Character table for a nonabelian group of order 55

Let G be a group of order 55, generated by two elements x, y with the relations $x^{11} = 1$, $y^5 = 1$, $yxy^{-1} = x^3$. Let's skip the verification that this group actually has order 55.

The Sylow theorems tell us that the 11-subgroup $< x >$ is normal, and that there are 11 conjugate 5-subgroups, one of which is $<y>$.

The centralizer $Z(x)$ of x contains $< x >$ and it is not the whole group. So $Z(x) =< x >$, and the counting formula shows that $|C(x)| = 5$. Similarly, $|C(y)| = 11$.

We can list the elements in the conjugacy class $C(x)$. The relation $yxy^{-1} = x^3$ shows that x^3 is in that class. Then $yx^3y^{-1} = x^9$ is also in the class. Continuing this way, we find that

$$C(x) = \{x, x^3, x^9, x^5, x^4\}. \tag{1}$$

The other powers of x not equal to 1 form the conjugacy class of x^2:

$$C(x^2) = \{x^2, x^6, x^7, x^{10}, x^8\}. \tag{2}$$

On the other hand, any two of the 11 Sylow 5-subgroups intersect in the identity. So each of those subgroups contains an element in the conjugacy class of y. The classes of y^2, y^3, y^4 also have order 11 each. The class equation of G is

$$55 = 1 + 5 + 5 + 11 + 11 + 11 + 11. \tag{3}$$

So there are seven irreducible characters.

Since $<x>$ is a normal subgroup, we may form the quotient group $G/<x>= \overline{G}$. Counting shows that $\overline{G}$ has order 5. So $\overline{G}$ is generated by the residue of any element not in $<x>$, such as the residue $\overline{y}$ of y.

The canonical homomorphism $\pi : G \to \overline{G}$ is useful for determining some characters of G. Namely, if $\overline{\rho} : \overline{G} \to GL(V)$ is any representation of $\overline{G}$, we obtain a representation of G as $\rho = \overline{\rho} \circ \pi$.

Since $\overline{G}$ is abelian, its irreducible characters are one-dimensional, and there are five of them. Since $\overline{y}$ has order 5, the value of a one-dimensional character on $\overline{y}$ must be a fifth root of unity, a power of $\zeta = e^{2\pi i/5}$. There are five such powers and five irreducible characters, so the table for $\overline{G}$ is determined.

size	(1)	(1)	(1)	(1)	(1)
elt	1	$\overline{y}$	$\overline{y}^2$	$\overline{y}^3$	$\overline{y}^4$
$\overline{\chi}_1$	1	1	1	1	1
$\overline{\chi}_2$	1	ζ	ζ^2	ζ^3	ζ^4
$\overline{\chi}_3$	1	ζ^2	ζ^4	ζ	ζ^3
$\overline{\chi}_4$	1	ζ^3	ζ	ζ^4	ζ^2
$\overline{\chi}_5$	1	ζ^4	ζ^3	ζ^2	ζ

Each of these characters $\overline{\chi}_i$ determines a character χ_i of the group G, and χ_i takes the value 1 on the elements in the kernel of π, i.e., on the subgroup $<x>$.

The first five rows in the character table below have been determined, and the formula $|G| = d_1^2 + \cdots + d_7^2$ and the fact that d_i divides $|G|$ show that the dimensions of the remaining two characters are 5.

size	(1)	(5)	(5)	(11)	(11)	(11)	(11)
elt	1	x	x^2	y	y^2	y^3	y^4
χ_1	1	1	1	1	1	1	1
χ_2	1	1	1	ζ	ζ^2	ζ^3	ζ^4
χ_3	1	1	1	ζ^2	ζ^4	ζ^3	ζ
χ_4	1	1	1	ζ^3	ζ	ζ^4	ζ^2
χ_5	1	1	1	ζ^4	ζ^3	ζ^2	ζ
χ_6	5	u	v	?	?	?	?
χ_7	5	?	?	?	?	?	?

Next, we determine the values labeled u and v: since χ_6 is a five-dimensional character and since x has order eleven, $\chi_6(x)$ is a sum of five 11th roots of unity, powers of $\eta = e^{2\pi i/11}$. Here the fact that we know the conjugacy class explicitly is very useful. Namely, ρ_{x^3} is a conjugate of ρ_x, so it has the same eigenvalues. One the other hand, $\rho_{x^3} = \rho_x^3$. Therefore its eigenvalues are the cubes of the eigenvalues of ρ_x. So if η^ν is an eigenvalue of ρ_x, $\eta^{3\nu}$ is also an eigenvalue. Looking at this, one sees that there are only three possibilities for $\chi_6(x)$: It must be equal to one of: $5 = 1+1+1+1+1$, or $\alpha = \eta + \eta^3 + \eta^9 + \eta^5 + \eta^4$, or $\beta = \eta^2 + \eta^6 + \eta^7 + \eta^{10} + \eta^8$. Now the value 5 would contribute $5^2 \cdot 5 = 125$ to $\sum \overline{\chi_6(g)}\chi_6(g)$. But this is a sum of positive terms totalling $|G| = 55$, because $\langle \chi_6, \chi_6 \rangle = 1$. So $\chi_6(x) = 5$ is too large, and therefore $u = \alpha$ or $u = \beta$. Since there are two characters of dimension 5, both values must be taken on. The bottom rows of the character table are

size	(1)	(5)	(5)	(11)	(11)	(11)	(11)
elt	1	x	x^2	y	y^2	y^3	y^4
χ_6	5	α	β	0	0	0	0
χ_7	5	β	α	0	0	0	0