Quantum Steenrod operations ← part joint work with Wilkins

Recall

$H^*(B\mathbb{Z}/p;\mathbb{F}_p)$

$/\!/$

$$H^*_{\mathbb{Z}/p}(\text{point};\mathbb{F}_p) = \mathbb{F}_p[[t,\theta]] \leftarrow \left[\begin{array}{ll} |t|=2, & |\theta|=1 \\ \underline{p=2}: & \theta^2=t \\ \underline{p>2}: & \theta t = t\theta,\ \ \theta^2=0 \end{array}\right.$$

is one-dimensional in each degree ≥ 0.

The ordinary Steenrod operation

$$St: H^k(M;\mathbb{F}_p) \longrightarrow \left(H^*(M;\mathbb{F}_p)[[t,\theta]]\right)^{kp}$$

$$St(x) = x^p + (\text{terms with a } t \text{ or } \theta)$$

alternatively,

$$St(x) = \pm\, t^{\frac{p-1}{2}|x|}\, x + (\text{terms in } H^*(M),\ * > |x|).$$

$\longrightarrow$ for $p=2$, $t^{1/2}=\theta$.

From now on, let M be a closed monotone symplectic manifold. The quantum Steenrod operation is originally due to Fukaya

$$QSt : H^k(M;\mathbb{F}_p) \longrightarrow \left(H^*(M;\mathbb{F}_p)[[t,\theta]]\right)^{pk}$$

graded mod 2, with curves in class A contributing with grading $-2c_1(A)$

$$QSt(x) = \underbrace{x * \cdots * x}_{p} + (\text{terms with } \theta \text{ or } t)$$

p-fold quantum product

<u>Example</u> $M = S^2$, $p=2$.

$$QSt(1) = 1, \quad QSt(\underset{H^2(M;\mathbb{F}_p)}{\text{point}}) = \overbrace{t\,\text{point}}^{St} + \overbrace{1}^{\text{point} * \text{point}}$$

more computation: Wilkins' papers

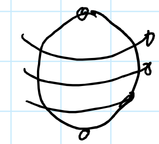
Application to Hamiltonian dynamics

Suppose $H_*(M)$ is torsion-free. A Hamiltonian diffeomorphism $\phi: M \to M$ is called a "nondegenerate pseudorotation" if

- the only periodic points of ϕ are fixed points
- $\phi(x) = x \Rightarrow D\phi_x$ has no $\sqrt{1}$ as eigenvalues

  x nondegenerate as a periodic point
- $|\mathrm{Fix}\,\phi| = \operatorname{rank} H_*(M)$

Ex.



irrational rotation of $M = S^2$

Thm (Salamon-Zehnder) $c_1(M) = 0$, then M can't admit pseudo-rotations.

Thm (Ginzburg-Gürel) $c_1(M) = -[\omega_M]$, then M can't admit pseudo-rotations.

Theorem (Cineli–Ginzburg–Gürel; Shelukhin)

Suppose that for some p, $QSt([\text{point}]) = St([\text{point}])$.

Then M can't admit a pseudorotation.

Hence, we should focus on manifolds with lots of rational curves (rational curves through every point).

$QSt([\text{point}]) = t^2[\text{point}]$

Example (S.-Wilkins) The cubic surface ($\mathbb{CP}^2$ blown up at 6 points, with its monotone symplectic form) does not admit a pseudorotation.

$p=2$ computation

Example

T^4 blown up at a point has no pseudorotations

Example

$\mathbb{CP}^2$ blown up at ≤ 3 points, with $[\omega_M] = c_1(M)$, is toric and hence has a pseudorotation

Formal structure (this description is inaccurate in many ways)

Gromov-Witten theory

$$\begin{tikzcd} & \overline{\mathcal{M}}_{0,p+1}(M;A) \arrow[dl, "\text{evaluate at } p"'] \arrow[dr, "\text{evaluate, remaining}"] & \\ M^p & & M \times \overline{\mathcal{M}}_{0,p+1} \end{tikzcd}$$

Deligne-Mumford

$$\begin{tikzcd} & H^*(\overline{\mathcal{M}}_{0,p+1}(M;A)) \arrow[dr, "\text{“pushforward”}"] & \\ H^*(M)^{\otimes p} \arrow[ur, "\text{pullback}"] & & H^*(M) \otimes H^*(\overline{\mathcal{M}}_{0,p+1}) \end{tikzcd}$$

Sym_p-equivariant theory (permuting p marked points)

local $\mathbb{F}_p$-coefficients

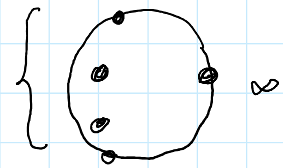
$$\begin{tikzcd} & H^*_{Sym_p}(\overline{\mathcal{M}}_{0,p+1}(M;A);\pm 1) \arrow[dr, "\text{“pushforward”}"] & \\ H^*_{Sym_p}(M^p;\pm 1) \arrow[ur, "\text{pullback}"] & & H^*_{Sym_p}(M \times \overline{\mathcal{M}}_{0,p+1};\pm 1) \arrow[d, equal] \\ H^*(M;\mathbb{F}_p) \arrow[u, "\text{equivariant } p\text{-th power}"] \arrow[rr, dashed, "\text{quantum Steenrod}"] & & H^*(M;\mathbb{F}_p) \otimes H^*_{Sym_p}(\overline{\mathcal{M}}_{0,p+1};\pm 1) \end{tikzcd}$$

This means that operations are parametrized by $H_*^{Symp}(\overline{\mathcal{M}}_{0,p+1}; \pm 1)$, which is unfortunately unknown. But we have the unique non-free orbit $O \subset \overline{\mathcal{M}}_{0,p+1}$, with stabilizer $\mathbb{Z}/p$,

to me

and correspondingly can specialize the operations to

p-th roots of 1



non-free orbit

QSt = equivariant GW-theory of this curve.

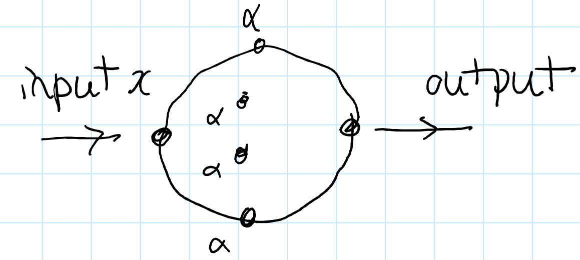
$$H^*(M;\mathbb{F}_p) \longrightarrow H^*(M;\mathbb{F}_p) \otimes H^*_{Symp}(O_p; \pm 1)$$

QSt

$$\cong H^*(M;\mathbb{F}_p) \otimes H^*_{\mathbb{Z}/p}(\text{point}) = H^*(M;\mathbb{F}_p)[[t,\theta]].$$

By localization, $H^*_{Symp}(O)$ is "most of" $H^*_{Symp}(\overline{\mathcal{M}}_{0,p+1})$.

Looking at the geometry, we see that there is a further natural operation,



$$Q\Sigma : H^*(M;\mathbb{F}_p)\otimes H^*(M;\mathbb{F}_p) \longrightarrow H^*(M;\mathbb{F}_p)[[t,\theta]]$$

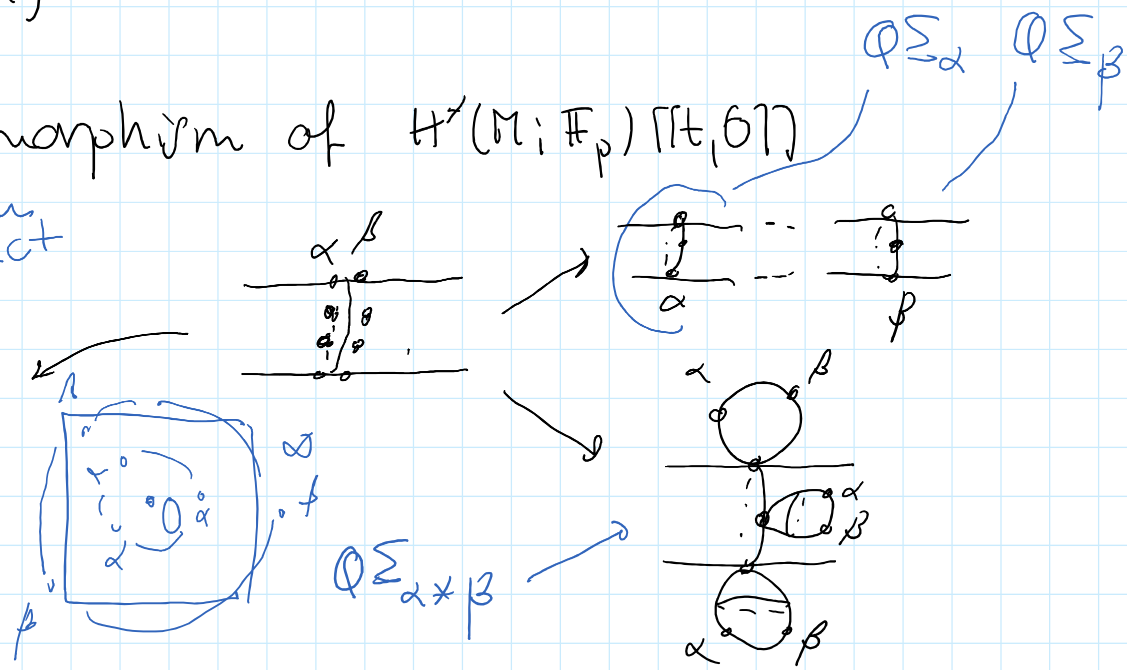
$$(\alpha, x) \longmapsto Q\Sigma_\alpha(x)$$

Let's extend $Q\Sigma$ to an endomorphism of $H^*(M;\mathbb{F}_p)[[t,\theta]]$

$$\begin{cases} Q\Sigma_\alpha(1) = QSt(\alpha) \\ Q\Sigma_\alpha \circ Q\Sigma_\beta = Q\Sigma_{\alpha * \beta} \end{cases}$$

quantum product

$$\Rightarrow \quad Q\Sigma_\alpha(QSt(\beta)) = QSt(\alpha * \beta)$$



Theorem (S-Wilkins) For any α, $Q\Sigma_\alpha$ is covariantly constant with respect to the quantum connection.

For that to make sense, we define our cohomology over a ring

$$\sum_{\substack{A \in H_2(M;\mathbb{Z}) \\ A=0 \text{ or } \omega_M(A)>0}} c_A q^A, \qquad c_A \in \mathbb{F}_p$$

Any $\beta \in H^2(M;\mathbb{Z})$ gives an operation

$$\partial_\beta (q^A) = (\beta \cdot A)\, q^A$$

The quantum connection is

$$\nabla_\beta = t\partial_\beta + \beta * \cdot$$

+ classical Steenrod

Cor Gromov-Witten theory determines $Q\Sigma$ up to the ideal formed by q^A, $A \neq 0$, $A\cdot\beta \equiv 0$ (p) for all β. $\to$ $\partial_\beta q^A = 0$ $\forall \beta$

First case that eludes computation: double covers of (-1)-curves in 4d (p=2).

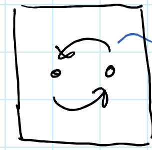
Homological algebra

Recall that quantum Steenrod operation on $H^k(M;\mathbb{F}_p)$ are parametrized by $H_*^{Sym_p}(\overline{\mathcal{M}}_{0,p+1};(-1)^k)$.

There is a classical analogue, Cohen's computation of the equivariant homology of configuration space. Essentially the only interesting class is

$$H_{p-1}^{Sym_p}(\mathrm{Conf}_p(\mathbb{C});-1) \cong \mathbb{F}_{p-1}.$$

Example $p=2$, $H_1\left(\frac{\mathrm{Conf}_2(\mathbb{C})}{Sym_2};\mathbb{F}_p\right)$ contains the cycle



move points until they exchange position

Now,

$$\begin{array}{c}
H_{p-1}^{Sym_p}(\mathrm{Conf}_p(\mathbb{C});-1) \\
\downarrow \\
H_{p-1}^{Sym_p}(\overline{\mathcal{M}}_{0,p+1};-1) \\
\uparrow \\
H_{p-1}^{\mathbb{Z}/p}(\text{point};\mathbb{F}_p)
\end{array}$$

hit the same image

$= H_{p-1}^{Sym_p}(0;-1)$

The consequence is that

$$Q\Xi = t^{\frac{p-1}{2}}\text{-coefficient of } QSt,$$

acting on $H^{odd}(M;\mathbb{F}_p)$

(if $p>2$)

has an elementary meaning in homological algebra, as part of the operations carried by any (E_2)-algebra $\mathbb{F}_p$, such as the Hochschild cohomology of an algebra over $\mathbb{F}_p$.

Example A algebra over $\mathbb{F}_p$; if $D: A \to A$ is a derivation, then so is

$$D^p = D \circ \cdots \circ D.$$

$$[D] \in HH^1(A) \longmapsto [D^p] \in HH^1(A)$$

For $p=3$,

$$D^3(ab) = D^2(aD(b) + D(a)b)$$

$$= D\big(D(a)D(b) + aD^2(b) + D(a)D(b) + D^2(a)b\big)$$

$$= aD^3(b) + D^3(a)b + \cancel{3D^2(a)D(b)} + \cancel{3D(a)D^2(b)}.$$

For symplectic geometry, this means that if $\mathcal{F}(M)$ is the Fukaya category over $\mathbb{F}_p$, then

$$
\begin{tikzcd}
H^{odd}(M;\mathbb{F}_p) \arrow[r, "Q\Xi"] \arrow[d, "CO", "\substack{\text{closed-}\\\text{open map}}"'] & H^{odd}(M;\mathbb{F}_p) \arrow[d, "CO"] \\
HH^{odd}(\mathcal{F}(M)) \arrow[r, "\substack{\text{algebraic}\\\text{operation}}"'] & HH^{odd}(\mathcal{F}(M)).
\end{tikzcd}
$$

Does not use the Calabi-Yau structure of the Fukaya category

There is an explicit expression for the algebraic operations (Tourtchine)

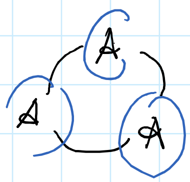
This particular operation is easy to construct on (non-S^1-equivariant) symplectic cohomology or string topology

"Exercise" Look at superpotentials with 1d critical locus.

Different homological algebra interpretation $A = A_\infty$-algebra / $\mathbb{F}_p$

$$HH_*^{\mathbb{Z}/p}(A) \cong HH_*^{S^1}(A) \otimes_{\mathbb{F}_p[[t]]} \mathbb{F}_p[[t,\theta]]$$

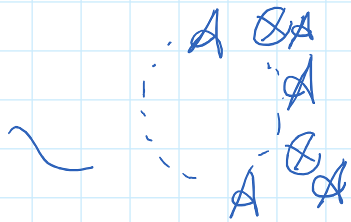
$\mathbb{Z}/p$-equivariant version of Hochschild homology, easy to construct through cyclic tensor product of bimodules



$\mathbb{Z}/p$ acts

(negative) cyclic homology

acting



We take the p-fold module structure of $HH^*(A)$ acting on $HH_*(A)$, and that is naturally $\mathbb{Z}/p$-equivariant (think of it as endomorphisms of the bimodule A acting on the cyclic tensor product)

The outcome is expected to be $\mathcal{A} = \mathrm{Fuk}(M)$

$$\begin{array}{ccc}
H^*(M) \otimes H^*(M)[[t,\theta]] & \xrightarrow{Q\Sigma} & H^*(M)[[t,\theta]] \\
\Big\downarrow {\scriptstyle OC \otimes OC_{S^1}} & & \Big\downarrow {\scriptstyle OC_{S^1}} \\
\boxed{HH^*(\mathcal{A}) \otimes HH_*^{S^1}(\mathcal{A})[\theta]} & \xrightarrow[\text{structure}]{\mathbb{Z}/p\text{-equivariant module}} & \boxed{HH_*^{S^1}(\mathcal{A})[\theta]}
\end{array}$$

$$\qquad // \; HH_*^{\mathbb{Z}/p}(\mathcal{A})$$

The image of $1 \in H^0(M)$ under OC_{S^1} describes the Calabi-Yau structure of the Fukaya category (Ganatra). We can specialize to that class to get

$$HH^*(\mathcal{A}) \longrightarrow HH_*^{S^1}(\mathcal{A})[\theta]$$

If we use the CY structure and a splitting of the Hodge-de Rham spectral sequence, we can get "algebraic Steenrod operations" on $HH^*(\mathcal{A},\mathcal{A}) \cong HH_*(\mathcal{A},\mathcal{A})$.

What might this look like concretely?

On categories $D^b Coh(X)$, $p > \dim(X)$,

$HH^0(X) \cong H^0(X, \mathcal{O}_X) \ni f$ acts on the de Rham complex by

$$\eta \longmapsto f^p \eta$$

(note $d(f^p \eta) = f^p d\eta + \cancel{p f^{p-1} df \wedge \eta}$)

If we have $\xi \in H^0(X, T_X) \subseteq HH^1(X)$, that should act on the de Rham complex (enhanced with t's and θ) by operations of degree p.

$\underline{p=2}$ $\quad \xi^2$ = square of our vector field as a derivation

$$\eta \longmapsto \Big(\overbrace{L_\xi \iota_\xi \eta + \iota_{\xi^2} \eta}^{\text{commutes with de Rham } d}\Big)\theta .$$

after applying $d \circ - \circ d$

$$L_\xi L_\xi \eta \qquad\qquad L_{\xi^2} \eta .$$