LEFSCHETZ FIBRATIONS IN SYMPLECTIC TOPOLOGY

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0 BACKGROUND

I CLASSIFICATION

II STABILIZATION

III POSITIVITY

largely based on work of Auroux, Baykur, Giroux-Pardon, Maydanskiy, Murphy ...

... and of course this inspiring gentleman

0 BACKGROUND

Low-dimensional situation

We consider

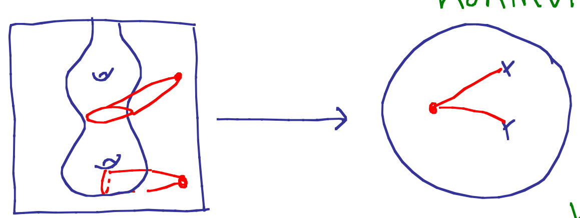
$$\pi : E \longrightarrow D$$

2-dim. disc

where the fibre F is an oriented compact surface with $\partial F \neq \emptyset$.

π has finitely many critical points, modelled on the simplest complex critical point, $\pi(x_1, x_2) = x_1^2 + x_2^2$.

After choosing paths



homologically nontrivial

we get vanishing cycles $V_1, \ldots, V_m \subseteq F$.

The $(V_1, \ldots, V_m)$ are unique up to Hurwitz moves

$$(V_1, \ldots, \boxed{V_i, V_{i+1}} \ldots) \longmapsto (V_1, \ldots, \boxed{\tau_{V_i}(V_{i+1}), V_i}, \ldots)$$

The boundary monodromy is

$$\tau_{V_1} \cdots \tau_{V_m} \in \mathrm{Diff}(F, \partial F).$$

Arbitrary dimensions

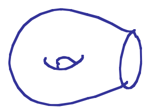
F is now a compact symplectic manifold of dimension $2n-2$, of a specific kind (a Liouville domain).

E is similarly a Liouville domain of dimension $2n$. The vanishing cycles are Lagrangian spheres in F.

I CLASSIFICATION ISSUES

which we can't solve, obviously

Simplest example (Auroux 2013)

Fibre $F =$  $T^2 \setminus \{\text{disc}\}$

$H_1(F) \cong \mathbb{Z}^2$, nonseparating closed curves are determined by their (primitive) homology classes.

braid group

$\pi_0(\mathrm{Diff}(F, \partial F)) \cong \widetilde{SL}_2(\mathbb{Z}) \cong Br_3$

Two vanishing cycles (V_1, V_2), whose homology classes form a matrix $A = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}$. Classification is up to the left action of $SL_2(\mathbb{Z})$, and

Hurwitz
$$A \longmapsto \begin{pmatrix} a_{21} - \det(A)\, a_{11} & a_{11} \\ a_{22} - \det(A)\, a_{12} & a_{12} \end{pmatrix}$$

Invariants

- $|\det(A)| = |H_1(E)|$

- The boundary monodromy is an invariant up to conjugation. In particular, its trace is an invariant. Here, the boundary monodromy $M : H_1(F) \circlearrowleft$ is

$$\left(\mathbb{1} - \begin{pmatrix} a_{11} \\ a_{12} \end{pmatrix} \otimes \begin{pmatrix} a_{12} & -a_{11} \end{pmatrix}\right) \cdot \left(\mathbb{1} - \begin{pmatrix} a_{21} \\ a_{22} \end{pmatrix} \otimes \begin{pmatrix} a_{22} & -a_{21} \end{pmatrix}\right)$$

$$= \begin{pmatrix} 1 - a_{11}a_{12} & a_{11}^2 \\ -a_{12}^2 & 1 + a_{12}a_{11} \end{pmatrix} \begin{pmatrix} 1 - a_{21}a_{22} & a_{21}^2 \\ -a_{22}^2 & 1 + a_{22}a_{21} \end{pmatrix}$$

$$= \begin{pmatrix} (1 - a_{11}a_{12})(1 - a_{21}c_{22}) - a_{11}^2 a_{22}^2 & * \\ * & (1 + a_{12}a_{11})(1 + a_{22}a_{21}) - a_{12}^2 a_{21}^2 \end{pmatrix}$$

hence its trace is

$$2 + 2a_{11}a_{12}a_{21}a_{22} - a_{11}^2 a_{22}^2 - a_{12}^2 a_{22}^2$$
$$= 2 - \det(A)^2.$$

<u>However</u>, $|\det(A)|$ is not a complete invariant of our Lefschetz fibration. After left multiplication with $SL_2(\mathbb{Z})$, one has

coprime:

$$A = \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix} \qquad ax + by = 1.$$

where $0<b = \det(A)$, and a matters only mod b. The Hurwitz move plus $SL_2(\mathbb{Z})$-action is

$$A \mapsto \begin{pmatrix} a-b & 1 \\ b & 0 \end{pmatrix} \overset{SL_2(\mathbb{Z})}{\sim} \begin{pmatrix} 1 & x \\ 0 & -b \end{pmatrix} \overset{\text{orientation change}}{\sim} \begin{pmatrix} 1 & -x \\ 0 & b \end{pmatrix}$$

This is $a \mapsto -\frac{1}{a}$ on $(\mathbb{Z}/b)^\times$.

This has $\geq \frac{\phi(b)-1}{2}$ orbits, which are distinct Lefschetz fibrations. Of course, it could still be the case that they are distinguished by boundary monodromies.

<u>Example</u> • $A = \begin{pmatrix} 1 & \boxed{2} \\ 0 & 5 \end{pmatrix}$ yields the boundary monodromy

$$M = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -9 & 4 \\ -25 & 11 \end{pmatrix} = \begin{pmatrix} -34 & 15 \\ -25 & 11 \end{pmatrix}$$

• $\tilde{A} = \begin{pmatrix} 1 & \boxed{3} \\ 0 & 5 \end{pmatrix}$ correspondingly yields

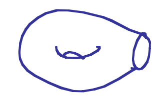
$$\tilde{M} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -14 & 9 \\ -25 & 16 \end{pmatrix} = \begin{pmatrix} -39 & 25 \\ -25 & 16 \end{pmatrix}$$

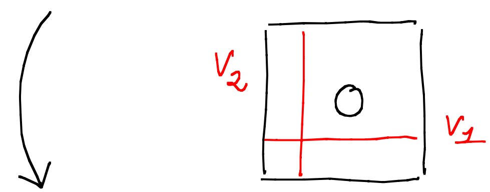
These are actually <u>conjugate</u>, and the same holds in $SL_2(\mathbb{Z})$.

Outcome: We have two 4 dim. Lefschetz fibrations whose total spaces E, $\tilde{E}$ are Liouville domains with $\partial E \cong \partial \tilde{E}$ as contact 3-manifolds. The Lefschetz fibrations are distinct, but:

Open question Is $E \cong \tilde{E}$ as a symplectic 4-manifold?

Remark There is always a diffeomorphism of total spaces which reverses the symplectic form, and takes $(a,b) \longmapsto (b-a, b)$. This applies to $(2,5) \mapsto (3,5)$.

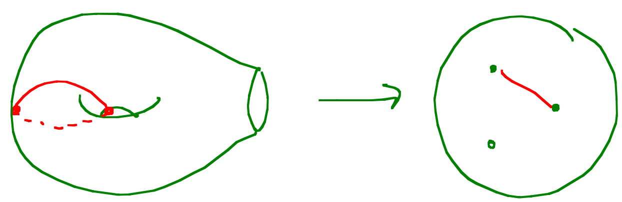
Contrast Classical result (Mandelbaum–Moishezon 1980): take a Lefschetz fibration with $F =$ , and boundary monodromy = k-fold rotation along ∂F (some $k \geq 1$). Then, the fibration has $12k$ vanishing cycles, and is uniquely determined: after Hurwitz moves, the vanishing cycles are $(V_1, V_2, V_1, \ldots)$



In particular, the previous two examples become isomorphic after adding (the same set of) additional vanishing cycles!

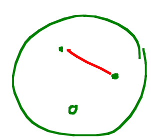
Higher dimensions Idea: think of F = as a double branched cover

$$\mathbb{C}^2 \supset F = \{x^2 + y^3 = 1\} \xrightarrow{y} \mathbb{C}$$

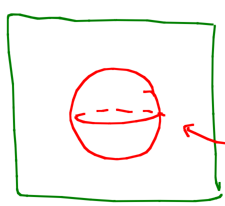


The preimage of a path in $\mathbb{C}$ joining two branch points is a closed curve $V \subseteq F$. Now, take instead

$$\mathbb{C}^3 \supset F = \{x_1^2 + x_2^2 + y^3 = 1\} \longrightarrow$$

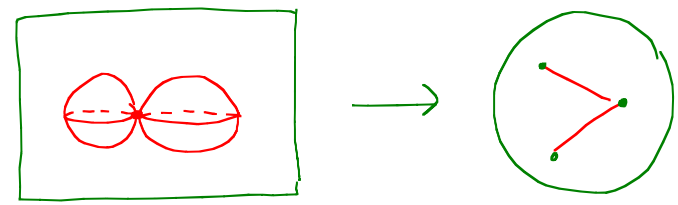


(A_2) Milnor fibre



Lagrangian sphere V

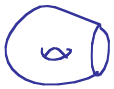
- $H_2(F) \cong \mathbb{Z}^2$
- $\pi_0(\mathrm{Symp}(F, \partial F)) \cong \widetilde{SL_2(\mathbb{Z})} \cong Br_3$ generated by Dehn twists along

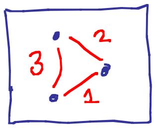


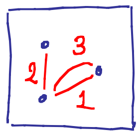
However, the classical (differential) topology of these manifolds is quite poor:

$$\begin{array}{ccc} Br_3 & \xrightarrow{\cong} & \pi_0(\mathrm{Symp}(F, \partial F)) \\ \downarrow & & \downarrow \\ \mathrm{Sym}_3 & \longrightarrow & \pi_0(\mathrm{Diff}(F, \partial F)) \\ & \underset{\cong}{\searrow} \quad \mathrm{Aut}(H_2(F), \cdot) \quad \swarrow & \end{array}$$

Example

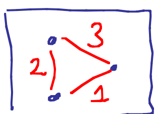
$F =$  , 3 vanishing cycles

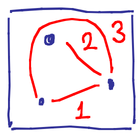
- 

$\sim$ Hurwitz 

$\sim$ cancellation

Total space $E \cong \{x_1^2 + x_2^2 + y^2 = 1\}$
$\cong T^*S^2$, intersection form (-2)

- 

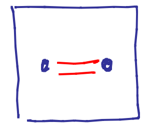
$\sim$ Hurwitz 

(no cancellation possible in this case)

Total space $E \cong \{x^2 + yz^2 = 1\}$
(called a Danielewski surface),
intersection form (-4)

Same example in higher dimension

$F = (A_2)$ Milnor fibre, 3 vanishing cycles

 (I did not explain cancellation...)

Total space $E \cong \{x_1^2 + x_2^2 + x_3^2 + y^2 = 1\}$
$\cong T^*S^3$ (intersection form is trivial)

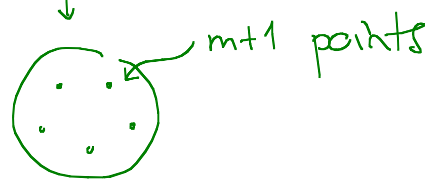
Total space $E \cong \{x_1^2 + x_2^2 + yz^2 = 1\}$
is diffeomorphic to $T^*S^3 = S^3 \times \mathbb{R}^3$

Thm (Maydanskiy 2009) $E \not\cong T^*S^3$ as a symplectic manifold.

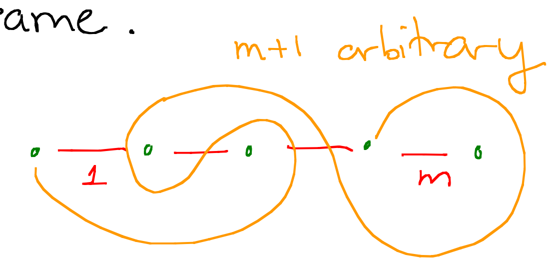
More generally, for $m \geq 2$ consider

$$F = \{x_1^2 + x_2^2 + x_3^{m+1} = 1\}$$

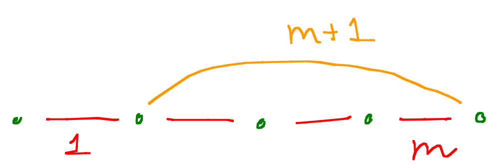
$$\downarrow$$



Let's take $m+1$ vanishing cycles, of which the first m are always the same.



For

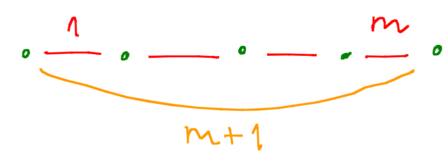


we get total space $E \cong T^*S^3$.

<u>Thm</u> (Maydanskiy-Seidel 2009) For all but $\binom{m+1}{2}$ choices of the last vanishing cycle, $E \not\cong T^*S^3$ symplectically.

But what is E then? The underlying differentiable manifold is always T^*S^3.

<u>Thm</u> (Murphy 2014) For the following choice of vanishing cycles,



the total space E is a <u>flexible Stein manifold</u>

completely determined by homotopical invariants

Corollary The affine complex 3-folds

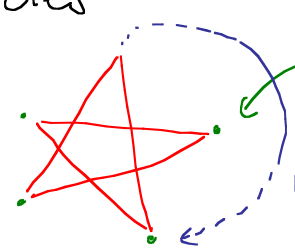
$$\mathbb{C}^4 \supseteq \{ xy^a + z_1^2 + z_2^2 = 1 \}, \quad a \geq 2$$

are all symplectically isomorphic.

Question How about the following (for coprime a, b):

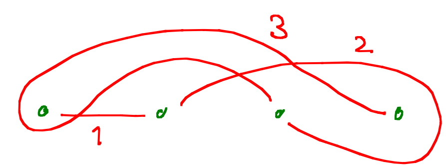
$$\{ x^a y^b + z_1^2 + z_2^2 = 1 \} \quad ?$$

For $F = \{ x^{a+b} + z_1^2 + z_2^2 = 1 \}$, this corresponds to a choice of vanishing cycles



The most ill-guided conjecture in symplectic topology:

Conjecture (Seidel, 2006) Pick a "random" choice of m vanishing cycles in $F = \{ x^m + z_1^2 + z_2^2 = 1 \}$ such that the endpoints of the associated paths form a chain:



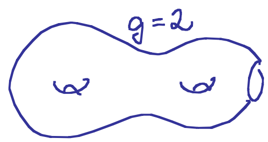
Then, with "very high probability", the resulting E is a nonstandard symplectic structure on $\mathbb{R}^6$.

The opposite seems much more likely!

II STABILIZATION PROCESSES

which possibly no one other than Denis Auroux understands, even in the lowest dimension

Take a Lefschetz fibration with

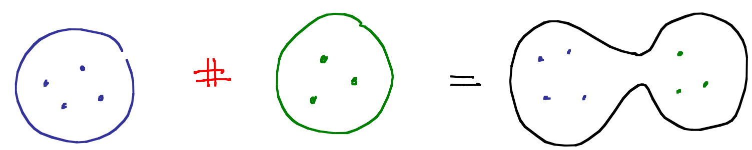
- fibre $F = \Sigma_{g,1}$ 

- boundary monodromy equals the k-fold rotation of ∂F
- d vanishing cycles
- total space E with signature s

<u>Theorem</u> (Auroux 2004) Any two such fibrations with the same (g, k, d, s) become isomorphic after sufficiently many iterations of a fixed <u>stabilization</u> process.

The stabilization process adds a fixed pattern of vanishing cycles, and can be thought of as a <u>fibre connect sum</u>:

$$\begin{array}{ccc} E & & E^{std} \\ \downarrow \pi & \# & \downarrow \pi^{std} \\ D & & D \end{array}$$



The idea behind Auroux's theorem is to ensure that all relations in the mapping class group can be "realized by Hurwitz moves".

Corollary There is an explicit sequence of 4dim Lefschetz fibrations, such that any other fibration can be embedded into one belonging to that sequence.

Corollary There is an explicit sequence of 4dim Stein domains, such that every Stein domain can be symplectically embedded into one belonging to our sequence.

Theorem (Giroux-Pardon) Any Stein domain is the total space of a Lefschetz fibration

These domains are the 4-dimensional counterpart of a "surface of genus $\gg 0$". Let's imprecisely call them "large".

Question Is there a stable classification of fibrations whose fibre is a "large" 4-dimensional Stein domain?

Hope Such "large" Stein domains can be constructed in any dimension (maybe using Lefschetz fibrations as a tool).

III POSITIVITY

Recent progress in
low-dimensional case
(Baykur and collaborators)

Classical low-dimensional situation

Theorem Let F be a surface (as usual, compact, oriented, $\partial F \neq \emptyset$). In $\mathrm{Diff}(F, \partial F)$, a product of Dehn twists is never isotopic to the identity.

(Smith 1999)

Proof sketch for $g(F) \geq 2$. Take the closed surface $F \subseteq S$, $* \in S \setminus F$, and

$$\pi_0(\mathrm{Diff}(F, \partial F)) \longrightarrow \pi_0(\mathrm{Diff}(S, *))$$

$$\mathrm{Homeo}^+(S^1)$$

pass to universal cover, extend to boundary at ∞

Dehn twists "turn S^1 to the right".

Arbitrary dimensions

Theorem (S, unpublished) Let F be a Liouville domain (assume for simplicity that $H^1(F, \partial F) = 0$). In $\mathrm{Symp}(F, \partial F)$, a product of Dehn twists is never isotopic to the identity.

Proof sketch Suppose the contrary,

$$\tau_{V_1} \cdots \tau_{V_r} \simeq id.$$

Then, the V_i are vanishing cycles for a Lefschetz fibration that extends over the two-sphere, $\pi : E \longrightarrow S^2$.

By looking at pseudo-holomorphic sections, this can't have critical points.

<u>Quantitative positivity</u> In the case of $\pi_0(\mathrm{Diff}(F,\partial F)) \cong \widetilde{SL_2(\mathbb{Z})}$, $F =$,

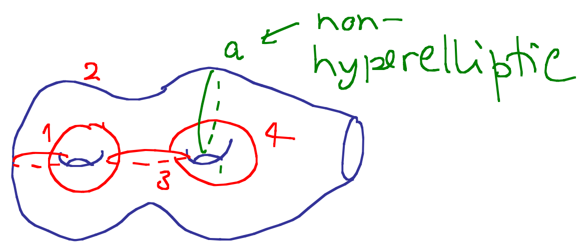
we have a homomorphism

$$\widetilde{SL_2(\mathbb{Z})} \longrightarrow \mathbb{Z}$$

$$\text{any Dehn twist} \longmapsto 1$$

which gives us a precise measure of positivity (same for all braid groups)

Take

$F =$ 

and define (in shorthand notation) two elements of $\pi_0(\mathrm{Diff}(F,\partial F))$ as follows.

$$\phi = 4\,3\,2\,1\,1\,2\,3\,4\,4\,a\,3\,4$$

$$\psi = 1\,2\,3\,1\,2\,3\,2\,1\,3\,2$$

<u>Theorem</u> (Baykur) ϕ is conjugate to $\psi^{-1}\phi$, say

$$\phi = \alpha(\psi^{-1}\phi)\alpha^{-1}$$

Hence $\psi(\alpha^{-1}\phi\alpha) = \phi$; "the number of Dehn twists is not preserved".

<u>Corollary</u> We get Lefschetz fibrations with arbitrarily big $b_2(E)$ and the same boundary monodromy.

<u>Conjecture</u> The same is true for "large" higher-dim. Liouville domains.