Journal of Pure and Applied Algebra 73 (1991) 307–314
North-Holland 

# On the Hilbert function of a graded Cohen–Macaulay domain

Richard P. Stanley*

*Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA*

Communicated by A. Blass
Received 7 May 1990
Revised 6 August 1990

*Abstract*

Stanley, R.P., On the Hilbert function of a graded Cohen–Macaulay domain, Journal of Pure and Applied Algebra 73 (1991) 307–314.

A condition is obtained on the Hilbert function of a graded Cohen–Macaulay domain $R = R_0 \oplus R_1 \oplus \cdots$ over a field $R_0 = K$ when $R$ is integral over the subalgebra generated by $R_1$. A result of Eisenbud and Harris leads to a stronger condition when char $K = 0$ and $R$ is generated as a $K$-algebra by $R_1$. An application is given to the Ehrhart polynomial of an integral convex polytope.

## 1. Introduction

By a *graded algebra* over a field $K$, we mean here a commutative $K$-algebra $R$ with identity, together with a vector space direct sum decomposition $R = \coprod_{i \geq 0} R_i$, such that: (a) $R_i R_j \subseteq R_{i+j}$, (b) $R_0 = K$ (i.e., $R$ is *connected*), and (c) $R$ is finitely-generated as a $K$-algebra. $R$ is *standard* if $R$ is generated as a $K$-algebra by $R_1$, and *semistandard* if $R$ is integral over the subalgebra $K[R_1]$ of $R$ generated by $R_1$. The *Hilbert function* $H(R, \cdot)$ of $R$ is defined by $H(R, i) = \dim_K R_i$, for $i \geq 0$, while the *Hilbert series* is given by

$$F(R, \lambda) = \sum_{i \geq 0} H(R, i)\lambda^i .$$

There has been considerable recent interest in the connections between the behavior of $H(R, i)$ and the structure of $R$. In particular, Hilbert functions of the following classes of standard graded algebras have been completely characterized: (a) arbitrary [11, Theorem 2.2] (essentially a result of Macaulay), (b) Cohen–

*Partially supported by NSF grant #DMS 8401376.

Macaulay, or more generally, of fixed depth and Krull dimension [11, Corollaries 3.10 and 3.11] (again essentially due to Macaulay), (c) complete intersections [11, Corollary 3.4] (again Macaulay, and also independently, Gröbner), and (d) reduced (i.e., no nonzero nilpotents) [1]. Partial results have been achieved for Gorenstein rings [11, Theorem 4.1; 9]. One class of rings conspicuously absent from the above list is the (integral) domains. Some results in this direction are due to Roberts and Roitman [10]. In particular, they obtain [10, Theorem 4.5] a strong restriction on the Hilbert function of a standard graded domain of Krull dimension one, viz., if the function $\Delta H(R, i) := H(R, i) - H(R, i-1)$ starts to decrease strictly, then it strictly decreases until reaching 0. Moreover, they show [10, p. 103], based on an idea of A. Geramita, that for any $d \geq 0$ there does not exist a graded domain $R$ of Krull dimension $d$ and Hilbert series

$$F(R, \lambda) = \frac{1 + 2\lambda + \lambda^2 + \lambda^3}{(1-\lambda)^d}\,. \tag{1}$$

(They assume that $R$ is standard, but their proof does not use this fact.) Moreover, there do exist reduced Cohen–Macaulay standard graded algebras $R$ with this Hilbert series when $d \geq 1$.

Our main result (Theorem 2.1) will be a condition on the Hilbert function (or Hilbert series) of a semistandard Cohen–Macaulay domain $R$. We point out how further results follow from work of Eisenbud and Harris [2] related to Castelnuovo theory when $R$ is standard and char $K = 0$. Finally in Section 4 we give an application to the Ehrhart polynomial of a convex polytope.

## 2. Semistandard Cohen–Macaulay domains

Let $R$ be a semistandard graded $K$-algebra of Krull dimension $d$. Let $K[R_1]$ be the subalgebra of $R$ generated by $R_1$, so $K[R_1]$ is a standard graded $K$-algebra. Since $R$ is integral over $K[R_1]$ it follows that $R$ is a finitely-generated $K[R_1]$-module. Hence by well-known properties of Hilbert series we have

$$F(R, \lambda) = \frac{h_0 + h_1\lambda + \cdots + h_s\lambda^s}{(1-\lambda)^d}\,,$$

for certain integers $h_0, \ldots, h_s$ satisfying $\sum h_i \neq 0$ and $h_s \neq 0$. We call the vector $h(R) := (h_0, \ldots, h_s)$ the *h-vector* of $R$.

**Theorem 2.1.** *Suppose $R$ is a semistandard graded Cohen–Macaulay domain with $h(R) = (h_0, \ldots, h_s)$. Then*

$$h_0 + h_1 + \cdots + h_i \leq h_s + h_{s-1} + \cdots + h_{s-i} \tag{2}$$

*for all $0 \leq i \leq s$.*

**Proof.** Let $\Omega(R)$ denote the canonical module of $R$ (see [4]), which exists since $R$ is Cohen–Macaulay. $\Omega(R)$ has the structure $\Omega(R) = \Omega(R)_0 \oplus \Omega(R)_1 \oplus \cdots$ of a finitely-generated graded $R$-module with Hilbert series

$$F(\Omega(R), \lambda) = \frac{h_s + h_{s-1}\lambda + \cdots + h_0\lambda^s}{(1-\lambda)^d}. \tag{3}$$

(See the proof of Theorem 4.4 of [11]. The integer $q$ of [11, equation (12)] may be chosen arbitrarily by shifting the grading of $\Omega(R)$; we choose $q$ so that (3) above is valid.) Pick an element $0 \neq u \in \Omega(R)_0$. Since $R$ is a domain, $\Omega(R)$ is a torsion-free $R$-module. (In fact, $\Omega(R)$ is isomorphic to an ideal of $R$ [4, Corollary 6.7].) Hence as $R$-modules we have $uR \cong R$.

We now use the following result from [8, Exercise 14(2) on p. 103] (in the special case $I = R_+$). Let $0 \to A \to B \to C \to 0$ be an exact sequence of graded $R$-modules, with $R_+A \neq A$, $R_+B \neq B$, $R_+C \neq C$. (If $A$, $B$, $C$ are finitely-generated, then these last conditions are equivalent to $A \neq 0$, $B \neq 0$, $C \neq 0$.) Assume depth $B >$ depth $C$. Then depth $A = 1 +$ depth $C$.

Apply this result to the exact sequence

$$0 \to uR \to \Omega(R) \to \Omega(R)/uR \to 0. \tag{4}$$

Since $R \neq 0$, we always have $uR \cong R \neq 0$ and $\Omega(R) \neq 0$. Thus if $\Omega(R)/uR \neq 0$, then

$$\operatorname{depth} uR = 1 + \operatorname{depth} \Omega(R)/uR.$$

Now $\operatorname{depth} uR = d$ since $uR \cong R$ and $R$ is Cohen–Macaulay. Hence either $\Omega(R) = uR$, or $\operatorname{depth} \Omega(R)/uR = d - 1$. But since $\Omega(R)$ is isomorphic to a nonzero ideal of the domain $R$, it follows that $\dim \Omega(R)/uR < \dim R = d$. Therefore, we have

$$\Omega(R) = uR, \quad \text{or} \quad \dim \Omega(R)/uR = \operatorname{depth} \Omega(R)/uR = d - 1. \tag{5}$$

In the latter case we have that $\Omega(R)/uR$ is Cohen–Macaulay of Krull dimension $d - 1$.

*Note.* (5) can also be obtained from the long exact sequence of some depth-sensitive functor such as local cohomology (with respect to the ideal $R_+ = R_1 \oplus R_2 \oplus \cdots$ of $R$), applied to the short exact sequence (4).

If $\Omega(R) = uR$, then $\Omega(R) \cong R$ so $R$ is Gorenstein. In this case we have $h_i = h_{s-i}$ [11, Theorem 4.1], so (2) holds with equality. Hence assume $\Omega(R)/uR \neq 0$. We may tensor the $R$-module $M = \Omega(R)/uR$ with an infinite extension field of $K$ without altering the Cohen–Macaulay property, the Krull dimension, or the Hilbert series. Thus assume that $K$ is infinite. Let $R' = R/(\operatorname{Ann} M)$, where $\operatorname{Ann} M = \{x \in R: xM = 0\}$. Since $K$ is infinite, the subalgebra $K[R_1']$ of $R'$

generated by $R_1'$ has a homogeneous system of parameters (h.s.o.p.) $\theta_1, \ldots, \theta_{d-1}$ of degree one. Since $R$ is integral over $K[R_1]$, it follows that $\theta_1, \ldots, \theta_{d-1}$ is an h.s.o.p. for $R'$. Any h.s.o.p. for $R/(\text{Ann } M)$ is an h.s.o.p. for $M$, so $\theta_1, \ldots, \theta_{d-1}$ is an h.s.o.p. for $M$.

Let $N = M/(\theta_1 M + \cdots + \theta_{d-1}M)$. Since $M$ is Cohen–Macaulay we have [11, Corollary 3.2]

$$F(M, \lambda) = \frac{F(N, \lambda)}{\prod_{i=1}^{d-1} (1 - \lambda^{\deg \theta_i})} = \frac{F(N, \lambda)}{(1-\lambda)^{d-1}} .$$

Thus the polynomial $F(N, \lambda) = \sum k_i \lambda^i$ has nonnegative coefficients. But

$$\begin{aligned} F(M, \lambda) &= F(\Omega(R), \lambda) - F(uR, \lambda) \\ &= \frac{h_s + h_{s-1}\lambda + \cdots + h_0\lambda^s}{(1-\lambda)^d} - \frac{h_0 + h_1\lambda + \cdots + h_s\lambda^s}{(1-\lambda)^d} \\ &= \frac{k_0 + k_1\lambda + \cdots + k_{s-1}\lambda^{s-1}}{(1-\lambda)^{d-1}} . \end{aligned}$$

An easy computation shows that

$$k_i = (h_s + h_{s-1} + \cdots h_{s-i}) - (h_0 + h_1 + \cdots + h_i) ,$$

and the proof follows. □

**Note.** The module $M = \Omega(R)/uR$ has the interesting property that it is a 'Gorenstein module' in the sense that $\Omega(M) \cong M$, where $\Omega(M)$ is the canonical module of $M$ as defined, e.g., in [12, equation (15)].

## 3. Some further results

For the sake of completeness we mention the following easy and well-known result. Geometrically, it asserts when $R$ is standard that an irreducible projective variety of dimension zero over an algebraically closed field consists of a single point.

**Proposition 3.1.** *Let $R$ be a graded domain of Krull dimension one over an algebraically closed field $K$. Then $R$ is isomorphic to the monoid algebra $K[\Gamma]$ of some (additive) submonoid $\Gamma$ of $\mathbb{N} = \{0, 1, 2, \ldots\}$. In other words, $R$ is isomorphic to a graded subalgebra of the polynomial ring $K[x]$ (with the standard grading $\deg x = 1$), i.e., a subalgebra generated (or spanned) by monomials. In particular, if $R$ is semistandard, then $R \cong K[x]$.*

**Proof.** It clearly suffices to show that $H(R, i) = 0$ or 1 for every $i \geq 0$. Suppose $H(R, i) \geq 2$. Let $u, v \in R_i$ be linearly independent. Since $\dim R = 1$, $u$ and $v$ satisfy a nontrivial homogeneous polynomial equation $P(u, v) = 0$. Since $K$ is algebraically closed, $P(u, v)$ factors into linear factors $\alpha u + \beta v$. Since $R$ is a domain, at least one of these factors must be zero, contradicting the linear independence of $u$ and $v$. □

Of course Proposition 3.1 fails for $K$ nonalgebraically closed, e.g., $R = \mathbb{R}[x, y]/(x^2 + y^2)$.

Now assume $R$ has Krull dimension at least two. If $L$ is a purely transcendental extension field of $L$, then $R \otimes_K L$ will be a graded $L$-algebra which preserves such properties of $K$ as being standard, semistandard, Cohen–Macaulay, and a domain, as well as the Hilbert function, depth, and Krull dimension. (For all these properties except being a domain, $L$ can be any extension field of $K$.) Thus in the proof of Corollary 3.3 below it is valid to replace $K$ by a purely transcendental extension field.

Insofar as Hilbert functions of standard graded domains $R$ of Krull dimension at least two are concerned, Bertini's theorem from algebraic geometry (see [15, p. 68] and also [3, Chapter II, Theorem 8.18 and Remark 8.18.1]) tells us that we may assume $\dim R = 2$. For completeness we state a weak form of this result in the following algebraic form.

**Proposition 3.2.** *Let $R$ be a standard graded domain of Krull dimension at least three over an infinite field $K$. Then there exists a parameter $\theta$ of degree one (i.e., $\theta \in R_1$ and $\dim R/\theta R = \dim R - 1$) such that if $S = R/\theta R$, then $S/H^0(S)$ is a domain. Here*

$$H^0(S) = \{x \in S : xS_+^n = 0 \text{ for some } n \geq 1\},$$

*the 0th local cohomology module of $S$ (with respect to the irrelevant ideal $S_+$).* □

**Corollary 3.3.** *Let $R$ be a standard Cohen–Macaulay graded domain of Krull dimension $d \geq 2$. Then the h-vector $h(R)$ is the h-vector of a standard Cohen–Macaulay graded domain of Krull dimension two.*

**Proof.** Extend the field $K$ by a purely transcendental extension field if necessary. By Proposition 3.2 there is a regular sequence $\theta_1, \ldots, \theta_{d-2} \in R_1$ for which $R/(\theta_1 R + \cdots + \theta_{d-2} R)$ is a standard Cohen–Macaulay graded domain of Krull dimension two. But for any graded algebra $A$, if $\theta \in A_i$ is a non-zero-divisor, then $F(A/\theta A, \lambda) = (1 - \lambda^i) F(A, \lambda)$. Hence $R$ and $R/(\theta_1 R + \cdots + \theta_{d-2} R)$ have the same $h$-vector, as desired. □

Finally we mention how a result of Eisenbud and Harris leads to some results related to Theorem 2.1 when $R$ is standard and char $K = 0$.

**Proposition 3.4.** *Let $R$ be a standard graded Cohen–Macaulay domain of Krull dimension $d \geq 2$ over a field $K$ of characteristic 0. Let $h(R) = (h_0, h_1, \ldots, h_s)$, where $h_s \neq 0$. Let $m \geq 0$ and $n \geq 1$, with $m + n < s$. Then*

$$h_{m+1} + h_{m+2} + \cdots + h_{m+n} \geq h_1 + h_2 + \cdots + h_n .$$

**Proof.** The quantity $h_{\Gamma}(n)$ of [2, Chapter 3] is equal, in our notation, to $h_0 + \cdots + h_n$. Moreover, the degree $d$ in [2] is our $h_0 + \cdots + h_s$. Corollary 3.5 of [2] asserts that

$$h_{\Gamma}(m+n) \geq \min(d, h_{\Gamma}(m) + h_{\Gamma}(n) - 1) ,$$

so in our notation,

$$h_0 + \cdots + h_{m+n} \geq \min(h_0 + \cdots + h_s, h_0 + \cdots + h_m + h_1 + \cdots + h_n) ,$$

since $h_0 = 1$. This is easily seen to be equivalent to the desired result. □

For instance, if $n = 1$ in Proposition 3.4, we obtain $h_1 \leq h_i$ for $1 \leq i \leq s-1$. In particular, if $R$ is Gorenstein (so $h_i = h_{s-i}$) and $s \leq 5$, then $h(R)$ is unimodal. It is not known whether $h(R)$ is unimodal for any standard Cohen–Macaulay (or Gorenstein) graded domain $R$ (see [14, Conjecture 4(a)], [5, Conjecture 1.5]). If $R$ is just assumed to be standard Gorenstein (but not a domain), then $h(R)$ need not be unimodal [11, p. 70]. If $R$ is assumed to be a semistandard Gorenstein graded domain, then again $h(R)$ need not be unimodal, as shown by the example

$$R = K[y, x_1x_2y, x_1x_3y, x_2x_3y, x_1x_2x_3y^2]$$

(with the grading given by $\deg x_1^{a_1}x_2^{a_2}x_3^{a_3}y^b = b$), where $h(R) = (1, 0, 1)$. A related conjecture of Hibi [5, Conjecture 1.4] states that $h_0 \leq h_1 \leq \cdots \leq h_{[s/2]}$ and $h_i \leq h_{s-i}$ for all $0 \leq i \leq [s/2]$, when $R$ is a standard Cohen–Macaulay graded domain. We also do not know whether Proposition 3.4 continues to hold for arbitrary fields $K$. It would be interesting to investigate to what extent the techniques of [2] can be used to obtain additional results about Hilbert functions of standard graded domains.

## 4. An example: The Ehrhart polynomial

In this section we will give a combinatorially interesting example of a semistandard Cohen–Macaulay graded domain. Let $\mathcal{P}$ be a $d$-dimensional convex polytope in $\mathbb{R}^n$ with integer vertices. Let $R_{\mathcal{P}}$ be the subalgebra of

$$K[x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}, y]$$

generated by all monomials

$$x_1^{a_1} \cdots x_n^{a_n} y^b \quad \text{with } b \geq 1 \text{ and } \frac{1}{b}(a_1, \ldots, a_n) \in \mathcal{P}\,.$$

In fact, $R_{\mathcal{P}}$ as a $K$-vector space has a basis consisting of these monomials together with 1. Define a grading on $R_{\mathcal{P}}$ by setting $\deg x_1^{a_1} \cdots x_n^{a_n} y^b = b$. Thus the Hilbert function $H(R_{\mathcal{P}}, j)$ is equal to the number of points $\alpha \in \mathcal{P}$ satisfying $j\alpha \in \mathbb{Z}^n$, or in other words

$$H(R_{\mathcal{P}}, j) = \#(j\mathcal{P} \cap \mathbb{Z}^n)\,.$$

Then $H(R_{\mathcal{P}}, j)$ is a polynomial function of $j$ of degree $d$, known as the *Ehrhart polynomial* of $\mathcal{P}$ and denoted $i(\mathcal{P}, j)$. For an introduction to Ehrhart polynomials, see [13, pp. 235–241].

Since $\deg H(R_{\mathcal{P}}, j) = d$ it follows that $\dim R_{\mathcal{P}} = d + 1$. Moreover, it is easy to see that $R_{\mathcal{P}}$ is normal, so by a theorem of Hochster [7] $R_{\mathcal{P}}$ is Cohen–Macaulay. Trivially $R_{\mathcal{P}}$ is a domain. Finally, the subalgebra $K[(R_{\mathcal{P}})_1]$ contains the monomials $x_1^{a_1} \cdots a_n^{a_n} y$ for which $(a_1, \ldots, a_n)$ is a vertex of $\mathcal{P}$. It then follows easily from the convexity of $\mathcal{P}$ that $R_{\mathcal{P}}$ is integral over $K[(R_{\mathcal{P}})_1]$. Hence $R_{\mathcal{P}}$ is semistandard. Thus from Theorem 2.1 we obtain the following proposition:

**Proposition 4.1.** *Let $\mathcal{P}$ be a convex d-polytope in $\mathbb{R}^n$ with integer vertices. Let $i(P, j)$ denote its Ehrhart polynomial, and write*

$$\sum_{j \geq 0} i(\mathcal{P}, j)\lambda^j = \frac{h_0 + h_1\lambda + \cdots + h_s\lambda^s}{(1-\lambda)^{d+1}}\,, \tag{6}$$

*where $h_s \neq 0$. (Since $i(\mathcal{P}, j)$ is a polynomial for* all *$j$ we have $s \leq d$.) Then*

$$h_0 + h_1 + \cdots + h_i \leq h_s + h_{s-1} + \cdots + h_{s-i}$$

*for all $0 \leq i \leq s$.* □

The algebra $R_{\mathcal{P}}$ need not be standard, e.g., when $\mathcal{P}$ is the simplex with vertices $(0,0,0)$, $(1,1,0)$, $(1,0,1)$, $(0,1,1)$. For this example $R_{\mathcal{P}}$ is just the ring $R$ mentioned at the end of Section 3, so $h(R) = (h_0, \ldots, h_s) = (1, 0, 1)$.

In [6, Theorem 1] Hibi obtains the additional inequality

$$h_0 + h_1 + \cdots + h_{i+1} \geq h_d + h_{d-1} + \cdots + h_{d-i}\,,$$

$0 \leq i \leq d$, where $(h_0, \ldots, h_s)$ is given by (6) (and where we set $h_{s+1} = h_{s+2} = \cdots = h_d = 0$). Such an inequality exists because one can describe an explicit ideal $I$ of $R_{\mathcal{P}}$ for which $I \cong \Omega(R)$ and then apply an argument to $R_{\mathcal{P}}/I$ similar to what was done in the proof of Theorem 2.1 to $\Omega(R)/uR$.

## Acknowledgment

I am grateful to David Eisenbud for some helpful discussions.

## References

[1] A.V. Geramita, P. Maroscia and L. Roberts, The Hilbert function of a reduced $k$-algebra, J. London Math. Soc. (2) 28 (1983) 443–452.
[2] J. Harris (with D. Eisenbud), Curves in Projective Space, Séminaire de Mathématiques Supérieures (Les Presses de l'Université de Montréal, Montréal, 1982).
[3] R. Hartshorne, Algebraic Geometry (Springer, Berlin, 1977).
[4] J. Herzog and E. Kunz, eds., Der kanonische Modul eines Cohen–Macaulay-Rings, Lecture Notes in Mathematics 238 (Springer, Berlin, 1971).
[5] T. Hibi, Flawless $O$-sequences and Hilbert functions of Cohen–Macaulay integral domains, J. Pure Appl. Algebra 60 (1989) 245–251.
[6] T. Hibi, Some results on the Ehrhart polynomial of a convex polytope, Discrete Math. 83 (1990) 119–121.
[7] M. Hochster, Rings of invariants of tori, Cohen–Macaulay rings generated by monomials, and polytopes, Ann. Math. 96 (1972) 318–337.
[8] I. Kaplansky, Commutative Rings (Allyn & Bacon, Boston, MA, 1970).
[9] P. Kleinschmidt, Über Hilbert-Funktionen graduierter Gorenstein-Algebren, Arch. Math (Basel) 43 (1984) 501–506.
[10] L.G. Roberts and M. Roitman, On Hilbert functions of reduced and of integral algebras, J. Pure Appl. Algebra 56 (1989) 85–104.
[11] R. Stanley, Hilbert functions of graded algebras, Adv. in Math. 28 (1978) 57–83.
[12] R. Stanley, Linear diophantine equations and local cohomology, Invent. Math. 68 (1982) 175–193.
[13] R. Stanley, Enumerative Combinatorics, Vol. 1 (Wadsworth and Brooks/Cole, Monterey, CA, 1986).
[14] R. Stanley, Log-concave and unimodal sequences in algebra, combinatorics, and geometry, in: M.F. Capobianco et al., eds., Graph Theory and Its Applications: East and West, Annals of the New York Academy of Sciences 576 (New York Academy of Sciences, New York, 1989) 500–535.
[15] O. Zariski, Pencils on an algebraic variety and a new proof of a theorem of Bertini, Trans. Amer. Math. Soc. 50 (1941) 48–70.