IHES, June 2014

NONCOMMUTATIVE GEOMETRY OF LEFSCHETZ PENCILS

Paul Seidel, MIT

Pencils of hypersurfaces

X smooth projective variety $/\mathbb{C}$

$L \to X$ line bundle

$s_0, s_\infty \in \Gamma(L)$ sections, whose common zero locus B has codimension ≥ 2

"base locus"

$\pi = \frac{s_0}{s_\infty} : X \dashrightarrow \mathbb{P}^1$ has "fibres"

$X_z = \{ s_0(x) = z\, s_\infty(x) \} \supseteq B$

also use $W = \frac{s_0}{s_\infty} : X \setminus X_\infty \longrightarrow \mathbb{C}$

with fibres $X_z \setminus B$

(Symplectic) topology

Vanishing cycles

Monodromy

Fukaya category

(does not use complex structure)

Algebra(ic geometry)

(also over $\mathbb{K} \neq \mathbb{C}$)

Deformation theory

Gauß-Manin connection

Coherent sheaves

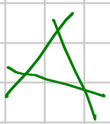
Classical example of Mirror Symmetry

TOPOLOGY

$X = \mathbb{CP}^2$, $L = K_X^{-1} = \mathcal{O}(3)$

$s_\infty = x_0 x_1 x_2$,

$s_0 =$ generic cubic $\pitchfork X_\infty$

$X_\infty =$  chain of 3 $(+1)$-curves

$B = 9$ points

$X_t = T^2$ a torus

$X \setminus X_\infty = \mathbb{C}^* \times \mathbb{C}^*$

W has 9 nondegenerate critical points

$\xleftrightarrow{\text{mirror}}$

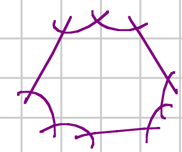
ALGEBRA

action of $\mathbb{Z}/3 \subseteq SL_3(\mathbb{C})$

$X^\vee =$ minimal resolution of $\mathbb{CP}^2/(\mathbb{Z}/3)$, $L^\vee = K_{X^\vee}^{-1}$

$s_\infty^\vee = x_0 x_1 x_2$

$s_0^\vee = x_0^3 + x_1^3 + x_2^3$

$X_\infty^\vee =$  chain of 9 $(-1)/(-2)$-curves

$B^\vee = 3$ points

$X_t^\vee =$ elliptic curve (torus)

$X^\vee \setminus X_\infty^\vee = \mathbb{C}^* \times \mathbb{C}^*$

$W^\vee(z_1, z_2) = z_1 + z_2 + \frac{1}{z_1 z_2}$ has 3 nondegenerate critical points

Homological Mirror Symmetry involves equivalences of categories

- $\mathcal{F}(W)$, the Fukaya category associated to $W : X \setminus X_\infty \longrightarrow \mathbb{C}$
- $\mathcal{F}(X_z \setminus B)$, the Fukaya category of the fibre minus base locus
- $\mathcal{F}(X_z, B)$, the relative Fukaya category. The parameter q counts intersections with B.

two q's are related by a nontrivial change of variables

- $D(X^\vee)$, the bounded derived category of coherent sheaves
- $\mathrm{Perf}(X^\vee_\infty) \subseteq D(X^\vee_\infty)$, the category of perfect complexes
- $\mathrm{Perf}(\mathcal{X}^\vee_\infty) \leftarrow \mathcal{X}^\vee_\infty$ is a scheme over $\mathbb{C}[[q]]$, namely the fibre of the pencil over a formal disc near $\infty \in \mathbb{P}^1$ ($\longleftrightarrow q=0$). This is a formal one-parameter deformation of $X^\vee_\infty$.

- $\mathcal{F}(X_z)$ (or a suitable full subcategory, where q is the "Novikov parameter")

- $\mathcal{W}(X \setminus X_\infty)$, $\mathcal{W}$ = "wrapped"

- $\mathcal{F}_z(X)$ with "bulk parameter" $z \in \mathbb{C}$

in most cases, certain formal completions are necessary, which we have omitted

- ??

- $D(\mathcal{X}^\vee_{\infty,*})$, where $\mathcal{X}^\vee_{\infty,*}$ is the generic fibre of our deformation. This is over $\mathbb{C}((q))$.

- $D(X^\vee \setminus X^\vee_\infty)$

$\cong \mathrm{Sing}(W^{-1}(z))$
$= D/\mathrm{Perf}$

- $LG(W-z)$, the Landau-Ginzburg category, $z \neq \infty$. Zariski-locally, objects are 2-periodic "complexes"

$$\cdots \to \mathcal{E}_0 \xrightarrow{d_0} \mathcal{E}_1 \xrightarrow{d_1} \mathcal{E}_0 \xrightarrow{d_0} \mathcal{E}_1 \to \cdots$$

$$d_0 d_1 = (W-z) \cdot \mathrm{id} = d_1 d_0$$

- $D(X^\vee_z)$ or $\mathrm{Perf}(X^\vee_z)$, $z \neq \infty$

The same, with names (not a complete account)

Kontsevich, S. • $\mathcal{F}(W)$, the Fukaya category associated to $W : X \setminus X_\infty \longrightarrow \mathbb{C}$	Auroux–Katzarkov–Orlov $\longleftrightarrow$	Verdier • $D(X^\vee)$, the bounded derived category of coherent sheaves
Fukaya • $\mathcal{F}(X_z \setminus B)$, the Fukaya category of the fibre minus base locus	folk? $\longleftrightarrow$	Grothendieck? • $\mathrm{Perf}(X_\infty^\vee) \subseteq D(X_\infty^\vee)$, the category of perfect complexes
S. • $\mathcal{F}(X_z, B)$, the relative Fukaya category. The parameter q counts intersections with B.	Kontsevich Polishchuk–Zaslow $\longleftrightarrow$	• $\mathrm{Perf}(\mathcal{X}_\infty^\vee) \leftarrow \mathcal{X}_\infty^\vee$ is a scheme over $\mathbb{C}[[q]]$, namely the fibre of the pencil over a formal disc near $\infty \in \mathbb{P}^1$ ($\longleftrightarrow q=0$). This is a formal one-parameter deformation of $X_\infty^\vee$.

Fukaya-Oh-Ohta-Ono

- $\mathcal{F}(X_z)$ (or a suitable full subcategory, where q is the "Novikov parameter") $\longleftrightarrow$ Kontsevich, Polishchuk-Zaslow
- $D(\mathcal{X}^\vee_{\infty,*})$, where $\mathcal{X}^\vee_{\infty,*}$ is the generic fibre of our deformation. This is over $\mathbb{C}((q))$.

Abouzaid-S.

- $\mathcal{W}(X \setminus X_\infty)$, $\mathcal{W}$ = "wrapped" $\longleftrightarrow$ Abouzaid-S. (partially, to be completed)
- $D(X^\vee \setminus X^\vee_\infty)$

Eisenbud, ... Buchweitz, Orlov

Fukaya, Oh

- $\mathcal{F}_z(X)$ with "bulk parameter" $z \in \mathbb{C}$ $\longleftrightarrow$ Cho, Abouzaid-Fukaya-Oh-Ohta-Ono
- $LG(W-z)$, the Landau-Ginzburg category, $z \neq \infty$. Zariski-locally, objects are 2-periodic "complexes"

$$\cdots \to \mathcal{E}_0 \xrightarrow{d_0} \mathcal{E}_1 \xrightarrow{d_1} \mathcal{E}_0 \xrightarrow{d_0} \mathcal{E}_1 \to \cdots$$

$$d_0 d_1 = (W-z)\cdot id = d_1 d_0$$

$\boxed{\text{In most cases, certain formal completions are necessary, which we have omitted}}$

- ??
- $D(X^\vee_z)$ or $Perf(X^\vee_z)$, $z \neq \infty$

Relations between categories

$$\begin{tikzcd}
\mathcal{F}(X_z) & & \mathcal{F}(X_z, B) \arrow[ll, Leftarrow] \\
\mathcal{F}(W) \arrow[rr, bend left] & & \mathcal{F}(X_z \setminus B) \arrow[ll, bend left] \arrow[u, dashed, Rightarrow, "\begin{array}{l}\text{disc count}\\ \text{in } (X_z, B)\end{array}"'] \\
\mathcal{W}(X \setminus X_\infty) \arrow[from=2-1, Rightarrow, "\begin{array}{l}\text{(Abouzaid-S.)}\\ \text{acceleration functor}\end{array}"] \arrow[d, dashed, Rightarrow, "\begin{array}{l}\text{(idea of Auroux et al.)}\\ \text{holomorphic disc count}\\ \text{in } (X, X_\infty)\end{array}"] & & \\
\mathcal{F}_z(X) & &
\end{tikzcd}$$

(in general, only a version of) $\mathcal{F}_z(X)$

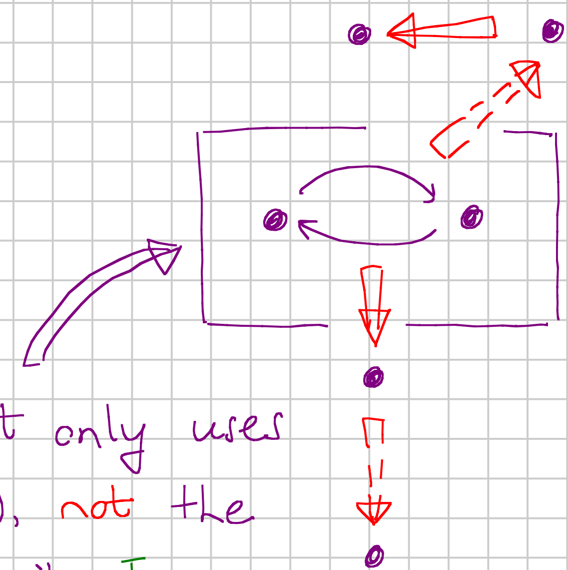
$$\begin{tikzcd}
D(\mathcal{X}^\vee_\infty, *) & & Perf(\mathcal{X}^\vee_\infty) \arrow[ll, Leftarrow, "\begin{array}{c}\text{invert}\\ \text{parameter}\end{array}"'] \\
D(X^\vee) \arrow[rr, bend left, "\text{restrict}"] & & Perf(X^\vee_\infty) \arrow[ll, bend left, "\text{include}"] \arrow[u, dashed, Rightarrow, "\text{deformation}"'] \\
D(X^\vee \setminus X^\vee_\infty) = D(X^\vee)/Perf(X^\vee_\infty) \arrow[from=2-1, Rightarrow, "\text{(categorical) localization}"] \arrow[d, dashed, Rightarrow, "\begin{array}{l}\text{deformation (with formal}\\ \text{parameter of degree 2),}\\ \text{then invert the}\\ \text{deformation parameter}\end{array}"] & & \\
LG(W - z) & &
\end{tikzcd}$$

Problem

The picture in the previous slide is fundamentally unsatisfactory –

- The different parts of the structure do not really determine each other

- What is the central object? Geometrically, it is the pencil itself, but none of the categories we have listed corresponds fully to it.



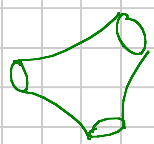
This part only uses $(\check{X}, \check{X}_\infty)$, not the full pencil. In (symplectic) topological terms, it only uses $W : X \setminus X_\infty \to \mathbb{C}$, ignoring the fact that this extends to a pencil over $\mathbb{CP}^1$.

A branched cover (in the manner of Abouzaid - Auroux - Gross - Katzarkov - Ruddat - ...)

$X = S^2$, $\pi : X \to S^2$ a generic degree 3 branched cover ($\infty \in S^2$ is not a branch point). Hence,

$X_\infty \cong \{1, 2, 3\} \cong X_z$

for generic z

$X \setminus X_\infty =$ 

$$\begin{array}{c} X \\ {\scriptstyle 3:1}\downarrow \\ \mathbb{C} \end{array}$$

W has 4 branch points

Branch data:

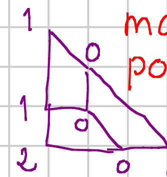
(1 2) (1 2) (2 3) (2 3)

HI DENIS!

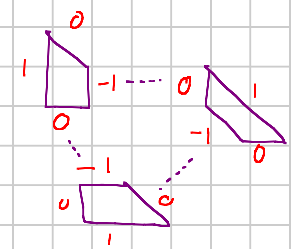
$\longleftrightarrow$ mirror

$X^\vee \xrightarrow{\mathcal{U}} \mathbb{C}$ is a 3-dimensional toric degeneration with general fibre $\mathbb{P}^2$ and special fibre $\cup_3 \mathbb{F}_1$

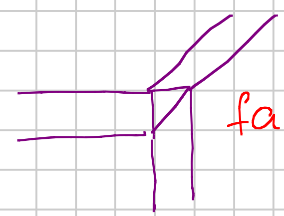
It still comes with $\pi^\vee : X^\vee \dashrightarrow \mathbb{P}^1$.



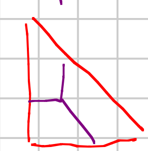
moment polytope $\longleftarrow$



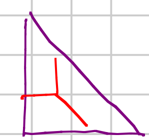
$L^\vee|_{\mathbb{F}_1} =$ pullback of $\mathcal{O}_{\mathbb{P}^2}(1)$ via the blowdown map $\mathbb{F}_1 \to \mathbb{P}^2$,



fan

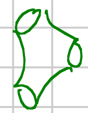


$X^\vee_\infty \cap \mathcal{U}^{-1}(0)$



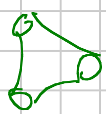
$X^\vee_0 \cap \mathcal{U}^{-1}(0)$

Categorical relationships

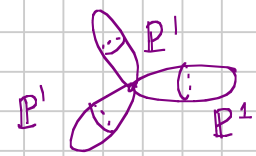
- $\mathcal{F}($ $W:$  $\longrightarrow \mathbb{C})$ is described by a quiver with relations

$$\begin{tikzcd} \bullet \arrow[r, bend left, "1"] \arrow[r, bend right, "2"'] & \bullet \arrow[r, "2"] & \bullet \arrow[r, bend left, "2"] \arrow[r, bend right, "3"'] & \bullet \end{tikzcd}$$

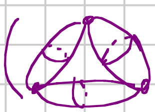
relations $ij = 0$ if $i \neq j$

- $\mathcal{W}($  $)$
- $\mathcal{F}(3 \text{ points}) = \bigoplus_3 \mathcal{F}(\text{point})$ is semisimple
- $\mathcal{F}(S^2, z)$ nonzero for $z = \pm 2$

- $LG(u) = Sing(u^{-1}(0))$. Thm (Orlov): after formal (Karoubi) closure, it depends only on the formal neighbourhood of the singular set



$\subseteq u^{-1}(0)$

- $LG(u \mid X^\vee \setminus X^\vee_\infty) = LG(\mathbb{C}^3 \xrightarrow{-x_1x_2x_3} \mathbb{C})$
- $LG(u \mid X^\vee_\infty) = Sing($  $)$. Use Orlov's thm + Knörrer periodicity
- $LG(u \mid X^\vee \setminus X^\vee_\infty + W^\vee - z) =$ $LG(-x_1x_2x_3 + x_1 + x_2 + x_3 - z : \mathbb{C}^3 \longrightarrow \mathbb{C})$

General algebraic framework

- In which sense can the previous algebro-geometric situation

$$\left(\begin{array}{ccc} X^\vee & \xrightarrow{u} & \mathbb{A}^1 \\ \downarrow \pi^\vee & & \\ \mathbb{P}^1 & & \end{array} \right)$$

be thought of as "a pencil in Landau-Ginzburg theory"?

- This question extends to other "deformations" of algebraic geometry (e.g. noncommutative)

Given an algebraic variety X and line bundle L, a divisor $s^{-1}(0)$, $0 \neq s \in H^0(X, L^{-1})$, gives rise to a dg scheme structure

$$\Lambda^*(L) = \mathcal{O}_X \oplus L[1] \longrightarrow X$$

differential is s

This differential graded scheme is quasi-isomorphic to the hypersurface $s^{-1}(0)$. One can think of a pencil analogously, as a family of dg scheme extensions of $\mathcal{O}_X$ by L, parametrized by two homogeneous coordinates.

Noncommutative (algebraic) geometry

Let $\mathcal{A}$ be an A_∞-algebra, and $\mathcal{P}$ an $\mathcal{A}$-bimodule which is invertible (for $\otimes_{\mathcal{A}}$). A noncommutative divisor is an A_∞-algebra structure on $\mathcal{B} = \mathcal{A} \oplus \mathcal{P}[1]$, such that

$$\mu_{\mathcal{B}}^{*} : \mathcal{A} \otimes \cdots \otimes \mathcal{A} \longrightarrow \mathcal{A}$$

is the given A_∞-structure on $\mathcal{A}$;

$$\mu_{\mathcal{B}}^{*} : \mathcal{A} \otimes \cdots \otimes \mathcal{P} \otimes \cdots \otimes \mathcal{A} \longrightarrow \mathcal{P}$$

is the given bimodule structure on $\mathcal{P}$.

Versions of this appear in the literature (e.g. Kontsevich-Vlassopoulos)

Don't know about A_∞-structures? Can use dg structures after quasi-isomorphic replacement ($\mathcal{P}$ should be K-projective as an $\mathcal{A}$-bimodule, as well as left and right K-flat as an $\mathcal{A}$-module). Then, $\mathcal{B}$ is an extension of the dga $\mathcal{A}$ by the dg bimodule $\mathcal{P}$, but not necessarily a square zero extension.

Analyzing a noncommutative divisor

Consider $\mathbb{B}$ as an $\mathbb{A}$-bimodule.
It fits into

$$0 \longrightarrow \mathbb{A} \longrightarrow \mathbb{B} \longrightarrow \mathcal{P}[1] \longrightarrow 0$$

whose boundary map

$$\sigma \in \mathrm{Hom}^0(\mathcal{P}, \mathbb{A}) \cong \mathrm{Hom}^0(\mathbb{A}, \mathcal{P}^{-1})$$

we call the first order part of the noncommutative divisor. The entire structure can be analyzed through "obstruction theory" in

$$\left\{ \mathrm{Hom}^{1-j}(\mathcal{P}^{\otimes_{\mathbb{A}} j}, \mathbb{A}) \leftarrow \mathrm{Hom}^{-j}(\mathcal{P}^{\otimes_{\mathbb{A}} j}, \mathbb{A}) \right\}$$

Associated structures

$\mathbb{A}$	The ambient space
$\mathcal{P}$	The line bundle
σ	Section of $\mathcal{P}^{-1}$
$\mathbb{B}$	The divisor by itself (forgetting about $\mathbb{A}$)
$\mathbb{A}\backslash\mathbb{B}$	The complement of the divisor (this is a localization construction, equivalent to a suitable Keller-Drinfeld quotient)

Noncommutative pencils

Set $V \cong \mathbb{C}^2$, $W = \mathrm{Hom}(V, \mathbb{C})$. Let's use an additional "weight" grading

$\mathcal{A}$ has weight 0

$\mathcal{P}$ has weight 1

V has weight 1

Definition A noncommutative (nc) pencil is a collection of maps

$$\phi^d : \mathcal{B}^{\otimes d} \longrightarrow \mathcal{B}[2-d] \otimes \mathrm{Sym}^*(V)$$

which preserve weights and specialize to n.c. divisors for $w \in W$

The ϕ^d are generalized A_∞ operations (in a dg context, one would just have the differential ϕ^1 and product ϕ^2). More precisely, they define a family of A_∞-algebra structures parametrized by $w \in W$ (and with additional properties).

Example Any pencil of hypersurfaces (in ordinary algebraic geometry) gives rise to a nc pencil, $\mathcal{A} \cong$ derived category.

Analyzing a noncommutative pencil

We now have a bimodule map $\mathcal{P} \to \mathcal{A} \otimes V$, or equivalently ($V = \mathbb{C}^2$), two bimodule maps

$$\sigma_0, \sigma_\infty : \mathcal{P} \longrightarrow \mathcal{A}$$

These form the first order part.

Deformation theory treatment – One can associate to $\mathcal{A}$ and $\mathcal{P}$ a bigraded dgla $\mathfrak{g}$, such that an nc divisor is an L_∞-homomorphism $\mathbb{C}[-1] \longrightarrow \mathfrak{g}$, and a nc pencil an L_∞-homomorphism $W[-1] \longrightarrow \mathfrak{g}$.

Associated structures

$\mathcal{B}_z$ fibres at $z \in \mathbb{P}(W) \cong \mathbb{CP}^1$

But, one can also take fibres at more interesting "points". For example, a formal disc around z yields a deformation $\hat{\mathcal{B}}_z$ of $\mathcal{B}_z$ (an A_∞-algebra over $\mathbb{C}[[q]]$)

Take $\mathcal{A} \backslash \mathcal{B}_\infty$ ($z = \infty \in \mathbb{CP}^1$). This has a formal deformation of degree 2, called the associated noncommutative Landau-Ginzburg model.

Symplectic geometry

X a symplectic manifold which is "Fano", $[\omega_X] = c_1(TX)$

$\pi : X \dashrightarrow \mathbb{CP}^1$ a symplectic Lefschetz pencil for $L = K_X^{-1}$, with "fibres" X_z, base B; assume X_∞ smooth

$c_1(X_z) = 0$

$W : X \setminus X_\infty \longrightarrow \mathbb{C}$ the associated Lefschetz fibration

$\omega_X | (X \setminus X_\infty)$ is exact

Theorem $\mathcal{A} = \mathcal{F}(W)$ can be equipped canonically with a noncommutative pencil.

What makes up the pencil?

$\mathcal{P} = \mathcal{A}^\vee$ the dual diagonal bimodule (no additional information)

σ_∞ exists for any Lefschetz fibration over $\mathbb{C}$

σ_0 uses the fact that the fibration extends to a Lefschetz pencil (it counts sections through the fibre at ∞)

Conjectural web of relationships

Fibres $\mathbb{B}_z$ have no geometric meaning ??

Fibre $\mathbb{B}_\infty$ at ∞ corresponds to $\mathcal{F}(X_z \setminus \mathbb{B})$

Noncomm. pencil associated to $X \dashrightarrow \mathbb{CP}^1$

$A \setminus \mathbb{B}_\infty$ corresponds to the wrapped Fukaya category $\mathcal{W}(X \setminus X_\infty)$

Fibre $\hat{\mathbb{B}}_\infty$ over a formal disc near ∞ corresponds to $\mathcal{F}(X_z, \mathbb{B})$ after a suitable change of variables

Fibre over a punctured formal disc near ∞ corresponds to $\mathcal{F}(X_z)$

The noncommutative LG model on $A \setminus \mathbb{B}_\infty$ corresponds to (a close relative of) $\mathcal{F}(X)$. After an obvious change of variables, get $\mathcal{F}_z(X)$

Actual state of the theory

known to first order (S)

large part proved (Abouzaid–S.)

Fibre B_∞ at ∞ corresponds to $\mathcal{F}(X_z \setminus B)$

Noncomm. pencil associated to $X \dashrightarrow \mathbb{C}P^1$

$A \setminus B_\infty$ corresponds to the wrapped Fukaya category $\mathcal{W}(X \setminus X_\infty)$

conjectural (some ideas)

conjectural

Fibre $\hat{B}_\infty$ over a formal disc near ∞ corresponds to $\mathcal{F}(X_z, B)$ after a suitable change of variables

This would imply that after change of variables, the A-model connection on $H^*(X_z)$ extends over a punctured $\mathbb{C}P^1$

The noncommutative LG model on $A \setminus B_\infty$ corresponds to (a close relative of) $\mathcal{F}(X)$. After an obvious change of variables, get $\mathcal{F}_z(X)$

Homological mirror symmetry for pencils

Symplectic Lefschetz pencil, $L = K_X^{-1}$, X_∞ could be singular, but still $[\omega_X] = c_1(X)$

$\longrightarrow$ noncommutative pencil, $\mathcal{A} = \mathcal{F}(W)$ or $\mathcal{A}^\vee = \mathcal{D}(X^\vee)$ $\longleftarrow$

Algebraic pencil, $L = K_X^{-1}$, X_∞ could be singular

We can consider other situations, but the algebraic formalism needs to be adapted accordingly

On this side, many generalizations can be admitted