PICARD-LEFSCHETZ THEORY AND HIDDEN GROUP ACTIONS

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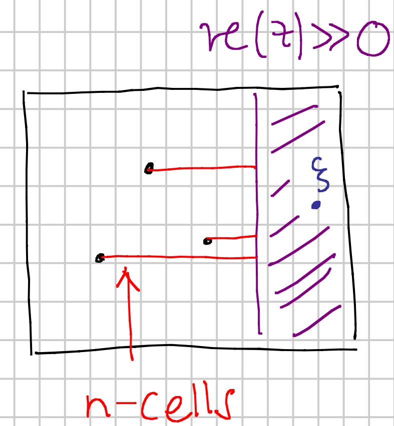
PICARD-LEFSCHETZ THEORY

X^n smooth affine algebraic variety / $\mathbb{C}$

$\pi : X \longrightarrow \mathbb{C}$ a LEFSCHETZ FIBRATION

- only nondegenerate critical points (at most 1 in each fibre, μ in total)
- no "critical points at ∞"



$Y = X_\xi$ fibre at some base point ξ, $re(\xi) \gg 0$

Then

$$\pi^{-1}(\{re(z) \gg 0\}) \underset{C^\infty}{\cong} \{re(z) \gg 0\} \times Y$$

Define

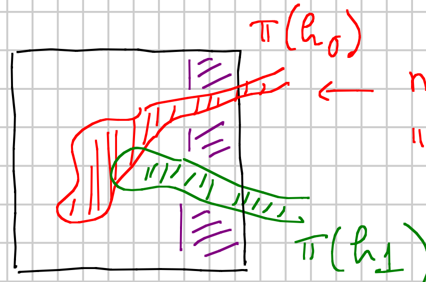
$$\boxed{H_\pi = H_n(X, \pi^{-1}\{re(z) \gg 0\}) \cong \mathbb{Z}^\mu}$$

PICARD–LEFSCHETZ THEORY AND INTERSECTION NUMBERS

$$\underset{\text{weak Lefschetz}}{0} \longrightarrow H_n(X) \longrightarrow H_\pi \xrightarrow{\partial} H_{n-1}(Y) \to H_{n-1}(X) \to \cdots$$

H_π carries an unsymmetric (nondegenerate) bilinear pairing, the VARIATION PAIRING

$$h_0 \bullet_\pi h_1 \in \mathbb{Z}$$

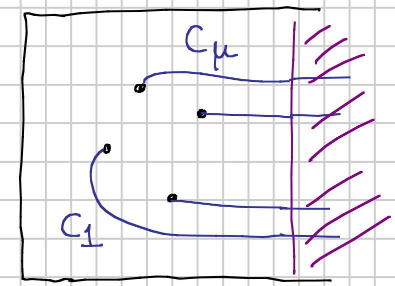


← move the first cycle "up", then intersect

PROPERTIES :

- on $H_n(X)$, it is the ordinary intersection pairing, written as $\bullet$
- $h_0 \bullet_\pi h_1 - (-1)^n h_1 \bullet_\pi h_0 = \partial h_0 \bullet \partial h_1$

COMPUTING INTERSECTION NUMBERS



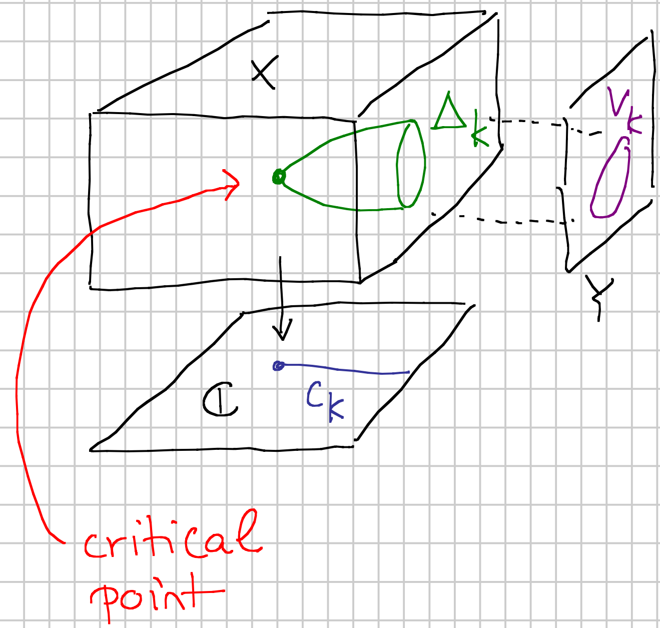
Choose a BASIS OF VANISHING PATHS $(c_1, \ldots, c_\mu)$ in $\mathbb{C}$

their LEFSCHETZ THIMBLES $(\Delta_1, \ldots, \Delta_\mu)$ in X

and vanishing cycles $(V_1, \ldots, V_\mu)$ in Y

The $[\Delta_k]$ form a basis of H_π, and:

$$\boxed{[\Delta_i] \cdot_\pi [\Delta_j] = \begin{cases} [V_i] \cdot [V_j] & i < j \\ 1 & i = j \\ 0 & i > j \end{cases}}$$



(STILL COMPUTING INTERSECTION NUMBERS)

Conversely:

$$\boxed{[V_i]\cdot[V_j] = [\Delta_i]\underset{\pi}{\bullet}[\Delta_j] - (-1)^n [\Delta_j]\underset{\pi}{\bullet}[\Delta_i]}$$

In particular, $[V_i]\cdot[V_i] = \begin{cases} 0 & n \text{ even} \\ 2 & n \text{ odd} \end{cases}$

CONVENTION (for intersection #s of half-dimensional cycles in $\dim_{\mathbb{C}}(X) = n$) differs from the standard one by $(-1)^{n(n+1)/2}$. Hence, if $L \subset X$ is TOTALLY REAL, $L\cdot L = \chi(L)$. Vanishing cycles are totally real spheres.

(ALMOST DONE COMPUTING INTERSECTION NUMBERS)

Even more explicitly, define $A, B \in \mathrm{Mat}_{\mu\times\mu}(\mathbb{Z})$

$$B_{ij} = V_i \cdot V_j$$

$$A_{ij} = \begin{cases} B_{ij} & i<j \\ 1 & i=j \\ 0 & i>j \end{cases} \quad \Rightarrow \quad B = A - (-1)^n A^*$$

If $L^n \subset X$ is oriented closed, $[L] \in H_\pi \longleftrightarrow \ell_L \in \mathbb{Z}^\mu$ satisfies

- $B\ell_L = 0$
- $\ell_L^* A \ell_L = L \bullet L$ $(= \chi(L)$ if L is totally real$)$

A contains a lot of topological information, e.g. the global monodromy on H_π is given by $(-1)^n (A^*)^{-1} A$.

ker(B) is the preimage of the null-space of $\cdot$ on $\mathrm{im}(\partial)$ (larger than $\ker(\partial)$).

EXAMPLE : DANIELEWSKI SURFACES AND GENERALIZATIONS

Fix coprime natural numbers (a,b), and set

$$X = X_{a,b} = \{ x^a y^b - 1 = z^2 \} \subset \mathbb{C}^3$$

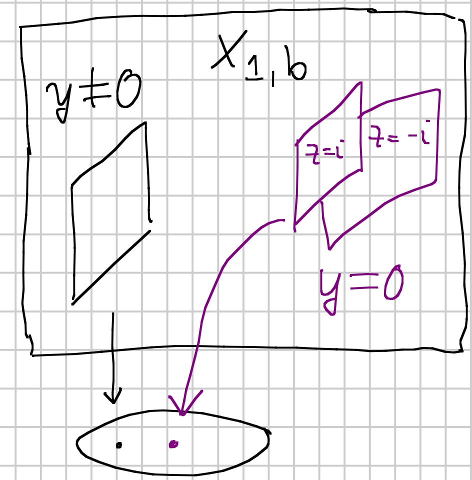
Double cover of $\mathbb{C}^2$ branched along $\mathbb{C}^*$

Topologically :

$(a,b) = (1,1)$	$a = 1,\ b > 1$
This is $T^* S^2$ (we'll exclude this case from now on : let $a < b$)	(Danielewski surface, two $\mathbb{C}^2$ glued together) Total space of the line bundle over S^2 with $e = -2b$

remove $y = 0$, $z = \pm i$

$$X_{1,b} \xrightarrow{y} \mathbb{C}$$



(not a Lefschetz fibration)

(EXAMPLE CONTINUES)

$1 < a < b$?

This is a simply-connected 4-manifold, $H_2(X) = \mathbb{Z}$, with boundary (at infinity) a Seifert fibered space.

Apply Picard-Lefschetz theory:

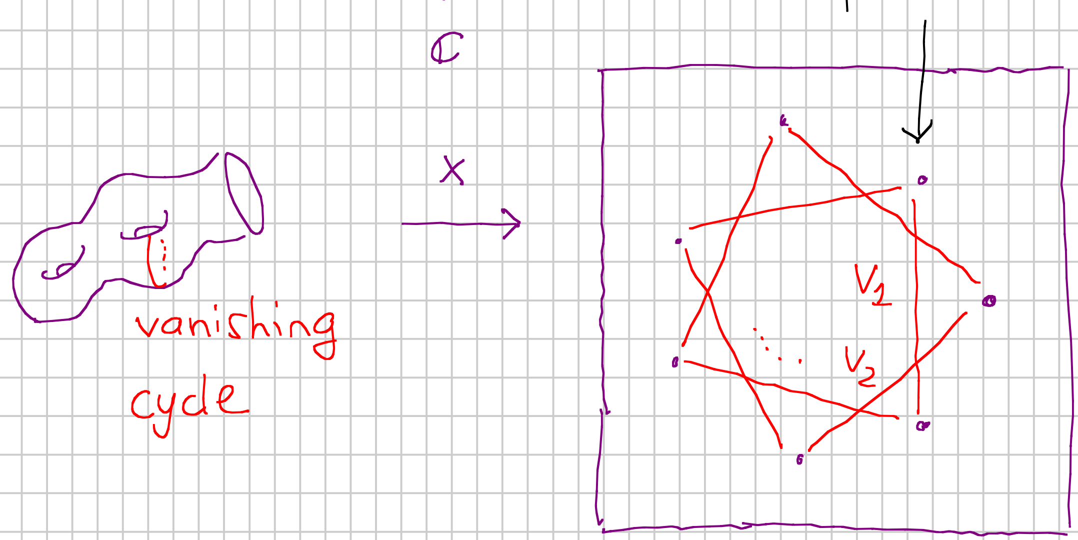
$$X = \{ x^a y^b - 1 = z^2 \}$$
$$\downarrow \pi = ax + by$$
$$\mathbb{C}$$

$\mu = a + b$

Vanishing cycles $(V_1, \dots, V_\mu)$ live on the hyperelliptic curve

$$\pi^{-1}(0) = \{ x^{a+b} \left(-\tfrac{a}{b}\right)^b - 1 = z^2 \}$$
$$\downarrow x$$
$$\mathbb{C}$$

branch points



$a=2, \ b=5$

(EXAMPLE ENDS – FOR NOW)

B is a skew-symmetric cyclic band matrix

$\mathbb{Z}/\mu$ symmetry on X rotates the base $\mathbb{C}$, does not preserve $\{re(z) \gg 0\}$

$$B = \begin{pmatrix} 0 & \overbrace{2 & \cdots & 2}^{a-1} & 1 & 0 & -1 \cdots -2 \\ -2 & 0 & \ddots & & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & & & & \\ -2 & \ddots & & & & & \\ -1 & & & & & & \end{pmatrix} \longrightarrow A = \begin{pmatrix} 1 & 2 \cdots & 1 & & -1 \cdots -2 \\ & 1 & \ddots & \ddots & \ddots \\ & & 1 & & \\ & 0 & & \ddots & \end{pmatrix}$$ upper triangular matrix

Its nullspace is spanned by $h = (1, \ldots, 1)$ (and if $a+b$ is odd, $\bar{h} = (1,0,1,0,\ldots)$).

$$\boxed{h^* A h = 2ab}$$ intersection pairing on $H_2(X) \cong \mathbb{Z}$

note sign conventions

HIGHER-DIMENSIONAL ANALOGUE

$$X_{a,b} = \{ x^a y^b - 1 = z_1^2 + \dots + z_{n-1}^2 \}$$

$$H_*(X_{a,b}) \cong H_*(S^n)$$

$$B = \begin{pmatrix} 1-(-1)^n & \overbrace{1+(-1)^n \cdots}^{a-1} & (-1)^n \cdots \\ -1-(-1)^n & \ddots & \\ \vdots & & \\ -1 & & \end{pmatrix}$$

For any n, the nullspace of B is generated by $h = (1, \dots, 1)$ (and if n is even, $\overline{h}$ as before).

$$\boxed{h^* A h = ab\left(1 + (-1)^n\right)}$$

COROLLARY n even, $L \subset X = X_{a,b}$ (closed oriented) totally real, then $\chi(L)$ is a multiple of $2ab$

What about n odd?

FACT If n is odd, $X_{1,b} \overset{C^\infty}{\cong} T^*S^n$ for all b, and this is compatible with the homotopy class of almost complex structures.

(Possibly, for all (a,b) as well)

Q-DEFORMATION $(n \geq 3)$ ← important

$B_q \in \mathrm{Mat}_{\mu\times\mu}(\mathbb{Z}[q, q^{-1}])$, in this case obtained from B by replacing $(-1)^n \mapsto q(-1)^n$:

$$B_q = \begin{pmatrix} 1-q(-1)^n & 1+q(-1)^n & \cdots & q(-1)^n \\ -1-q(-1)^n & \ddots & & \\ \vdots & & & \\ -1 & & & \end{pmatrix} \xrightarrow{\text{make upper triangular}} A_q = \begin{pmatrix} 1 & 1+q(-1)^n & \cdots & q(-1)^n \\ & 1 & & \\ & & \ddots & \\ & 0 & & \end{pmatrix}$$

Nullspace now generated by $h = (1, \ldots, 1)$ (no more $\bar{h}$).

Define a pairing on $H_{\pi,q} = \mathbb{Z}[q^{\pm 1}]^\mu$

$$(h_0, h_1) \longmapsto h_0^* A_q h_1 \in \mathbb{Z}[q^{\pm 1}]$$

$*$ includes substitution $q \mapsto q^{-1}$

$$\boxed{h^* A_q h = ab\left(1 + q(-1)^n\right)}$$

SYMPLECTIC TOPOLOGY

$X \subset \mathbb{C}^{n+1}$ has a (real) symplectic structure, ω_X = constant Kähler form

Consider half-dimensional sub-manifolds L which are LAGRANGIAN, hence, totally real

$$\omega_X|L = 0 \in \Omega^2(L).$$

EXAMPLE Lefschetz thimbles and vanishing cycles are Lagrangian.

character of a $\mathbb{G}_m$-action on $H^i(L;\mathbb{C})$

THEOREM Let $L \subset X_{a,b}$ be a connected closed Lagrangian submanifold with $H^1(L)=0$ and which is Spin. Then there is an associated

$$h_{L,q} \in H_{\pi,q} = \mathbb{Z}[q^{\pm 1}]^{\mu}$$

satisfying

- $B_q h_{L,q} = 0$
- $h_{L,q}^* A_q h_{L,q} = 1 + q(-1)^n + \sum_{i=1}^{n-1} (-1)^i X_i$

CONSEQUENCES

COR $X_{a,b}$ for any $(a,b) \neq (1,1)$ and any n, does not contain a Lagrangian sphere.

Compare:

THM (Maydanskiy-S.) $X_{1,b}$ is "empty"; it does does not contain any closed Lagrangian submanifold with $H^1(L)=0$ (or more generally, such that $HF^*(L,L) \neq 0$).

For n even, this follows from classical Picard-Lefschetz theory

ADDENDUM TO THEOREM Reducing $h_{L,q}$ to $q=1$ recovers

$$h_L = [L] \in H_T \cong \mathbb{Z}^\mu .$$

COR $L \subset X$ with $H^1(L)=0$, Spin, $[L]\neq 0$, then the total Betti number of L is $\geq 2ab$

SECOND EXAMPLE

$$X = \{ (xy^2-1)x = z^2 \}$$

$$\cup\!\!\!\!|$$

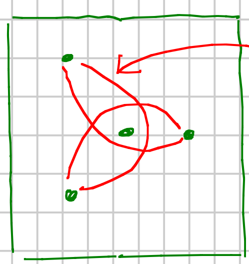
$$(\mathbb{C}^*)^2 = \left\{ \begin{array}{l} u_1 = (xy-z)^{-1} \\ u_2 = (xy+z)^{-1} \end{array} \right\}$$

$$\pi = x + 2y = u_1 + u_2 + \frac{1}{u_1 u_2}$$

Landau-Ginzburg mirror of $\mathbb{P}^2$

As before:

$$\pi^{-1}(0) \xrightarrow{X} \mathbb{C}$$



vanishing cycles

Higher-dimensional counterpart

$$X = \{ (xy^2-1)x = z_1^2 + \cdots + z_{n-1}^2 \}$$

$$H_*(X) = \begin{cases} \mathbb{Z} & * = 0, n-1, n \\ 0 & \text{otherwise} \end{cases}$$

- Intersection pairing on $H_n(X)$ vanishes for any n
- For n <u>even</u>, there is a Lagrangian $L \cong S^1 \times S^{n-1} \subseteq X$, $[L] \in H_n(X)$ nonzero; L monotone, $HF^*(L,L) \neq 0$ (at least for $n \gg 0$)

INTERSECTION NUMBERS AND Q-DEFORMATION

$$B = \begin{pmatrix} 1-(-1)^n & -1-2(-1)^n & 2+(-1)^n \\ 2+(-1)^n & 1-(-1)^n & -2-(-1)^n \\ -1-2(-1)^n & 1+2(-1)^n & 1-(-1)^n \end{pmatrix}$$

$\uparrow$ $q=1$

Nullspace of B is generated by $h = (1,1,1)$

$$B_q = \begin{pmatrix} 1-q(-1)^n & -q^{-1}(-1)^n-1-(-1)^n q & 1+q(-1)^n+q^2 \\ 1+q(-1)^n+q^2 & 1-q(-1)^n & -q^{-1}(-1)^n-1-(-1)^n q \\ -q^{-1}(-1)^n-1-(-1)^n q & 1+q^{-1}(-1)^n+q(-1)^n & 1-q(-1)^n \end{pmatrix}$$

$$\boxed{\det(B_q) \neq 0}$$

COROLLARY X (in any dimension n) does not contain a Lagrangian sphere.

FLOER COHOMOLOGY

To a symplectic manifold such as X, one can associate a dg category over $\mathbb{C}$, the FUKAYA CATEGORY $\mathcal{F}(X)$.

For convenience: other coefficient fields are possible (and useful in other circumstances)

OBJECTS Closed Lagrangian submanifolds $L \subseteq X$ with certain additional conditions ($H^1(L) = 0$ and L Spin will be sufficient).

MORPHISMS

FLOER COHOMOLOGY

$$H^*(\hom_{\mathcal{F}(X)}(L_0, L_1)) = HF^*(L_0, L_1).$$

Properties:

"categorified" intersection number

- $\chi(HF^*(L_0, L_1)) = L_0 \cdot L_1$
- $HF^*(L, L) \cong H^*(L; \mathbb{C})$

CATEGORICAL SYMMETRIES

Suppose that $\mathcal{F}(X)$ carries an ACTION OF $\mathbb{G}_m = \mathbb{C}^*$ (this is a purely algebraic notion). Then, any L with $H^1(L)=0$ can be made into an EQUIVARIANT OBJECT. For two such objects, $HF^*(L_0, L_1)$ inherits an induced $\mathbb{G}_m$-action. In particular, for $L_0 = L_1 = L$, $H^*(L;\mathbb{C})$ becomes a representation of $\mathbb{G}_m$.

$\mathbb{G}_m$-actions are compatible with multiplicative structures

DEFINTION A dilating $\mathbb{G}_m$-action on X is a $\mathbb{G}_m$-action on $\mathcal{F}(X)$ such that for any L with $H^1(L)=0$, the induced $\mathbb{G}_m$-action on

$$HF^n(L,L) \cong H^n(L;\mathbb{C}) \cong \mathbb{C}$$

has weight 1.

Q-INTERSECTION NUMBERS

Compare S'-Solomon 2010, Abouzaid-Smith 2013 for alternative approaches

Suppose: X has dilating $\mathbb{C}_m$-action, L_0, L_1 equivariant Lagrangian submf.

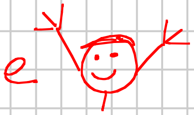
DEF

$$\boxed{L_0 \bullet_q L_1 = \mathrm{Str}\big(HF^*(L_0,L_1) \circlearrowleft q \big) \in \mathbb{Z}[q,q^{-1}]}$$

PROPERTIES

- $L_1 \bullet_q L_0 = q(-1)^n \left(L_0 \bullet_q L_1 \right)^*$, where $*$ is $q \mapsto q^{-1}$
- If L is a rational homology sphere, $L \bullet_q L = 1 + (-1)^n q$
- If $L_0 \cap L_1 = \emptyset$, $L_0 \bullet_q L_1 = 0$
- Invariant under Hamiltonian isotopies
- q-Picard-Lefschetz formula

DILATING GROUP ACTIONS AND PICARD-LEFSCHETZ THEORY

The last slide 

EXAMPLE $X = \mathbb{C}^*$ admits a dilating $\mathbb{G}_m$-action

EXAMPLE $X = \{ x^m - 1 = z_1^2 + \cdots + z_n^2 \}$ admits a dilating $\mathbb{G}_m$-action if $n \geq 2$ (for any m)

THEOREM $\pi : X \longrightarrow \mathbb{C}$ a Lefschetz fibration with fibre Y ($n = \dim_{\mathbb{C}} X \geq 3$). If Y admits a dilating $\mathbb{G}_m$-action, then so does X.

$$\boxed{\text{MAIN THEOREM The } q\text{-version of Picard-Lefschetz theory } (A_q, B_q) \text{ and their implications) applies in this situation.}}$$

Interesting open issues:

- Do it without "categorifying"
- q-monodromy

$$(-1)^n (A_q^*)^{-1} A_q$$