. This shows that $T - \lambda I = 0$, hence that $T = \lambda I$. $\square$

We go back to the definition of a G-invariant transformation T. If we write the relation (3.1) out displaying the two representations, it becomes $T\rho_g(v) = \sigma_g T(v)$, or

$$\sigma_g^{-1} T \rho_g = T. \tag{3.4}$$

The diagram below illustrates the relationship between an arbitrary operator T and the operator $\sigma_g^{-1} T \rho_g$:

$$\begin{array}{ccc} V & \xrightarrow{\sigma_g^{-1} T \rho_g} & W \\ {\scriptstyle \rho_g}\downarrow & & \downarrow{\scriptstyle \sigma_g} \\ V & \xrightarrow{T} & W \end{array} . \tag{3.5}$$

If a linear transformation T is not G-invariant, then $\sigma_g^{-1} T \rho_g \neq T$ for at least one g. However, we can create a G-invariant transformation from any linear transformation T by averaging. We define

$$\widetilde{T} = \tfrac{1}{|G|} \textstyle\sum_g \; \sigma_g^{-1} T \rho_g. \tag{3.6}$$

Let's verify that $\widetilde{T}$ is G-invariant. Since g is the summation index, we write invariance as $\sigma_h^{-1} \widetilde{T} \rho_h = \widetilde{T}$ for h in G. We set $g' = gh$ and we note that $Gh = G$. As g runs over the group, so does g'. Then

$$\sigma_h^{-1} \widetilde{T} \rho_h = \tfrac{1}{|G|} \textstyle\sum_g \; \sigma_h^{-1} \sigma_g^{-1} \widetilde{T} \rho_g \rho_h = \tfrac{1}{|G|} \textstyle\sum_{g'} \; \sigma_{g'}^{-1} \widetilde{T} \rho_{g'} = \widetilde{T}. \tag{3.7}$$

Of course we have no guarantee that the transformation $\widetilde{T}$ will be different from zero. In fact, Schur's Lemma tells us that it must be zero if ρ and σ are non-isomorphic irreducible representations.

4. G-invariant matrices

Let $R : G \to GL_n$ and $S : G \to GL_m$ be matrix representations. An $m \times n$ matrix M is G-invariant if it has the property

(4.1) $$M = {S_g}^{-1} M R_g$$

for every g in G.

The averaging process will produce a G-invariant matrix starting with any matrix M:

(4.2) $$\widetilde{M} = \tfrac{1}{|G|} \textstyle\sum_g S_g^{-1} M R_g$$

is G-invariant. The proof is analogous to (3.7).

Suppose that we are given the representations ρ and σ, and that we also have bases for V and W. Let R and S be the matrix representations obtained from ρ and σ using these bases, and let M be the matrix of T. Then T is G-invariant if and only if M is G-invariant. And if M is the matrix of T, then $\widetilde{M}$ will be the matrix of $\widetilde{T}$.

Lemma 4.3. *With R, S as above, let Φ be the operator on the space $\mathbb{C}^{m\times n}$ of $m \times n$ matrices defined by $\Phi(M) = \widetilde{M}$.*
(a) Φ is a projection operator, and its image is the space of G-invariant matrices.
(b) Let R and S be the matrix representations associated to representations ρ and σ with resepct to some bases. If ρ and σ are non-isomorphic irreducible representations, then $\operatorname{trace}\Phi = 0$. *If ρ is irreducible, $\rho = \sigma$ and $R = S$, then* $\operatorname{trace}\Phi = 1$.

Proof. (a) It follows from its definition (4.2) that Φ is compatible with addition and scalar multiplication of matrices, so it is a linear operator. Moreover, $\Phi(\widetilde{M}) = \widetilde{M}$, so $\Phi^2 = \Phi$. The image of Φ is the space of G-invariant matrices.

(b) According to Lemma 2.1, the trace of Φ is equal to the dimension of the space of G-invariant operators. By Schur's Lemma, the only invariant operator is 0 if ρ and σ are non-isomorphic irreducible representations. Therefore the trace of Φ is zero in that case. Schur's Lemma also tells us that the space of G-invariant linear operators has dimension 1 if $\rho = \sigma$ and ρ is irreducible. In that case, the trace is 1. □

5. Proof of the orthogonality relations

We let R, S be matrix representations that correspond to two representations ρ, σ by choices of bases, and as before, we denote the characters of ρ, σ by χ, χ' respectively. Then we have the operator Φ, defined as in Lemma 4.3. Our plan is to prove the orthogonality relations by reinterpreting the trace of Φ. We will show that

(5.1) $$\operatorname{trace}\Phi = \langle \chi, \chi' \rangle \;=\; \tfrac{1}{|G|}\overline{\chi'(g)}\chi(g),$$

and when we have done this, the orthogonality relations (1.1a,b) will follow from Lemma 4.3b.

The linear operator Φ is an average of the operators φ_g defined by

(5.2) $$\varphi_g(M) = S_g^{-1} M R_g :$$

(5.3) $$\Phi = \tfrac{1}{|G|} \textstyle\sum_g \varphi_g.$$

Since trace is a linear function,

(5.4) $$\operatorname{trace}\Phi = \tfrac{1}{|G|} \textstyle\sum_g \operatorname{trace}\varphi_g.$$

We will prove (5.1) by showing that

(5.5) $$\operatorname{trace}\varphi_g = \overline{\chi'(g)}\chi(g).$$

This is accomplshed by the next two lemmas:

Lemma 5.6. *Let χ' be the character of a representation $\sigma : G \to GL(W)$ of a finite group G, and let S_g be the matrix of σ_g with respect to a basis of W. Then $\overline{\chi'(g)} = \chi'(g^{-1})$.*

Lemma 5.7. *Let A and B be $m \times m$ and $n \times n$ complex matrices, and let f be the linear operator on the space $\mathbb{C}^{m\times n}$ of complex $m \times n$ matrices that is defined by $f(M) = AMB$. The trace of f is the product $(\operatorname{trace} A)(\operatorname{trace} B)$.*

Let's assume the lemmas for the moment. By definition, $\chi(g) = \operatorname{trace} R_g$, and $\chi'(g) = \operatorname{trace} S_g$. Moreover, $S_g^{-1} = S_{g^{-1}}$ because $S : G \to GL_m$ is a homomorphism. Lemma 5.6 tells us that $\overline{\chi'(g)} = \chi'(g^{-1}) = \operatorname{trace} S_g^{-1}$. Then if we apply Lemma 5.7, we find that

$$\operatorname{trace} \varphi_g = (\operatorname{trace} S_{g^{-1}})(\operatorname{trace} R_g) = \overline{\chi'(g)}\chi(g). \tag{5.8}$$

Then

$$\operatorname{trace} \Phi = \tfrac{1}{|G|} \textstyle\sum_g \operatorname{trace} \varphi_g = \tfrac{1}{|G|} \sum_g \overline{\chi'(g)}\chi(g) = \langle \chi', \chi \rangle. \tag{5.9}$$

So the proof of the orthogonality relations will be complete, once the lemmas are proved.

Proof of Lemma 5.6. By definition, $\chi'(g)$ is the sum of the eigenvalues of σ_g or of S_g, say $\chi'(g) = \lambda_1 + \cdots + \lambda_m$. The eigenvalues of $S_{g^{-1}}$ are the inverses λ_i^{-1}. Now since g is an element of a finite group, it has finite order. Therefore S_g is a matrix of finite order, i.e., $(S_g)^r = I$ for some $r > 0$. Then $(\lambda_i)^r = 1$ for each i. This implies that λ_i are complex numbers of absolute value 1. And if λ is a complex number of absolute value 1, then $\lambda^{-1} = \overline{\lambda}$. Therefore $\chi'(g^{-1}) = \lambda_1^{-1} + \cdots + \lambda_m^{-1} = \overline{\lambda}_1 + \cdots + \overline{\lambda}_m = \overline{\chi'(g)}$. □

Proof of Lemma 5.7. We want to show that for any $m \times m$ matrix A and any $n \times n$ matrix B, the trace of the operator $f(M) = AMB$ on $\mathbb{C}^{m\times n}$ is equal to $(\operatorname{trace} A)(\operatorname{trace} B)$. One can order the $m \times n$ matrix units e_{ij} in some way and write down the matrix of f. This is boring and notationally confusing.

The operator f depends linearly on the matrix A. By this we mean that if $A = A_1 + A_2$, then with the obvious notation, $f = f_1 + f_2$. And if $A = cA'$, then $f = cf'$. Since trace is a linear function, $\operatorname{trace} f$ is also linear: $\operatorname{trace} f = \operatorname{trace} f_1 + \operatorname{trace} f_2$ and $\operatorname{trace} f = c \operatorname{trace} f'$. Then since any matrix A is a linear combination $\sum m_{ij}e_{ij}$ of $m \times m$ matrix units, it is enough to prove the lemma in the case that A is itself a matrix unit. Similarly, we may assume that B is an $n \times n$ matrix unit.

We note that $f^2(M) = A^2MB^2$.

We'll distinguish three cases.

Case 1: $A = e_{ii}$ and $B = e_{jj}$. Then $f(M) = e_{ii}Me_{jj} = m_{ij}e_{ij}$, and $f^2 = f$. Here f is a projection operator, and its image is the one dimensional space spanned by e_{ij}, so $\operatorname{trace} f = 1$. Also $(\operatorname{trace} A)(\operatorname{trace} B) = 1 \cdot 1 = 1$. The lemma is verified in this case.

Case 2: $A = e_{ik}$ and $i \neq k$. In this case $A^2 = 0$, so $f^2 = 0$; f is a nilpotent operator. The trace of a nilpotent operator is zero, so $\operatorname{trace} f = 0$. Since $\operatorname{trace} A = 0$ too, the lemma is verified in this case as well.

Case 3: $B = e_{j\ell}$ and $j \neq \ell$. The reasoning is analogous to that of Case 2. □