# Symplectic topology and q-intersection numbers

joint work with R. Bezrukavnikov and J. Solomon

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intersection number $L_0 \cdot L_1$, an integer* $\xleftarrow{q=1}$ improved or <u>q-intersection number</u> $L_0 \cdot_q L_1$, a function of a formal variable $q$

* $(-1)^{n(n-1)/2}$ times the standard convention

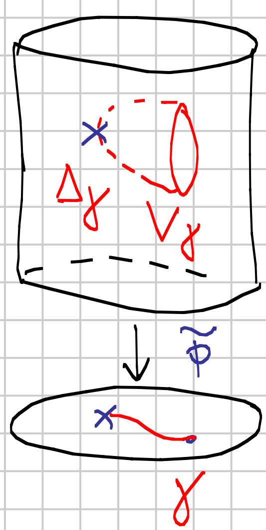
Here, $M^{2n}$ is an oriented smooth manifold, and $L_0^n, L_1^n$ are closed oriented submanifolds. In many situations, one would like to enhance the ordinary intersection number to a q-version. Obviously, this doesn't always work!

There is a classical <u>topological</u> approach based on infinite cyclic coverings, where $\mathbb{Z}[q, q^{-1}]$ is the group ring of the covering group.

<u>Sample application</u> (Givental) Take a polynomial $p : \mathbb{C}^{n+1} \longrightarrow \mathbb{C}$ with an isolated critical point at the origin. To simplify the exposition, assume $p$ is weighted homogeneous. The Milnor fibre is

$$M = p^{-1}(t)$$

for $t \neq 0$. $H_n(M) \cong \mathbb{Z}^{\mu}$ is generated by <u>vanishing cycles</u>. To see those, <u>replace</u> $p$ with a perturbation $\tilde{p}$.



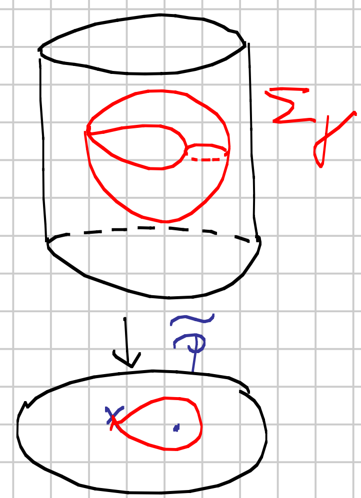

$\gamma$ path from $t$ to a critical value of $\tilde{p}$

$V_\gamma$ sphere in $\tilde{p}^{-1}(t) = M$

$\Delta_\gamma$ ball in $(\mathbb{C}^{n+1}, M)$, the <u>Lefschetz thimble</u>

Take $C = \mathbb{C}^{n+1} \setminus M$, and $\tilde{C} \to C$ its obvious $\mathbb{Z}$-covering.
Given $\gamma$, we have two ways of getting an associated cycle in $\tilde{C}$:

- lift $\Delta_\gamma \setminus V_\gamma$ to $\tilde{\Delta}_\gamma$ (yields a properly embedded open ball)
- double the path, which yields an immersed sphere $\Sigma_\gamma \subset C$, and lift that to $\tilde{\Sigma}_\gamma$:




If $\sigma : \tilde{C} \to \tilde{C}$ is the deck transformation, then Givental <u>defines</u>

$$V_{\gamma_0} \cdot_q V_{\gamma_1} = \sum_{k \in \mathbb{Z}} q^k \, \sigma^k(\tilde{\Delta}_{\gamma_0}) \cdot \tilde{\Sigma}_{\gamma_1}$$

He proves that this is well-defined and refines the standard intersection form.

$$
\begin{tikzcd}[column sep=huge, row sep=large]
\boxed{L_0 \cdot L_1 \in \mathbb{Z}} & \boxed{L_0 \cdot_q L_1 \text{, a formal function of } q} \arrow[l, dashed, "q=1"'] \\
\boxed{HF^*(L_0, L_1) \text{, Lagrangian Floer cohomology}} \arrow[u, "\text{Euler } \chi"] & \boxed{\begin{array}{l}\text{additional grading} \\ \text{(or similar structure)} \\ \text{on Floer cohomology}\end{array}} \arrow[l, dashed, "\text{forget}"'] \arrow[u, dashed, "\text{improved Euler } \chi"]
\end{tikzcd}
$$

This is <u>what we really want</u> (for $M$ a symplectic manifold, and $L_0, L_1$ Lagrangian submanifolds). One could follow Givental's approach for Milnor fibres, but his construction leaves $M$ and its asymmetry becomes problematic in the categorified world.

Mirror symmetry inspiration: Take $X$ a smooth algebraic variety, and consider compactly supported coherent sheaves (or more generally objects of $D^b_c(X)$).

coherent with compact support

By Hirzebruch-Riemann-Roch,

$$\chi\left(\mathrm{Ext}^*_X(\mathcal{E}_0, \mathcal{E}_1)\right) = \langle \mathcal{E}_0 | \mathcal{E}_1 \rangle = \int_X ch(\mathcal{E}_0^*)\, ch(\mathcal{E}_1)\, Td(X)$$

If $X$ is Calabi-Yau, the right hand side is the graded symmetric Mukai pairing. This can be seen directly or follows from Serre duality

$$\mathrm{Ext}^*(\mathcal{E}_1, \mathcal{E}_0) \cong \mathrm{Ext}^{n-*}(\mathcal{E}_0, \mathcal{E}_1 \otimes K_X)^\vee$$

Now suppose that $X$ carries a holomorphic $\mathbb{C}^*$-action, and consider equivariant sheaves. Then $\mathrm{Ext}^*(\mathcal{E}_0, \mathcal{E}_1)$ is a representation of $\mathbb{C}^*$, hence decomposes into weight spaces $\mathrm{Ext}^*(\mathcal{E}_0, \mathcal{E}_1)^k$. Define

$$\langle \mathcal{E}_0 | \mathcal{E}_1 \rangle_q = \sum_k q^k\, \chi\left(\mathrm{Ext}^*(\mathcal{E}_0, \mathcal{E}_1)^k\right).$$

## Modifications of the "equivariant" picture

Instead of a $\mathbb{C}^*$-action, let's take a holomorphic vector field $Z$. This induces elements $\phi^0_{\mathcal{E}} \in \mathrm{Ext}^1_X(\mathcal{E},\mathcal{E})$ for each $\mathcal{E}$. Instead of asking for equivariance, we only assume that these are zero. Then, there is an endomorphism

$$\tilde{\phi}^1_{\mathcal{E}_0,\mathcal{E}_1} : \mathrm{Ext}^*_X(\mathcal{E}_0,\mathcal{E}_1) \longrightarrow \mathrm{Ext}^*_X(\mathcal{E}_0,\mathcal{E}_1)$$

This may not be diagonalizable, but its eigenvalues still decompose $\mathrm{Ext}^*_X(\mathcal{E}_0,\mathcal{E}_1) = \bigoplus_{\lambda\in\mathbb{C}} \mathrm{Ext}^*_X(\mathcal{E}_0,\mathcal{E}_1)^\lambda$. We have $q$-Mukai pairings

$$\begin{aligned}\langle \mathcal{E}_0 | \mathcal{E}_1 \rangle_q &= \sum_\lambda q^\lambda \cdot \chi\left(\mathrm{Ext}^*_X(\mathcal{E}_0,\mathcal{E}_1)^\lambda\right) \\ &= \mathrm{Str}\left(\exp\left(\log(q)\, \tilde{\phi}^1_{\mathcal{E}_0,\mathcal{E}_1}\right)\right)\end{aligned}$$

Actually, $\tilde{\phi}^1_{\mathcal{E}_0,\mathcal{E}_1}$ is not strictly unique.

The difference between two choices is the sum of composition maps with elements of $\mathrm{Ext}^0_X(\mathcal{E}_k, \mathcal{E}_k)$ (on the left and right, respectively).

__Example__ If $\mathrm{Ext}^0_X(\mathcal{E}_k, \mathcal{E}_k) = \mathbb{C} \cdot \mathrm{Id}$, then $\tilde{\Phi}^1_{\mathcal{E}_0, \mathcal{E}_1}$ is unique up to constant multiples of the identity. If in addition $\mathcal{E}_0 = \mathcal{E}_1 = \mathcal{E}$, then $\tilde{\Phi}^1_{\mathcal{E}, \mathcal{E}}$ is independent of any choices.

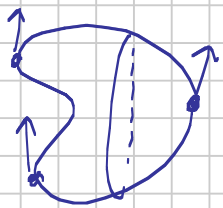
From now on, fix a holomorphic volume form $\eta$. The situation where $L_z \eta = 0$ is not so interesting from a mirror symmetry viewpoint, so __let's assume that__ $L_z \eta = \eta$. Then, Serre duality shows that

$$\langle \mathcal{E}_1 | \mathcal{E}_0 \rangle_q = q^{-1} (-1)^n \langle \mathcal{E}_0 | \mathcal{E}_1 \rangle_{q^{-1}}$$

__Example__ If $\mathcal{E}$ is __spherical__, then $\langle \mathcal{E} | \mathcal{E} \rangle_q = 1 + q(-1)^n$.

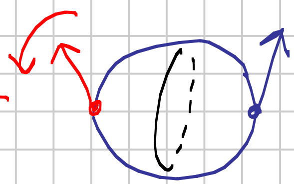
# Reformulation as extended TCFT associated to $(X,\eta)$

(Costello, Hopkins–Lurie, Caldararu–Willerton)

inputs  outputs (at least one) $\longrightarrow$ map $HH^*(X) \otimes HH^*(X) \longrightarrow HH^*(X)$

where by HKR $HH^*(X) \cong H^*(X, \Lambda^* TX)$, so $Z \in HH^1(X,X)$.

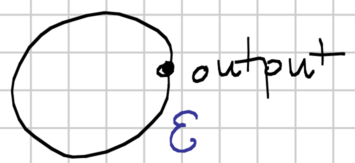
One also has operations for families of surfaces:

tangent vector moves  tangent vector is fixed $\longrightarrow$ BV operator $\Delta$ of degree $-1$

$$\begin{array}{ccc} HH^*(X) & \cong & H^*(X, \Omega^*_X) \\ \Delta \downarrow & & \downarrow d \text{ algebraic de Rham} \\ HH^*(X) & \cong & H^*(X, \Omega^*_X) \end{array}$$

$$\boxed{\begin{array}{l} L_Z \eta = \eta \text{ means } d i_Z \eta = \eta, \\ \text{hence } \quad \Delta Z = 1. \end{array}}$$

One further considers surfaces with boundaries marked with objects of $D^b_c(X)$.




$1 \in \mathrm{Ext}^0_X(\mathcal{E}, \mathcal{E})$

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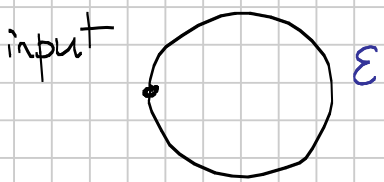


$\int : \mathrm{Ext}^n_X(\mathcal{E}, \mathcal{E}) = \mathrm{Ext}^n_X(\mathcal{E}, \mathcal{E} \otimes K_X) \longrightarrow \mathbb{C}$
(ok since $\mathcal{E}$ has compact support)

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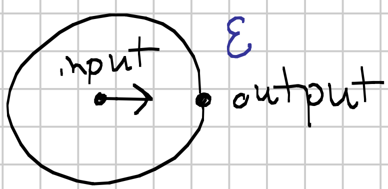


$HH^*(X, X) \longrightarrow \mathrm{Ext}^*_X(\mathcal{E}, \mathcal{E})$

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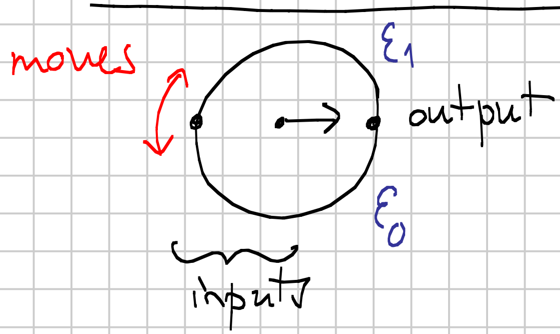
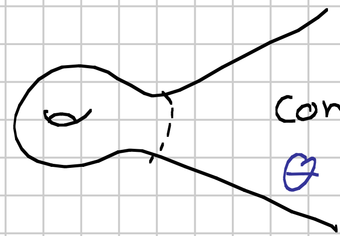



Assuming we have killed boundary contributions, this yields

$HH^*(X, X) \longrightarrow \mathrm{End}_{\mathbb{C}}(\mathrm{Ext}^*_X(\mathcal{E}_0, \mathcal{E}_1))[-1]$

(not strictly unique)

Symplectic topology $(M, \omega = d\theta)$ a finite type complete Liouville manifold



cone is $[0,\infty) \times \partial M$

$\theta = e^r \alpha$, $\alpha$ contact one-form

Take $H \in C^\infty(M, \mathbb{R})$, $H(r,x) = e^r$ on the cone. Symplectic cohomology (Cieliebak-Floer-Hofer-Wysocki, Viterbo) used here is

$$SH^*(M) = \varinjlim_{\lambda} HF^*(\lambda H)$$

We additionally assume that $c_1(M) = 0$, so that $SH^*$ is $\mathbb{Z}$-graded. Together with the standard Lagrangian Floer cohomology groups $HF^*(L_0, L_1)$ ($L_k \subset M$ compact, exact, spin, graded) this forms a naturally occurring example of an extended TCFT. In particular, we have $1 \in SH^0(M)$ as well as $\Delta : SH^*(M) \longrightarrow SH^{*-1}(M)$.

Let $SC^*(M)$ be the chain complex underlying $SH^*(M)$.

**Definition** A Liouville element is a cocycle $b \in SC^1(M)$ such that $B = [b] \in SH^1(M)$ satisfies $\Delta B = 1$.

We have open-closed string maps

$$\phi^0 : SC^*(M) \longrightarrow CF^*(L,L)$$
$$\phi^1 : SC^*(M) \longrightarrow \mathrm{End}(CF^*(L_0, L_1))[-1]$$
$$\vdots$$

satisfying

$\phi^0$ is a chain map

$\phi^1$ is a chain homotopy between left and right multiplication with $\phi^0$

...

(in fact, all of them together form a chain map from $SC^*(M)$ to the Hochschild cochain complex of the Fukaya category).

__Definition__ A __b-equivariant__ Lagrangian submanifold is a pair $(L, \beta)$ where $\beta \in CF^0(L,L)$ satisfies $d\beta = \phi^0(b)$.

More precisely, two choices of $\beta$ are considered equivalent if they differ by something nullhomologous; so, the __obstruction__ is $[\phi^0(b)] \in HF^1(L,L) = H^1(L;\mathbb{C})$, and the "__freedom of choice__" is $HF^0(L,L) = H^0(L;\mathbb{C})$. To simplify things, we will assume from now on that $L$ is __connected__.

Given $(L_0, \beta_0)$ and $(L_1, \beta_1)$, we can correct $\phi^1(b)$ to get an endomorphism $[\tilde{\phi}^1]$ of $HF^*(L_0, L_1)$, induced by

$$a \longmapsto \phi^1(b)(a) \pm \beta_1 a \pm a \beta_0$$

This yields $HF^*(L_0,L_1) = \bigoplus_{\lambda \in \mathbb{C}} HF^*(L_0,L_1)^\lambda$

__Definition__

$$L_0 \cdot_q L_1 = \mathrm{Str}\left(e^{\log(q)[\tilde{\phi}^1]}\right) = \sum_\lambda q^\lambda \chi\left(HF^*(L_0,L_1)^\lambda\right).$$

use $\Delta[b]=1$

<u>Easy properties</u> • $L_1 \cdot_q L_0 = (-1)^n q\,(L_0 \cdot_{q^{-1}} L_1)$.

• If $H^*(L;\mathbb{C})$ has a single generator (as a ring) of degree $k|n$, then $L \cdot_q L = 1 + (-1)^k q^{k/n} + (-1)^{2k} q^{2k/n} + \dots + (-1)^n q$.

• Change the equivariant structure $\beta_k \longmapsto \beta_k + r_k \cdot 1_{L_k}$, denoted by $L_k \longmapsto L_k\langle r_k\rangle$, yields $L_0\langle r_0\rangle \cdot_q L_1\langle r_1\rangle =$
$= q^{r_1 - r_0}(L_0 \cdot_q L_1)$.

• Invariant under Hamiltonian (exact Lagrangian) isotopies.

<u>Picard-Lefschetz</u> If $\tau_{L_1}$ is the Dehn twist along a Lagrangian sphere $L_1$,

$$\tau_{L_1}(L_0) \cdot_q L_2 = L_0 \cdot_q L_2 + (-1)^{n+1} q^{-1} (L_0 \cdot_q L_1)(L_1 \cdot_q L_2)$$

ordinary grading equivariant structure

Note $\tau_L(L) = L[1-n]\langle 1\rangle$, so q-intersection numbers feel the nontriviality of $\tau_L^2$ in even dimensions.

__Existence of Liouville elements__ (one could use more general classes in $SH^1$ as well, but these may not give rise to nontrivial q-intersection numbers)? On the __negative side__:

__Lemma__ (Viterbo, Abbondandolo-Schwarz) (i) If $L$ is a $K(\pi,1)$, then $M = T^*L$ does __not__ admit a Liouville element.

(ii) If $M$ admits a Liouville element, it can't contain any exact Lagrangian $K(\pi,1)$ submanifolds.

__Lemma__ (Bourgeois-Oancea) Let $\partial M$ be the contact boundary at $\infty$ of $M$. If $M$ admits a Liouville element, then $[M] \in H_{2n}(M, \partial M; \mathbb{C})$ lies in the image of the SFT map

$$CH_*^{lin}(\partial M) \longrightarrow H_*(M, \partial M; \mathbb{C})[u^{-1}]$$

(The converse is probably not true)

On the positive side:

<u>Lemma</u> (same people) (i) $T^*S^r$, $r \geq 2$, has a Liouville element.
(ii) If $L$ is so that there is a nonzero degree map $\tilde{L} \times S^r \longrightarrow L$, $r \geq 2$, then it has a Liouville element.

This applies for instance to $L = \mathbb{C}P^r = Sym^r(S^2)$.

To construct other examples, use any approach which (partially) computes $SH^*$, such as Oancea, McLean, ...

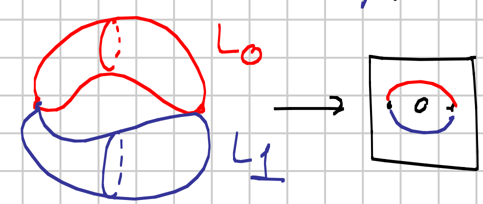
<u>Lemma</u> Let $\pi : M \longrightarrow \mathbb{C}$ be an exact Lefschetz fibration.
If the fibre admits a Liouville element, then so does the total space.

<u>Example</u> $M = \{ x_1^2 + x_2^2 + x_3^2 + x_4^m + 1 = 0 \} \subset \mathbb{C}^4$, $m \geq 2$

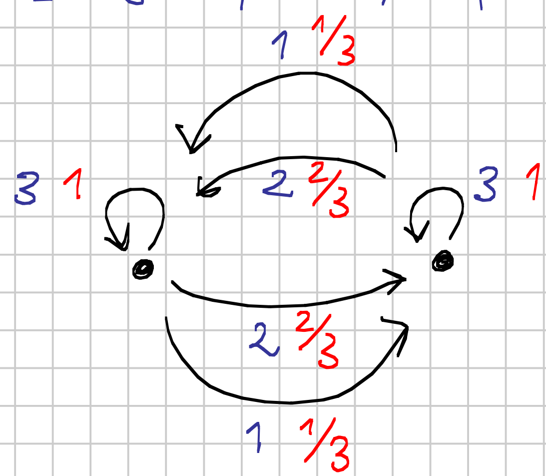
<u>Example</u> (This is the mirror of $\mathrm{Tot}(\mathcal{O}(-1)\oplus\mathcal{O}(-1)\to\mathbb{CP}^1)$)

Take

$$M = \{ x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1,\ x_4 \neq 0 \}$$




Draw $HF^*(L_i,L_j)$:




numbers in blue are degrees

numbers in red are "weights"

Such homogeneity properties have strong implications for the underlying cochain level structures:

<u>Lemma</u> $\mathcal{A}$ an $A_\infty$-algebra. If there is $b \in HH^1(\mathcal{A},\mathcal{A})$ such that $[b^0] = 0 \in H^1(\mathcal{A})$ and which induces the Euler derivation (weight $k$ on the degree $k$ part) $\Rightarrow$ $\mathcal{A}$ is formal.