Categorical Dynamics

Mordell lecture, 2012

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Automorphisms of symplectic manifolds

$(M^{2n}, \omega \in \Omega^2(M))$ a closed symplectic manifold. We have

$$\begin{array}{ccccc} C^\infty(TM) & \text{vector fields} & \xrightarrow{\cong} & C^\infty(T^*M) \cong \Omega^1(M) & \text{one-forms} \\ Z & & \longmapsto & -i_Z\omega & \end{array}$$

$$\begin{array}{ccccc} L_Z\omega = 0, & \text{symplectic} & \longleftrightarrow & d(-i_Z\omega) = 0, & \text{closed} \\ Z & \text{Hamiltonian} & \longleftrightarrow & \text{exact} & \end{array}$$

$\boxed{\begin{array}{l}\underline{\text{Theorem}}\text{ (Ono)} \\ \Gamma \text{ is discrete}\end{array}}$

Define

- $\mathrm{Symp}(M)$ symplectic automorphism group
  $\triangledown$
- $\mathrm{Symp}_0(M)$ connected component of the identity
  $\triangledown$
- $\mathrm{Ham}(M)$ Hamiltonian diffeomorphisms
- $\mathrm{Flux}(M) = \dfrac{\mathrm{Symp}_0(M)}{\mathrm{Ham}(M)}$.

Formally, $L\mathrm{Flux}(M) = \dfrac{\text{closed}}{\text{exact}} \cong H^1(M;\mathbb{R})$

In fact, $\mathrm{Flux}(M) \cong H^1(M;\mathbb{R})/\Gamma$.

Additional structure

For any $\phi \in \mathrm{Symp}(M)$ we have its Floer cohomology $HF^*(\phi)$, a finite-dimensional $\mathbb{Z}/2$-graded vector space over a (specific) field $\mathbb{K}$. This "categorifies" the Lefschetz number

$$\chi(HF^*(\phi)) = L(\phi) = \mathrm{Str}(\phi_* : H_*(M) \circlearrowleft)$$

$L(\phi)$ is not interesting for $\phi \in \mathrm{Symp}_0(M)$, but $HF^*(\phi)$ is. More structure:

- $HF^*(\mathrm{id}) \cong H^*(M;\mathbb{K}) \ni 1$ unit
- $HF^*(\phi) \otimes HF^*(\psi) \longrightarrow HF^*(\phi\psi)$ product
- $HF^*(\phi\psi\phi^{-1}) \cong HF^*(\psi)$ conjugation isomorphisms
- $HF^*(\phi) \ni u_\phi$ if $\phi \in \mathrm{Ham}(M)$ continuation elements (not always canonical)

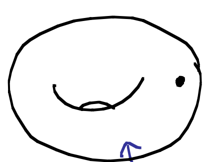
$$\boxed{\begin{array}{l}\textbf{Lemma} \text{ Multiplication with } u_\phi \\ \text{yields} \\ \quad HF^*(\psi) \cong HF^*(\phi\psi) \quad \forall \psi\end{array}}$$

$\longrightarrow$ Entire structure descends to $\mathrm{Flux}(M)$ (up to non-canonical isomorphisms)

Examples homotopy equivalence

$M = T^2$, $\omega = dp \wedge dq$

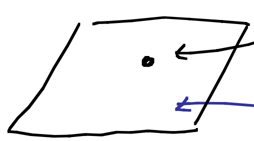
- $Symp(M) \simeq Diff^+(M) \simeq T^2 \rtimes SL_2(\mathbb{Z})$
- $Ham(M) \simeq$ point
- $Flux(M) \cong H^1(M; \mathbb{R}/\mathbb{Z})$



$\leftarrow HF^*(id) \cong H^*(M)$

all other HF^* vanish

$M = \Sigma_g$, $g \geq 2$, $Flux \cong H^1(M;\mathbb{R})$



$HF^*(id) \cong H^*(M)$

all others are Novikov homologies

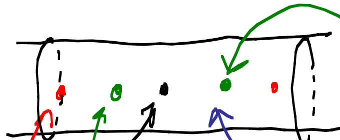
M = symplectic mapping torus of $f: F \to F$

$= \mathbb{R} \times S^1 \times F \,/\, (p,q,x) \sim (p-1, q, f(x))$

Assume that $H^1(F) = 0$ and that "HF^* distinguishes f from id"

Example F a surface, $[f]$ a non-trivial element of the mapping class group (of course, that violates the other assumption)

Then, $Flux(M) \cong \mathbb{R} \times \mathbb{R}/\mathbb{Z}$



$H^*(T^2) \otimes HF^*(f)$

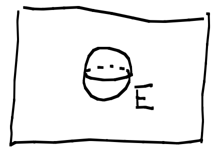
2C2E

all others zero

$H^*(M)$

$H^*(T^2) \otimes HF^*(f^{-1})$

More serious example



$M = T^4 \# \overline{\mathbb{CP}}^2$

$= $ abelian surface blown up once

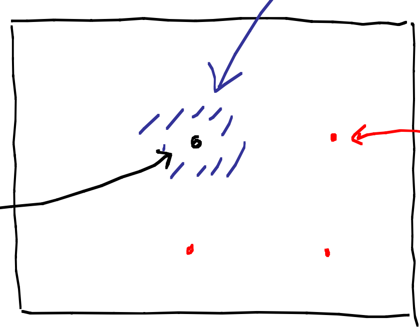
$= \mathrm{Sym}^2(\Sigma^2)$

with an appropriate symplectic form,

$\int_E \omega = \varepsilon > 0$ exceptional divisor

nearby $HF^*(\phi) \cong H^*(\text{point})$

$HF^*(\mathrm{id})$
$\cong$
$H^*(M)$



what happens at the lattice points $H^1(M;\mathbb{Z})$?

Picture of $\mathrm{Flux}(M) \cong \mathbb{R}^4$

Could be your thesis!

Lemma $\mathrm{Flux}(M) \cong H^1(M;\mathbb{R})$

Proof Take $(\varphi_t)_{0 \le t \le 1}$ a loop in $\mathrm{Symp}(M)$.

- The orbits $t \mapsto \varphi_t(x)$ are contractible. This is because the family $\varphi_t(E)$ of embedded spheres is contractible.
- Given a loop $\alpha : S^1 \to M$, we consider

$$T^2 \to M, \quad (s,t) \mapsto \varphi_t(\alpha(s))$$

This factors through $\pi_2(M)$, so $\int_{T^2} \omega = \int_{T^2} c_1 = 0$.

Finite-dimensional algebras

A = an associative algebra with unit, $\dim_{\mathbb{C}}(A) < \infty$

- $\mathrm{Aut}(A)$ automorphism group
- $\mathrm{Inn}(A) = \mathrm{im}(A^\times \to \mathrm{Aut}(A))$ inner automorphisms
- $\mathrm{Out}(A) = \mathrm{Aut}(A)/\mathrm{Inn}(A) \supseteq \mathrm{Out}^0(A)$ connected component of id

Definition For $\phi \in \mathrm{Aut}(A)$,

$$Z_\phi = \{x \in A \mid \phi(y)x = xy \ \forall y\}$$

- $Z_{id} = Z(A)$ center
- $Z_\phi \otimes Z_\psi \longrightarrow Z_{\phi\psi}$ product
- $Z_\phi \cong Z_{\psi\phi\psi^{-1}}$ conjugation

- If $\phi(x) = uxu^{-1}$ is inner, then $u \in Z_\phi$

Entire structure descends to $\mathrm{Out}(A)$, up to non-canonical isomorphism

Morita invariance

A (and B) algebras. $\mathrm{Mod}(A)$ = category of left A-modules, $\mathrm{Mod}(A,B)$ = category of bimodules.

Definition A and B are Morita equivalent if there are $P \in \mathrm{Mod}(A,B)$ and $Q \in \mathrm{Mod}(B,A)$ such that

$$P \otimes_B Q \cong A, \quad Q \otimes_A P \cong B \to \mathrm{Mod}(A) \cong \mathrm{Mod}(B)$$

One can introduce the Picard group

$$\mathrm{Pic}(A) = \text{bimodules invertible under } \otimes \,/\, \text{isomorphism}$$

This is obviously Morita invariant.

$$\mathrm{Out}(A) \hookrightarrow \mathrm{Pic}(A), \quad \phi \longmapsto \mathrm{Graph}(\phi)$$

and $\mathrm{Out}_0(A) \cong \mathrm{Pic}_0(A)$.

Example A and $\mathrm{Mat}_2(A)$ are Morita equivalent

$\mathrm{Aut}(A)$ is <u>not</u> a Morita invariant.

Theorem (Brauer) $\mathrm{Out}_0(A)$ is Morita invariant.

explains

$$Z_\phi = \mathrm{Hom}_{\mathrm{Mod}(A,A)}(A, \mathrm{Graph}(\phi))$$

product structure is the composition of morphisms

Derived Morita invariance

$\mathrm{DMod}(A)$ = derived category (chain complexes up to quasi-iso.)

Similarly $\mathrm{DMod}(A,B)$, have (derived) tensor product.

Definition A and B are derived Morita equivalent if there are $P \in \mathrm{DMod}(A,B)$ and $Q \in \mathrm{DMod}(B,A)$, such that

$$P \overset{L}{\otimes}_B Q \cong A, \quad Q \overset{L}{\otimes}_A P \cong B$$

$\hookrightarrow \mathrm{DMod}(A) \cong \mathrm{DMod}(B)$

Theorem (Rouquier) $\mathrm{Out}_0(A)$ is derived Morita invariant.

Example Let $A = \mathbb{C}[\vec{Q}]$ be the path algebra of the quiver

$$\underset{1}{\circ} \longrightarrow \underset{2}{\circ} \longrightarrow \underset{3}{\circ}$$

so

$$A = \overbrace{\mathbb{C}(1) \oplus \mathbb{C}(2) \oplus \mathbb{C}(3)}^{\text{idempotents}} \oplus \mathbb{C}(1\,2) \oplus \mathbb{C}(2\,3) \oplus \mathbb{C}(1\,2\,3)$$

and $B = A/(1\,2\,3)$. Then A and B are derived Morita equivalent.

Introduce $\mathrm{DPic}(A)$ = invertible objects in $\mathrm{DMod}(A,A)$ (derived Picard group). Then $\mathrm{Pic}(A) \subseteq \mathrm{DPic}(A)$ and

$$\mathrm{DPic}^0(A) \cong \mathrm{Pic}^0(A)$$

Corresponding infinitesimal statement:

$$L\mathrm{DPic}(A) = HH^1(A,A) = L\mathrm{Out}(A).$$

Algebraic geometry

X smooth projective variety / $\mathbb{C}$

- $\mathrm{Coh}(X)$ coherent sheaves
  ↳ $\mathrm{Pic}(X)$, invertible sheaves (line bundles) up to isomorphism
- $D^b\mathrm{Coh}(X)$ bounded derived category
  ↳ $\mathrm{DPic}(X)$, the derived Picard group

Fact (once $\mathrm{DPic}(X)$ has been equipped with a topology)

$$\mathrm{DPic}_0(X) \cong \mathrm{Aut}_0(X) \times \mathrm{Pic}_0(X)$$

Infinitesimally, this is because the Lie algebra of $\mathrm{DPic}(X)$ is

$$HH^1(X) = \mathrm{Ext}^1_{X \times X}(\mathcal{O}_\Delta, \mathcal{O}_\Delta)$$
$$\cong H^0(X, TX) \oplus H^1(X, \mathcal{O}_X)$$

(Hochschild-Kostant-Rosenberg)

$\boxed{\text{Corollary (Rouquier) } \mathrm{Aut}_0 \times \mathrm{Pic}_0 \text{ is a derived invariant (it depends only on } D^b\mathrm{Coh})}$

$\boxed{\text{Theorem (Popa, Schnell) The isogeny class of } \mathrm{Pic}_0(X) \text{ is a derived invariant.}}$

Non-vanishing loci

As usual, to each point of $DPic(X)$ one associates a graded vector space. Restricted to $DPic_0(X)$ these spaces are the fibres of a coherent sheaf. Concretely

- If $\varphi \in Aut(X)$, the vector space is

$$\mathrm{Ext}^*_{X\times X}(\mathcal{O}_\Delta, \mathcal{O}_{\Gamma_\varphi})$$

graph

Example If φ has nondegenerate fixed points,

$$\mathrm{Ext}^*_{X\times X}(\mathcal{O}_\Delta, \mathcal{O}_{\Gamma_\varphi}) \cong \bigoplus_{\varphi(x)=x} \mathbb{C}[-n]$$

- If $\mathcal{L}$ is an invertible sheaf, the vector space is

$$\mathrm{Ext}^*_{X\times X}(\mathcal{O}_\Delta, \mathcal{O}_\Delta \otimes \mathcal{L})$$
$$\Uparrow$$
$$H^*(X, \Lambda^* TX \otimes \mathcal{L})$$

Spectral sequence, seems to degenerate for $\mathcal{L} \in Pic^0(X)$

degree 0: $H^0(X, \mathcal{L}) \cong H^n(X, K_X \otimes \mathcal{L}^{-1})^\vee$

degree $2n$: $H^n(X, K_X^{-1} \otimes \mathcal{L}) \cong H^0(X, K_X^2 \otimes \mathcal{L}^{-1})^\vee$

We can consider the sheaves over $Pic^0(X)$ with fibre

$$H^*(X, K_X^d \otimes \mathcal{L})$$

Are they derived invariant? Partial answers from the formalism (see also Popa's work)

Abstract picture

$\mathcal{C}$ a differential graded category / $\mathbb{C}$

in

$Mod(\mathcal{C})$

$Mod(\mathcal{C},\mathcal{C})$ } suitable (dg derived) categories of modules and bimodules

$HH^*(\mathcal{C},\mathcal{C})$ Hochschild cohomology

$\hookrightarrow HH^1(\mathcal{C},\mathcal{C})$ Lie algebra

$Pic(\mathcal{C})$ (derived) Picard group

not easy to say exactly what kind of object this is. But we can define homomorphisms $G \longrightarrow Pic(\mathcal{C})$, G an algebraic group.

twisted Hochschild cohomology

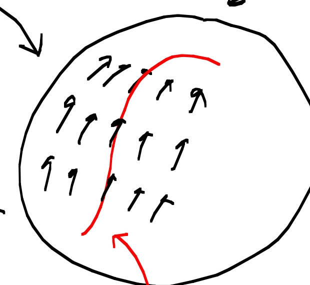
$HH^*(\mathcal{C},P) = Hom^*_{Mod(\mathcal{C},\mathcal{C})}(\mathcal{C},P)$ for $[P] \in Pic(\mathcal{C})$

In cloud-cuckoo land,

moduli space of objects of $Mod(\mathcal{C})$

$Z \in HH^1(\mathcal{C},\mathcal{C})$ defines a vector field

Integrating those vector fields gives the action of $Pic_0(\mathcal{C})$



use geometry of this space?

algebraic orbits easy to understand, what about the others?

$HH^*(\mathcal{C},P)$ is a kind of fixed point theory

Families of objects

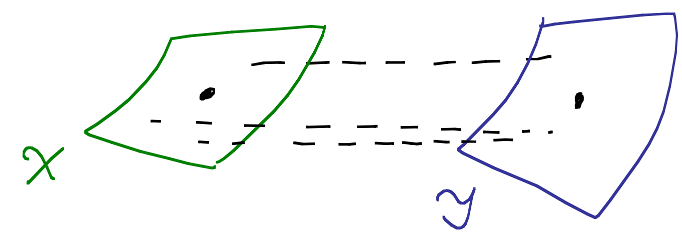
The "moduli space of objects" is based on the notion of algebraic family of objects in $\mathcal{C}$ (parametrized by some algebraic variety T).

More precisely (assuming $\mathcal{C}$ is proper) we arrange the definition so that:

> Fact: If $\mathcal{X}, \mathcal{Y}$ are families over T, then
>
> $$H^*(hom(\mathcal{X}, \mathcal{Y}))$$
>
> is a graded coherent sheaf on T

Non-Hausdorffness of the moduli space: suppose $T = \mathbb{A}^1$ for simplicity.

$$\mathcal{X}_t \cong \mathcal{Y}_t \quad \text{for all } t \neq 0$$



However, suppose the families are both driven by $Z \in HH^1(\mathcal{C}, \mathcal{C})$, so

$$\frac{d}{dt} \mathcal{X}_t = Z|_{\mathcal{X}_t} \in H^1(hom(\mathcal{X}_t, \mathcal{X}_t))$$

and also for $\mathcal{Y}$. Then $H^*(hom(\mathcal{X}, \mathcal{Y}))$ carries an algebraic connection

is locally free $\rightsquigarrow$ uniqueness

good notion of "algebraic orbit" of the action.

Missing foundations

The category associated to a symplectic manifold M is the Fukaya category $\mathrm{Fuk}(M)$, a dg category over a non-archimedean field $\mathbb{K}$

$$\underset{\text{Laurent series}}{\mathbb{C}((t))} \subseteq \mathbb{K}$$

Kontsevich
Soibelman
Fukaya

Algebraic geometry is not enough. We need to define $\mathrm{Pic}(C)$ as an analytic group over $\mathbb{K}$. Ideally, the $HF^*(\phi)$ should form a coherent analytic sheaf over $\mathrm{Pic}(C)$.

Missing notion of "analytic family of objects".

Example $M = T^2$,

$$\mathrm{Pic}(\mathrm{Fuk}(M)) = \text{Tate family of elliptic curves}$$

This is algebraic (an elliptic curve over $\mathbb{K}$)

Look - Mordell!

Fictitious example $c_1(M) = 0$, $b_1(M) = 1$,

$$\mathrm{Flux}(M) \cong H^1(M;\mathbb{R}) = \mathbb{R}.$$

Then one expects $\mathrm{Pic}(\mathrm{Fuk}(M))$ to be analytic,

$$\mathcal{O}(\mathrm{Pic}(\mathrm{Fuk}(M))) = \text{Tate algebra}$$

$$= \left\{ \sum_{k \in \mathbb{Z}} a_k z^k, \; a_k \in \mathbb{K}, \; \mathrm{val}(a_k) \longrightarrow \infty \text{ faster than linearly} \right\}$$

(with Hopf algebra structure)

Why did you have to sit through this?

(Answers may vary.)

In general, $HF^*(id) \cong H^*(M;\mathbb{K})$ but the left hand side is only $\mathbb{Z}/2$-graded. Similarly, $Fuk(M)$ only $\mathbb{Z}/2$-graded. So

$$LPic(Fuk(M)) \cong HH^{odd}(Fuk(M)) \xleftarrow[\text{often an } \cong]{} H^{odd}(M;\mathbb{K}).$$

The global object $Pic(Fuk(M))$ captures (classical) flux phenomena in its $H^1(M;\mathbb{K})$ part, but new phenomena for higher odd cohomology

new invariants of symplectic manifolds.

<u>Examples</u> Take M a symplectic mapping torus, $M \hookrightarrow \mathbb{CP}^r$ symplectically, $r \gg 0$ and blow it up to get N. Then

$$H^1(N) = 0, \quad \boxed{H^{\mathrm{codim}(M)-1}(N) \cong H^1(M)}.$$