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Some applications of algebra to combinatorics

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Abstract

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In extremal combinatorics, it is often convenient to work in the context of partially ordered sets.

First let us establish some notation and definitions. As general references on the subject of partially ordered sets we recommend [1; 28, Chapter 3].

1. Definitions

A *partially ordered set* (*poset*) is a set together with a binary relation which is reflexive, antisymmetric, and transitive. Let P be a (finite) *graded* poset, $P=P_0 \cup P_1 \cup \cdots \cup P_n$; P_i is the ith rank of P, and we let $p_i=|P_i|$ be the number of elements of rank i. Every maximal chain of P passes through exactly one element of each of the subsets P_i, starting from rank 0, and going up through rank 1, then rank 2, etc. The posets we will consider will be graded and each maximal chain will have length n (that is, $n+1$ elements).

The *rank generating function* of P is the polynomial $F(P,q)=\sum_{i=0}^{n} p_i q^i$, and it is a useful construct in the study of various properties of P. The poset properties in which we are interested here are: *rank symmetry* (i.e., $p_i=p_{n-i}$, for $i=0$ to n),

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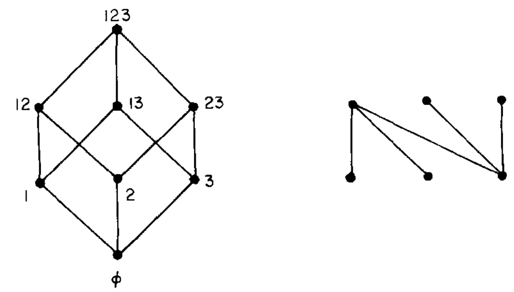

Fig. 1. The boolean algebra B_3 and an example of a non-Sperner poset.

rank unimodality (i.e., there is j such that $p_0 \le p_1 \le \cdots \le p_j \ge p_{j+1} \ge \cdots \ge p_n$), and the *Sperner property* (defined below). Of course, if a poset is both rank symmetric and rank unimodal, then the middle rank(s) achieve the maximum cardinality among all the ranks of P.

An *antichain* is a subset $A \subseteq P$ no two elements of which are comparable in P. Clearly, in a graded poset each rank P_i is an antichain and hence $\max_A |A| \ge \max_i p_i$, where A ranges over the antichains in P. If equality holds, then we say that the poset P is *Sperner*. Thus, in a Sperner poset the largest rank provides an antichain of maximum cardinality, but there may exist other antichains of maximum cardinality as well. So, if a poset P is known to be rank symmetric, rank unimodal, and Sperner, then we know that the maximum cardinality of an antichain in P is the cardinality of its middle rank(s). Figure 1 shows a poset of rank 1 which is not Sperner (the largest p_i equals 3 while the poset contains an antichain of cardinality 4) and the boolean algebra B_3 which is Sperner.

Many interesting problems can be formulated in terms of the Sperner property of some poset. An important example is the boolean algebra B_n, which is the poset of all subsets of an n-element set ordered by inclusion. In B_n the rank of an element is given by its cardinality, and there are $p_i = \binom{n}{i}$ elements of rank i. We have the rank generating function $F(B_n, q) = (1+q)^n$, and the rank symmetry and unimodality of B_n are obvious from well-known properties of binomial coefficients. It is not equally obvious whether B_n is Sperner. In fact, the origin of the terminology goes back to Emmanuel Sperner who proved in 1927 that:

Sperner's theorem 1.1. *The boolean algebra B_n is Sperner, for each $n \ge 1$.*

Sperner's theorem can be stated without reference to posets: given an n-element set, what is the maximum number of subsets you can select so that none of the subsets contains another? Sperner's result says that one cannot exceed $\binom{n}{[n/2]}$, which can be achieved by taking all the subsets of cardinality $[n/2]$.

There are many refinements and generalizations of the Sperner property, but we will keep things simple by considering only the Sperner property.

Next, we will relate the Sperner property to matchings. A useful working condition which implies that a rank unimodal poset P is Sperner is the existence of an order matching between any two consecutive levels. The map $\mu: P_i \to P_{i+1}$, or $\mu: P_{i+1} \to P_i$, is an *order matching* if μ is one-to-one and μ respects the order, i.e., $\mu(x) > x$, or $\mu(x) < x$, for all $x \in P$. In connection with the Sperner property we have the following simple proposition.

Proposition 1.2. *Suppose that in the poset P there exist order matchings* $P_0 \to P_1 \to \cdots \to P_j \leftarrow P_{j+1} \leftarrow \cdots \leftarrow P_n$. *Then P is rank unimodal and Sperner, with* $p_j = \max_i p_i$.

Proof. The unimodality property is clear from the definition of an order matching. The order matchings between successive ranks give rise to a partition of P into (disjoint) chains, each of which intersects P_j. Therefore, the number of chains is p_j. On the other hand, every antichain A intersects each chain in at most one point; hence, $|A| \leq p_j$, so P is Sperner. □

Now we will bring algebra into the picture, starting with linear algebra, and later building more algebraic machinery.

2. Linear algebra

Given a poset P, define $\mathbb{Q}P$ to be the vector space over the field $\mathbb{Q}$ of rational numbers (any other field could be used), consisting of formal linear combinations of elements of P with rational coefficients. Assume now that P is graded. Note that $\mathbb{Q}P$ is the direct sum of the subspaces spanned by the ranks of P, so we have $\mathbb{Q}P = \mathbb{Q}P_0 \oplus \mathbb{Q}P_1 \oplus \cdots \oplus \mathbb{Q}P_n$. If $x \in P$, we let $C^+(x) := \{y \in P\colon \operatorname{rank}(y) = \operatorname{rank}(x) + 1$ and $y > x\}$, that is, $C^+(x)$ denotes the set of elements in P which *cover* x. Similarly, we denote by $C^-(x)$ the set of elements which are covered by x, $C^-(x) := \{y \in P\colon \operatorname{rank}(y) = \operatorname{rank}(x) - 1$ and $y < x\}$.

The following is a key definition establishing a special kind of linear operator on the vector space $\mathbb{Q}P$ in which we will be interested. A linear operator $U: \mathbb{Q}P \to \mathbb{Q}P$ is *order raising* if $U(x) \in \mathbb{Q}C^+(x)$ for all $x \in P$. Thus, $U(x)$ is a linear combination of elements which cover x, and it is denoted U for *up*. We relate now the linear algebra with the Sperner property through the following proposition.

Proposition 2.1. *Let* $U: \mathbb{Q}P_i \to \mathbb{Q}P_{i+1}$ *be (the restriction to the ith rank of) an order raising operator. If U is one-to-one, then there exists an order matching* $\mu: P_i \to P_{i+1}$.

Note that this result gives us "more room to work" when we want to prove that a poset has an order matching. Instead of having to exhibit the actual matching of elements from consecutive ranks, we only need to exhibit an order raising operator mapping elements to linear combinations of elements from the next rank. This is easier because the set of linear combinations is much larger than the rank itself, so there are many more possibilities for an order raising operator than for an order matching.

Proof. Look at the matrix of the linear transformation U with respect to the bases P_i and P_{i+1}. Thus, it is a p_i by p_{i+1} matrix whose rows are indexed by the elements in $P_i=\{x_1,x_2,\dots,x_{p_i}\}$ and whose columns are indexed by the elements in $P_{i+1}=\{y_1,y_2,\dots,y_{p_{i+1}}\}$. Let r be the rank of this matrix. Since U is one-to-one, we have $r=p_i$, and let us assume that the row and column indexing is such that the r by r minor formed by the first r columns is not zero. In particular, there must exist a nonzero term in the expansion of this determinant. Permute the rows if necessary so that the diagonal term of this minor is nonzero, that is, each diagonal entry in the matrix is nonzero. But, if the ith diagonal entry is nonzero, it means that when U is applied to x_i, the element y_i appears with nonzero coefficient in $U(x_i)$. Since U is order raising this means further that x_i is covered by y_i, and that $\mu(x_i)=y_i$ gives an order matching $\mu: P_i\to P_{i+1}$. $\square$

2.1. *Applications*

Let us first apply this result to the boolean algebra. Alternatively, an order matching for B_n can be exhibited, by giving an explicit association of each element of B_n of rank k, $k<n/2$, with a particular element which covers it (or which it covers, if $k>n/2$). This however is not so easy. In the case of our approach, it will suffice to map each element of B_n of rank k, $k<n/2$, to a linear combination of *all* the elements which cover it. The linear algebra will do the work for us and supply an order matching, by ensuring that in the linear combinations associated with different elements of rank k, a different element has nonzero coefficient. We will need to prove only that the linear transformation is one-to-one if $k<n/2$ and onto if $k\geq n/2$.

Let us do the simplest thing and take $U:\mathbb{Q}B_n\to\mathbb{Q}B_n$ defined by $U(S)=\sum_{T\in C^+(S)} T$; that is, to each subset $S\in B_n$ we associate the sum of all the subsets which cover it. Let $U_j=U|_{(B_n)_j}$, i.e., U_j is the restriction of U to the jth rank of B_n.

Theorem 2.2. *With the notation established above, if $k<n/2$, then U_k is one-to-one, and dually, if $k\geq n/2$, then U_k is onto.*

The second part of the theorem follows from the first because B_n is self-dual, that is, there is a bijection $f: B_n\to B_n$, such that $S\subseteq T$ implies $f(T)\subseteq f(S)$; the complementation map serves as f.

So, in view of the duality, we can restate this theorem as: the incidence matrix between the k-element subsets and the $(k+1)$-element subsets of an n-element set has full rank. This result has many different proofs which have appeared in the literature; we will give here what we believe to be a particularly elegant new proof.

Proof. Define a second linear transformation, $D_j : \mathbb{Q}(B_n)_j \to \mathbb{Q}(B_n)_{j-1}$ (D for *down*), which maps a subset to the sum of the subsets covered by it and so is dual to U_j; namely, $D_j(S) = \sum_{T \in C^-(S)} T$, for each $S \in (B_n)_j$. The crucial observation for this proof is that U_j and D_{j+1} are adjoints (with respect to the bases P_j and P_{j+1}) since their matrices (with respect to these bases) are transposes of one another.

We claim that for each k,

$$D_{k+1}U_k - U_{k-1}D_k = (n-2k)I_k,$$

where I_k is the p_k by p_k identity matrix, and the linear transformations are multiplied from right to left. Indeed, apply the left-hand side to a generic k-element set S; now, in the resulting linear combination of k-element subsets, single out the coefficient of an arbitrary but fixed set $S' \in (B_n)_k$. The set S' will have coefficient equal to the number of ways in which it can be obtained from S by first adjoining and then deleting an element, minus the number of ways in which it can be obtained from S by first deleting and then adjoining an element. This is possible at all in precisely two situations: either $S = S'$ or else S and S' have $k-1$ elements in common. In the first situation the coefficient of S' is $(n-k)-k = n-2k$, because there are $n-k$ possibilities for an element to be adjoined to S and then removed, and k possibilities for an element to be removed from S and then added back. In the second situation the coefficient is $1-1=0$ because the element to be adjoined and then deleted as well as the element to be deleted and then adjoined are completely determined by S and S'. Hence, the claim is true.

Now, since U_{k-1} is the adjoint of D_k, the product $U_{k-1}D_k$ is a positive semidefinite matrix, and thus it has only nonnegative eigenvalues. If $k < n/2$, then the matrix $(n-2k)I_k$ is positive definite, so the sum $U_{k-1}D_k + (n-2k)I_k$ has only positive eigenvalues and therefore it is invertible. But, by the claim above, this expression equals $D_{k+1}U_k$. Finally, if the composition of the two operators is invertible, then the first one is one-to-one and the second is onto. Consequently, if $k < n/2$, then U_k is one-to-one, completing the proof of the first part of the theorem. A dual argument yields the existence of matchings between successive ranks above the middle rank of B_n. □

The previous theorem and two propositions give Sperner's theorem as a corollary. While there are other elegant proofs of the fact that B_n is Sperner, we will see that our algebraic approach to the Sperner property is justified by a number of other applications. Variations of the above application to the boolean algebra will provide results which are quite hard to obtain otherwise.

2.1.1. The vector space lattice

Perhaps the most straightforward variation is the application to the poset of subspaces of a finite dimensional vector space over a finite field. Thus, let us consider the field with q elements, $\mathbb{F}_q$, and the set $\mathbb{F}_q^n$ of all n-tuples of elements of $\mathbb{F}_q$; this forms a vector space of dimension n over the field $\mathbb{F}_q$. We will prove the Sperner property for the poset (in fact, lattice) $P = L_n(q)$ of all vector subspaces of $\mathbb{F}_q^n$, ordered by inclusion. The rank of a subspace W is its vector space dimension. This variation is a *q-analogue* of B_n, because many formulae pertaining to $L_n(q)$ (and which involve q) specialize to formulae for B_n when evaluated at $q = 1$.

Theorem 2.3. *The subspace lattice $L_n(q)$ is rank symmetric, rank unimodal, and Sperner.*

Proof. Rank symmetry is well known and easy to show. Rank unimodality follows from the argument below, though it too is well known and has a simple direct proof.

Just as we did for B_n, we consider the order raising and order lowering operators on $\mathbb{Q}P$,

$$U(W) = \sum_{Y \in C^+(W)} Y$$

and

$$D(W) = \sum_{Y \in C^-(W)} Y,$$

for all $W \in L_n(q)$. Their restrictions to the kth rank are denoted U_k and D_k. Through reasoning similar to that used in the case of B_n we obtain an analogous commutation relation

$$D_{k+1} U_k - U_{k-1} D_k = ([n-k]_q - [k]_q) I_k,$$

where $[j]_q = 1 + q + q^2 + \cdots + q^{j-1}$ is the q-analog of the integer j. Indeed, if W is a k-dimensional subspace, there are $q^n - q^k$ vectors which can be adjoined to W in order to increase the subspace dimension by 1, thus obtaining a subspace which contains W and consists of q^{k+1} vectors; however, exactly $q^{k+1} - q^k$ vectors give rise to the same $(k+1)$-dimensional space containing W; therefore there are $(q^n - q^k)/(q^{k+1} - q^k) = [n-k]_q$ distinct $(k+1)$-dimensional subspaces which contain W. On the other hand, there are $[k]_q$ distinct $(k-1)$-dimensional subspaces contained in W; $k-1$ independent vectors to span such a subspace can be chosen (in order) in $(q^k - 1)(q^k - q) \cdots (q^k - q^{k-2})$ ways, but the same subspace has $(q^{k-1} - 1)(q^{k-1} - q) \cdots (q^{k-1} - q^{k-2})$ different ordered bases, and the quotient of these two quantities gives the number of $(k-1)$-dimensional subspaces contained in W. For given positive integers k and n, the value of $[n-k]_q - [k]_q = (q^{n-k} - q^k)/(q-1)$ is positive if $k < n/2$, and by reasoning exactly as in the case of B_n we obtain that $D_{k+1} U_k$ is invertible, and hence U_k is one-to-one. Thus, there exists an order matching $P_k \to P_{k+1}$, if $k < n/2$. Similarly (or by using the fact that $L_n(q)$ is self-dual), we deduce the existence of an order matching $P_{k+1} \to P_k$ in the case when $k \geq n/2$. $\square$

So, our general set-up shows that the poset of subspaces of a finite dimensional vector space over $\mathbb{F}_q$ ordered by inclusion has the Sperner property. This result too has other proofs which are simpler than the algebraic proof we gave. The fact that U_k has full rank was first shown by Kantor [15] by a more complicated argument. We have mentioned that in the case of the boolean algebra B_n it is not so easy, but it is possible, to give an explicit order matching $P_k \to P_{k+1}$ for $k < n/2$. However, we believe that for $L_n(q)$ no explicit order matching is known.

Let us generalize the subspace lattice, at least when q is a prime, to the lattice of subgroups of a finite Abelian p-group.

2.1.2. The subgroup lattice of a finite Abelian p-group

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ be any sequence of positive integers, or we may assume that the integers are decreasingly ordered, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, so that we have an integer partition λ. Consider the group

$$G_\lambda = G_\lambda(p) = (\mathbb{Z}/p^{\lambda_1}\mathbb{Z}) \times (\mathbb{Z}/p^{\lambda_2}\mathbb{Z}) \times \cdots \times (\mathbb{Z}/p^{\lambda_n}\mathbb{Z}),$$

the direct product of cyclic p-groups. Such a group is an Abelian p-group of type λ. Conversely, a fundamental theorem about finite Abelian groups states that every finite Abelian p-group is isomorphic to a group of this form.

The poset to which we will apply our algebraic machinery next is the poset (in fact, lattice) of subgroups of G_λ, denoted $L_\lambda(p) = L(G_\lambda)$. In particular, if $\lambda = (1^n)$ (the notation 1^n indicates a sequence of n 1's), then we are looking at the direct product of n cyclic groups of order p, and its lattice of subgroups, $L_{(1^n)}(p)$, is isomorphic to the lattice of subspaces $L_n(p)$. The lattices $L_\lambda(p)$ form nice generalizations of the vector space lattices which we have discussed in our previous application.

We ask again, this time with regard to $L_\lambda(p)$, is this poset rank symmetric? Is it rank unimodal? Does it have the Sperner property?

Figure 2 shows the Hasse diagram of $L_{(2,1)}(2) = L(\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$, the lattice of subgroups of the product of a cyclic group of order $2^2 = 4$ with a cyclic group of order $2^1 = 2$. Clearly, it has an antichain of 4 elements and maximum rank size 3, so it is not Sperner.

For the special case when all numbers in the sequence λ are equal, the following conjecture was made (source unknown) several years ago, and is still open:

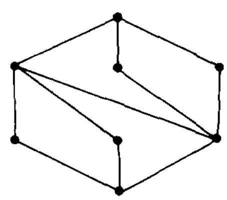

Fig. 2. The lattice $L_{(2,1)}(2)$.

Conjecture 2.4. *If $\lambda=(k,k,\dots,k)$, then L_λ has the Sperner property.*

Related to this conjecture are the questions of rank symmetry and rank unimodality. These have affirmative answers for *every* finite Abelian p-group.

Theorem 2.5 (Butler [6]). *The lattice L_λ is rank unimodal (and rank symmetric) for all λ.*

Butler's proof of rank unimodality uses the theory of symmetric functions; the rank symmetry is a well-known result. A very recent development is the following theorem of Regonati which is an elegant generalization of Butler's result. Its statement is rather technical, and we refer the reader to [28,1] for definitions omitted here. This theorem pertains to the class of finite modular lattices, which includes the lattices of subgroups in which we are interested here.

Theorem 2.6 (Regonati [25]). *The following conditions are equivalent for every finite modular lattice L:*

(i) *Every interval of rank 3 in L is rank symmetric.*
(ii) *Every interval in L is rank symmetric and rank unimodal.*
(iii) *$L=L_1\times\cdots\times L_k$, where each L_i is a lattice with the following two properties:*
  (α) *L_i is a so-called primary modular lattice, i.e., if x is a join irreducible element, then the interval $[\hat{0},x]$ is a chain;*
  (β) *There exists an integer $q=q(i)\geq 1$ such that if an interval $[x,y]$ of L_i is complemented, then the interval is a projective geometry of order q.*

This theorem has an elementary proof, not involving the theory of symmetric functions. While the equivalence of conditions (i) and (ii) is very surprising, condition (iii) removes some of the mystery of (i) and (ii). The third condition characterizes all the lattices satisfying (i), and then (ii) can be verified. It is well known [1,5] that every finite complemented modular lattice is a product of projective geometries and boolean algebras. So condition (iii)(β) says that the complemented intervals must be projective geometries of the *same* order, q. It also allows for $q=1$, in which case, complemented intervals are isomorphic to boolean algebras.

Let us return to the conjectured Sperner property of L_λ for $\lambda=(k,k,\dots,k)$. The case $k=1$ gives, as mentioned earlier, the subspace lattice $L_n(p)$, and the Sperner property holds as discussed in our previous application. We will now apply our algebraic approach to prove that the conjecture is true if $k=2$.

Theorem 2.7. *The lattice $L_{(2^n)}(p)$ of subgroups of the group $(\mathbb{Z}/p^2\mathbb{Z})^n$ is Sperner.*

Proof. We consider again the two operators U_j and D_j, and we obtain the relation

$$(D_{j+1}U_j-U_{j-1}D_j)(x)=(|C^+(x)|-|C^-(x)|)(x),$$

for every x of rank j. If x and y are elements of rank j, $x \neq y$, there is at most one way to go up one rank and then down one rank in order to get from x to y, else the least upper bound of x and y would not be well defined. Similarly, there is at most one way to go down one rank and then up one rank in order to get from x to y. It is immediate from the definition of modularity that going up, then down from x to y is possible if and only if going down and then up from x to y is possible. Thus, if the lattice is modular, then every element x is an eigenvector as above.

As before, we have the commutation relation $D_{j+1}U_j = U_{j-1}D_j + A$, only now A is a diagonal matrix rather than a multiple of the identity matrix. Now, if $k=2$ we can check that $|C^+(x)| - |C^-(x)| > 0$ for every subgroup x whose order is less than p^n, that is, which lies in the lower half of the lattice $L_{(2^n)}(p)$. Thus, when $j<n$, the diagonal matrix A is positive definite, and the proof of the existence of an order matching up to the middle rank is completed as in the case of the subspace lattice. Finally, duality gives an order matching for the upper half of the lattice. □

The conjecture is still open in the case $k=3$.

3. Group actions on posets

We now bring further algebraic machinery into the picture by tying in the linear algebra with group actions on a poset.

Let P be a (finite) poset, and $\mathrm{Aut}(P)$ be the group of automorphisms of P, i.e., order preserving bijections on P. Consider a group of automorphisms of P, $G \subseteq \mathrm{Aut}(P)$. The poset is partitioned into orbits under the action of G, and the orbit of an element $x \in P$ is $Gx = \{gx : g \in G\}$. Now, the *quotient poset* P/G consists of the orbits under G ordered as follows: $Gx \leq Gy$ in P/G if $x \leq y$ in P. In other words, one orbit is less than another if some element from the former is less than some element from the latter. It is easy to verify that this indeed defines a partial ordering on P/G.

Again, to keep things simple, we will look at applications to the boolean algebras. The automorphism group of B_n is isomorphic to the symmetric group S_n. S_n per-

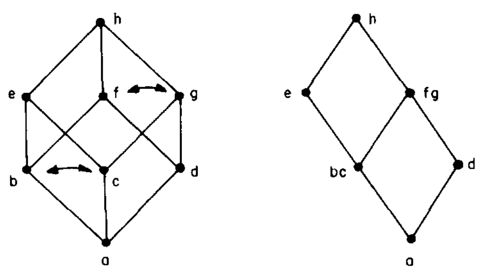

Fig. 3. Group action and quotient poset.

mutes the elements of the n-set, and this induces an action on the subsets of the n-element set. So, we consider the poset $P=B_n$ and a group of permutations $G\subseteq S_n$ acting on B_n; our goal is to establish the rank symmetry, rank unimodality, and Sperner property of the quotient poset B_n/G. Following this, we will specialize n and G and give a few concrete applications of the general result regarding quotient posets.

Figure 3 illustrates the action of $G=\{\mathrm{id},(1,2)\}\subseteq S_3$ on B_3 and shows the quotient poset B_3/G. Of course, $G\simeq\mathbb{Z}/2\mathbb{Z}$, and acts on B_3 by interchanging two pairs of elements: $b=\{1\}$, $c=\{2\}$, and $f=\{1,3\}$, $g=\{2,3\}$. So, $\{b,c\}$ and $\{f,g\}$ are the non-trivial orbits.

Theorem 3.1. *If $G\subseteq S_n$, then the quotient poset B_n/G is rank symmetric, rank unimodal, and Sperner.*

Proof. As before, we have the order raising operator $U:\mathbb{Q}B_n\to\mathbb{Q}B_n$, $U(x)=\sum_{y\in C^+(x)} y$. Note that in fact the action of the group G on B_n can be extended by linearity to all of $\mathbb{Q}B_n$. Indeed, let $\sum\alpha_x x$ be a typical element of $\mathbb{Q}B_n$ and let $g\in G$; we define

$$g\cdot\sum\alpha_x x=\sum\alpha_x(g\cdot x).$$

This is a standard way to convert a permutation representation to a linear representation. Let us make an elementary but crucial observation: the order raising operator U "commutes" with the action of G. Rigorously put, for all $x\in\mathbb{Q}B_n$, and all $g\in G$ we have $U(gx)=g(U(x))$. To prove this fact, it suffices to verify it for $x\in B_n$, since B_n is a basis for $\mathbb{Q}B_n$, and U is linear. If $x\in B_n$ and $g\in\mathrm{Aut}(B_n)$, then we have $C^+(gx)=g(C^+(x))$, and therefore, $U(gx)=\sum_{y\in C^+(gx)} y=\sum_{y\in g(C^+(x))} y=\sum_{z\in C^+(x)} gz=g\cdot\sum_{z\in C^+(x)} z=g(U(x))$.

Now we need to look at the fixed space of $\mathbb{Q}B_n$ under the action of G,

$$\mathbb{Q}B_n^G:=\{v\in\mathbb{Q}B_n: gv=v \text{ for all } g\in G\}.$$

What is the structure of this fixed space? Take $v\in\mathbb{Q}B_n^G$ and observe that elements in B_n which lie in the same orbit must have the same coefficient in v. Therefore, a basis for $\mathbb{Q}B_n^G$ is $\{\sum_{y\in Gx} y: Gx\in B_n/G\}$, and $\mathbb{Q}B_n^G$ can be identified with $\mathbb{Q}(B_n/G)$, the linear combinations of orbits, via $(\sum_{y\in Gx} y)\leftrightarrow Gx$. So we will prove our theorem by proving that $\mathbb{Q}B_n^G$ has the desired properties.

It follows from our earlier observation that U and G "commute", that U maps $\mathbb{Q}B_n^G$ to itself. Further, from the definition of the quotient poset B_n/G we see that the restriction of U to $\mathbb{Q}B_n^G$ is an order raising operator on the quotient poset B_n/G.

Now we are at the critical step in our argument. We know from our application of linear algebra to B_n that U is one-to-one on $\mathbb{Q}(B_n)_i$ if $i<n/2$. Since $\mathbb{Q}(B_n^G)_i$ is a subspace of $\mathbb{Q}(B_n)_i$ we see that the restriction $U|_{\mathbb{Q}(B_n^G)_i}$, is a one-to-one raising operator. Thus, we have an order matching up to the middle level of B_n^G. Through

a dual argument we obtain one for ranks $i > n/2$, and this shows that B_n/G is rank unimodal and Sperner.

We omit the argument that B_n/G is rank symmetric. It requires a separate (easy) proof that the action of any permutation on the i-element subsets is isomorphic to its action on the $(n-i)$-element subsets. □

3.1. Applications

Now let us look at some examples where our theorem about quotient posets is applied.

3.1.1. Graphs

First we present a straightforward example which is an application to graphs. We take $n = \binom{m}{2}$ for some positive integer m, and we regard the elements of B_n as the labeled simple graphs on m vertices as follows. Identify each of the $n = \binom{m}{2}$ elements of the underlying set of B_n with a different unordered pair of two distinct points out of a set of m points. In turn, such a pair represents an edge between the two points, and each labeled simple graph is identified with its set of edges. Moreover, the order relation in $B_{\binom{m}{2}}$ translates into the edge inclusion ordering on the set of labeled simple graphs on m points.

As the group G, take the symmetric group S_m, which permutes the m points. This action induces an action on the unordered pairs of points (edges), and thus we have the group $G = S_m$ acting on B_n. Since the action of S_m simply relabels the vertices, an orbit consists precisely of isomorpic graphs on the m vertices. Consequently, the quotient poset B_n/G is the subgraph ordering on the set of nonisomorphic (unlabeled) simple graphs on m vertices. See Fig. 4 for the Hasse diagrams of B_3/S_3 and B_6/S_4. While these are lattices, for $m = 5$ the quotient poset $B_{\binom{m}{2}}/S_m$ is no longer a lattice.

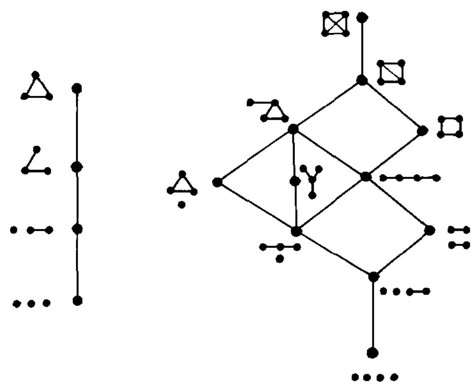

Fig. 4. The subgraph ordering on the unlabeled simple graphs on three and on four vertices.

By applying our previous theorem we obtain that the subgraph ordering on the set of unlabeled simple graphs on m vertices is rank symmetric (this can be easily seen directly by taking graph complements), rank unimodal (in other words, the count of nonisomorphic graphs on m vertices according to the number of edges increases until there are $\binom{m}{2}/2$ edges, then decreases; this was originally proved by Livingstone and Wagner [17]), and Sperner (in other words, what is the largest number of unlabeled simple graphs on m vertices, so that no graph is isomorphic to a subgraph of another? The answer is: you cannot do better than taking all the graphs having half the total possible number of edges: this is originally due to Pouzet and Rosenberg [23]). So all these results fall out of the algebraic machinery that we have set up.

It is easy to see that this type of argument will work equally well for other structures on vertex sets: directed graphs, posets, topologies, etc.: it is a very general approach. Now we want to look at a less obvious application, that is, we will look at a problem where it is not so apparent that the poset under investigation is the quotient poset of a boolean algebra.

3.1.2. *Integer partitions*

Consider the boolean algebra B_{mn}, and think of the underlying set of mn elements as an m by n rectangular array of cells. The group acting on subsets of cells will be $G=S_m \S S_n$, called the *wreath product* of S_m and S_n. This group permutes the n cells within each row independently, and permutes (interchanges) the m rows. Thus, the order of this group is $|G|=n!^m m!$. Given a set of cells, $T\in B_{mn}$, a canonical representative of its orbit under the action of G is the set of cells obtained by first moving the cells of T all the way to the left within each row, then permuting the rows so that the number of cells from each row which are contained in our set decreases. Figure 5 shows a set of cells and the canonical representative of its orbit.

An array of cells which are left justified and whose row lengths are decreasing is called a *Ferrers diagram* in the context of the theory of (integer) partitions; the lengths of the rows are the summands of the partition; the total number of cells in the diagram is the integer being partitioned. For example, in Fig. 5 we have the Ferrers diagram of the partition $\lambda=(3,2,2,1)$ of 8.

A moment's thought shows that each orbit contains exactly one Ferrers diagram, and that the quotient poset B_{mn}/G, denoted $L(m,n)$, is isomorphic to the set of Ferrers diagrams contained in an m by n rectangle, ordered by inclusion. The rank function is given by the number of cells in the diagram. See Figs. 6 and 7.

The poset $L(m,n)$ is a (distributive) lattice for any m and n, and is one of the most interesting posets. We have just shown that it is a quotient of a boolean algebra, so our general theorem says that it is rank symmetric, rank unimodal, and Sperner. Still, what are the cardinalities of the ranks of $L(m,n)$? This is a classical combinatorial problem: how many partitions of k have Ferrers diagram contained in an m by n rectangle? The answer, in the form of the rank generating function, goes back to Euler and Gauss:

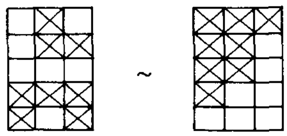

Fig. 5. The Ferrers diagram in the orbit of a set of cells.

$$F(L(m,n),q)=\begin{bmatrix} m+n \\ m \end{bmatrix}_q=\frac{[m+n]!}{[m]!\,[n]!},$$

where $[j]!=(1-q)(1-q^2)\cdots(1-q^j)$. If the expression $[{a \atop b}]_q$ is evaluated at $q=1$ we obtain the binomial coefficient $\binom{a}{b}$. It is a q-analogue of the binomial coefficient, called a *q-binomial coefficient*, and it appears in many combinatorial contexts. Note in particular that $F(L(m,n),1)=\binom{m+n}{m}$, which is the total number of elements in $L(m,n)$ (the jagged line which determines a Ferrers diagram contained in an m by n rectangle is a path of $m+n$ steps, of which m are horizontal and n are vertical).

The rank symmetry of $L(m,n)$ is easy to see directly, since complementing a Ferrers diagram with k cells with respect to the m by n rectangle gives a Ferrers diagram (upside down) with $mn-k$ cells. The rank unimodality of the quotient poset $L(m,n)$ constitutes a theorem which goes back to Sylvester in the nineteenth century, whose proof is not easy. Sylvester's proof uses invariant theory, a method similar to the techniques presented in the next part of this paper.

Corollary 3.2 (Sylvester [30]). *The q-binomial coefficient* $[{m+n \atop m}]_q$ *is a polynomial in the variable q and its coefficients form a unimodal sequence.*

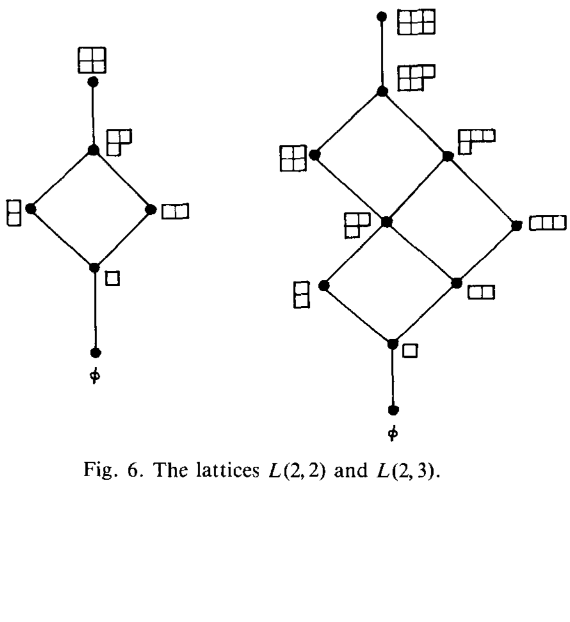

Fig. 6. The lattices $L(2,2)$ and $L(2,3)$.

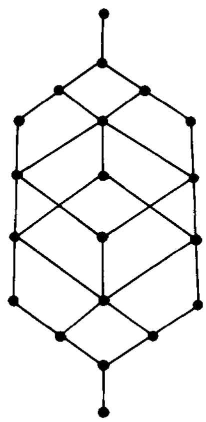

Fig. 7. The lattice $L(3,3)$.

Several algebraic proofs of this result are known and they are all somewhat related. An explicit one-to-one function between Ferrers diagrams of sizes k and $k+1$, $k<mn/2$, which constitutes the first combinatorial proof of the rank unimodality of $L(m,n)$, was given recently by O'Hara [21] (see also Zeilberger [31]). These injections however are not order matchings. There is no combinatorial proof yet of the existence of an order matching for $L(m,n)$, although the algebraic approach we used here guarantees that one exists. Also, there is no combinatorial proof of the Sperner property of $L(m,n)$, yet.

4. The Lie algebra sl(2, $\mathbb{C}$)

We continue with the topic of raising and lowering operators and their relevance to poset properties, this time from the point of view of the Lie algebra sl(2, $\mathbb{C}$). Much of what we present here is due to Robert Proctor.

First, what is sl(2, $\mathbb{C}$)? It is the set of all 2 by 2 matrices with complex entries, and whose trace, i.e., sum of the diagonal entries, is zero. It is a vector space of dimension 3 over the field of complex numbers, since the vector space of 2 by 2 matrices has dimension 4, and the linear condition that the trace be zero lowers the dimension by 1. Aside from the complex vector space structure, sl(2, $\mathbb{C}$) has a binary operation called the *bracket operation*, $[A,B]=AB-BA$, the commutator of the two matrices. With this bracket operation, the 2 by 2 complex matrices having trace zero form a *Lie algebra*. For our purposes it is not necessary that the reader be familiar with the theory of Lie algebras, but the interested reader is referred to Humphreys' book [12].

The connection between sl(2, $\mathbb{C}$) and our investigation of poset properties is based on raising and lowering operators. Let P be a graded poset, and consider now the vector space $\mathbb{C}P$, working with the complex rather than the rational field. Suppose that we have an order raising operator $U: \mathbb{C}P \to \mathbb{C}P$ and a lowering operator $D: \mathbb{C}P \to \mathbb{C}P$; thus, U and D are linear operators and, if $x \in P$, then $U(x)$ is a linear combination of elements which cover x, while $D(x)$ is a linear combination of elements whose rank is one unit less than that of x. More succinctly, $U(x) \in \mathbb{C}C^+(x)$ and $D(\mathbb{C}P_k) \subseteq \mathbb{C}P_{k-1}$. Unlike our earlier linear algebra framework, where U_k and D_{k+1} were adjoints with respect to the bases P_k and P_{k+1}, the sl(2, $\mathbb{C}$) set-up allows D to be any operator which lowers the rank by 1, so we are in a much more general situation.

We define a third linear transformation, $H: \mathbb{C}P \to \mathbb{C}P$, such that, if n is the height of P (i.e., the length of the longest chain in P), then $H(x) = (2i-n)x$ for every $x \in P_i$. Thus, every poset element is an eigenvector of H, with eigenvalue $2i-n$, where i is the rank of the element. Suppose that the three operators U, D, and H are related by $UD - DU = H$. Note that this means that on each $\mathbb{C}P_i$ the difference $DU - DU$ is a multiple of the identity, as in the case of our first linear algebra set-up. However, now D is not necessarily the adjoint of U with respect to the basis P. (It can be shown that U and D are adjoint with respect to *some* scalar product, but we will avoid using this fact.)

Under these assumptions, it can be verified that the set of all complex linear combinations of U, D, and H (the span of these three operators) is also closed under the bracket operation and, in fact, it is a Lie algebra isomorphic to sl(2, $\mathbb{C}$): $\operatorname{span}\{U, D, H\} \simeq \mathrm{sl}(2, \mathbb{C})$. A poset P for which such operators U, D, and H exist is called an sl(2, $\mathbb{C}$)-poset, and we have the following basic theorem (a stronger statement is true as well).

Theorem 4.1. *Every* sl(2, $\mathbb{C}$)*-poset is rank symmetric, rank unimodal, and Sperner.*

Proof. First, rank symmetry is easy to check.

We will show that P has an order matching, and hence, is rank unimodal and Sperner, by showing that $U: \mathbb{C}P_i \to \mathbb{C}P_{i+1}$ is one-to-one for $i < n/2$. A dual argument gives surjectivity for $i \geq n/2$. Note that we cannot use exactly the same proof as in our initial linear algebra set-up because now we do not know that U and D are adjoints. The proof we give here is somewhat simpler than Proctor's original proof.

Before proceeding with the proof, recall the following definition and fact from linear algebra. The *characteristic polynomial* of a linear transformation A is $\mathrm{ch}(A, \lambda) = \det(A - \lambda I)$; I is the identity matrix of the appropriate size. A general fact from linear algebra (which is not as well known as it should be) is that if V and W are two finite dimensional vector spaces and if $A: V \to W$ and $B: W \to V$ are two linear transformations, then the characteristic polynomials of AB and BA are related via

$$(-\lambda)^{\dim W} \mathrm{ch}(BA, \lambda) = (-\lambda)^{\dim V} \mathrm{ch}(AB, \lambda).$$

A better known fact is the special case when $V=W$, which asserts that the two characteristic polynomials are equal.

One final preparatory item pertains to notation: for *any* linear transformation $A:\mathbb{C}P\to\mathbb{C}P$, we denote by $A_i=A|_{\mathbb{C}P_i}$ the restriction of A to the vector subspace generated by the ith rank of P.

Now, we have the linear transformations U and D satisfying $DU_i=UD_i+(n-2i)I_i$, where the restriction DU_i of DU to $\mathbb{C}P_i$ is of course $D_{i+1}U_i$; similarly, $UD_i=U_{i-1}D_i$. Taking the characteristic polynomial on both sides,

$$\operatorname{ch}(DU_i,\lambda)=\operatorname{ch}(UD_i+(n-2i)I_i,\lambda).$$

The addition of $(n-2i)I_i$ to UD_i increases each eigenvalue by $n-2i$, so we have further

$$\operatorname{ch}(DU_i,\lambda)=\operatorname{ch}(UD_i,\lambda-(n-2i)).$$

Apply now to D_i and U_{i-1} on the right-hand side the fact from linear algebra mentioned above. We obtain the further equality

$$\operatorname{ch}(DU_i,\lambda)=(-\lambda+(n-2i))^{p_i-p_{i-1}}\operatorname{ch}(DU_{i-1},\lambda-(n-2i)).$$

Thus, we have expressed the characteristic polynomial of DU_i in terms of the characteristic polynomial of DU_{i-1}. It now follows easily by induction that all eigenvalues of DU_i are positive if $i<n/2$. □

Why is it useful to look at $\mathrm{sl}(2,\mathbb{C})$ when we want to do combinatorics? The reason is that algebraists have studied $\mathrm{sl}(2,\mathbb{C})$ and have determined many *linear representations* of it, i.e., linear operators on some vector space V whose span is a Lie algebra isomorphic to $\mathrm{sl}(2,\mathbb{C})$. So we can look at any one of these representations and see if it corresponds to a poset situation of the kind we have described. That is, we can look at any one of these representations and try to find a basis for V which is a poset P, and in terms of which three of the operators on V behave just like U, D and H with regard to P. If we succeed at this, then the theorem above applies and we know that the poset P is rank symmetric, rank unimodal, and Sperner. Note that while this plan of action may seem somewhat backward (we start in the realm of Lie algebras and *then* come upon a poset with interesting combinatorial properties), therein lies much of the power of this approach: it helps not only prove that a poset is interesting, but also discover interesting posets. Furthermore, in certain cases (such as the first two of our next examples) it may be possible, once we have obtained the poset P and the raising and lowering operators via Lie algebra information, to give an alternate, combinatorial proof without reference to $\mathrm{sl}(2,\mathbb{C})$ and its representation, although we do need the Lie algebra in order to discover the results. This was first done by Proctor. On the other hand, we will also see applications where no elementary proof of the identity $UD-DU=H$ is known and, at present, the use of Lie algebras seems to be essential.

4.1. Applications

A good source for representations of sl(2, ℂ) are the irreducible representations

$$\Phi : \mathscr{G} \to \mathrm{gl}(m, \mathbb{C})$$

of a complex semisimple Lie algebra $\mathscr{G}$ (familiarity with these technical terms is not needed in what follows) into gl(m, ℂ), the Lie algebra of all m by m complex matrices. These representations were classified through the efforts of Killing, Cartan, and others. Later Dynkin (with further work by Kostant) showed that $\mathscr{G}$ contains sl(2, ℂ) as a subalgebra in a certain canonical way, and that the representation Φ restricts to an "interesting" representation of sl(2, ℂ). Therefore we can go through the theory of representations of semisimple Lie algebras and see in what cases does the restriction of Φ to sl(2, ℂ) give combinatorially interesting results. We give the two most interesting examples we know.

4.1.1. 3-dimensional Ferrers diagrams

Take $\mathscr{G} = \mathrm{sl}(n, \mathbb{C})$, the Lie algebra formed by the n by n complex matrices having trace zero. For Φ we take a certain representation of it known as the rth fundamental representation, $1 \le r \le n-1$.

Upon analyzing the structure of this representation and understanding how it acts when restricted to sl(2, ℂ), it turns out that it has a basis which behaves like a poset. This basis was first described by Gelfand and Zetlin [10], while Proctor [24] observed its poset property. The poset is denoted $L(m, n, r)$ and consists of all 3-dimensional Ferrers diagrams contained in an m by n by r rectangular box, ordered by inclusion. Thus, $L(m, n, r)$ is an sl(2, ℂ)-poset and our general theorem tells us that it is rank symmetric, rank unimodal, and Sperner. At present no other proof of the fact that $L(m, n, r)$ is Sperner is known.

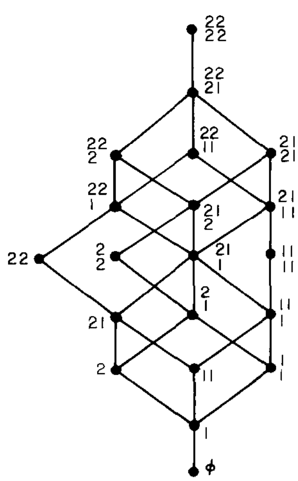

Fig. 8. The lattice $L(2, 2, 2)$.

Recall the poset $L(m,n)$, which we proved to be rank symmetric, rank unimodal, and Sperner. Just as its elements can be identified with Ferrers diagrams of integer partitions inside an m by n rectangle, the elements of $L(m,n,r)$ can be identified with Ferrers diagrams of *plane partitions*. We have $L(m,n) \simeq L(m,n,1)$, and $L(m,n,r)$ provides a very interesting generalization of $L(m,n)$.

The poset $L(m,n,r)$ can also be described as the lattice $J(\mathbf{m} \times \mathbf{n} \times \mathbf{r})$ of order ideals in the product poset of three chains, whose cardinalities are m, n, and r, respectively. These ideals can be represented by Ferrers diagrams of plane partitions. For example, Fig. 8 shows the poset $L(2,2,2)$ of the plane partitions contained in a 2 by 2 by 2 rectangular box ordered componentwise (equivalently, by inclusion). This poset is isomorphic to $J(\mathbf{2} \times \mathbf{2} \times \mathbf{2})$, the lattice of order ideals of the product of three chains each of two elements. This product, in turn, is the boolean algebra B_3, and so $L(2,2,2) \simeq J(B_3)$ is a free distributive lattice ([28, p. 158, Exercise 3.24] can be consulted with regard to free distributive lattices).

It is natural to ask now whether the 4-dimensional Ferrers diagrams fitting in a rectangular box form a Sperner poset. The Lie algebra techniques described here do not seem applicable. So, for $k \geq 4$, we have the open question: is $J(\mathbf{m}_1 \times \mathbf{m}_2 \times \cdots \times \mathbf{m}_k)$ Sperner? The rank unimodality of this poset is open as well.

A special case of interest is the case $m_i = 2$ for all i. That is, if $k \geq 4$, is the free distributive lattice on k generators, $\mathrm{FD}(k) := J(B_k)$, rank unimodal? Sperner? It is true that FD(4) is rank unimodal and perhaps it can be checked whether it is Sperner. Also, there is no nice formula for the number of elements of $\mathrm{FD}(k)$.

4.1.2. Partitions into distinct summands

We give now a second application of the representation theory of Lie algebras,

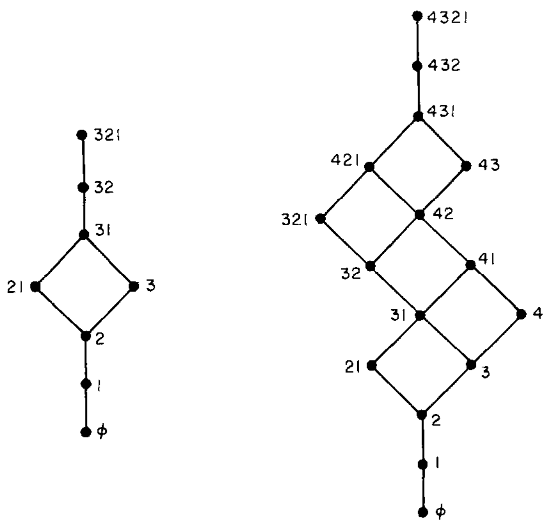

Fig. 9. The posets $M(3)$ and $M(4)$.

where we find a $\mathscr{G}$ and Φ which lead to a poset of combinatorial interest with a number theoretic application.

$\mathscr{G}$ will be a Lie algebra known as $so(2n+1,\mathbb{C})$, and the representation Φ will be a famous representation of degree $m=2^n$ discovered by É. Cartan, called the *spin representation*.

After going through the work of understanding how $sl(2,\mathbb{C})$ sits inside $so(2n+1,\mathbb{C})$ and how the spin representation restricts to $sl(2,\mathbb{C})$, we seek a basis for the restriction of the representation whose elements can be indexed by the elements of a poset. We get an $sl(2,\mathbb{C})$-poset, hence rank symmetric, rank unimodal, and Sperner, which we denote $M(n)$. Since the degree of the representation is $m=2^n$, it is not surprising that the elements of the poset $M(n)$ are the subsets of $[n]:=\{1,2,\ldots,n\}$. The order relation on $M(n)$ is the following: let $a=\{a_1,a_2,\ldots,a_k\}$ and $b=\{b_1,b_2,\ldots,b_j\}$ be two subsets of $[n]$, and assume that the elements in each of a and b are ordered in decreasing order; then $a\geq b$ in $M(n)$ if and only if $k\geq j$ and for each i, $1\leq i\leq j$, we have $a_i\geq b_i$. The Hasse diagrams of $M(3)$ and $M(4)$ appear in Fig. 9.

It is easy to see that in $M(n)$ the rank of a subset of $[n]$ is the sum of its elements. Therefore the cardinality of the ith rank of $M(n)$ is the number of subsets of $[n]$ whose elements add up to i. Alternately, it is the number of partitions of the integer i into distinct summands, no summand being larger than n. Therefore we have the rank generating function

$$F(M(n),q)=(1+q)(1+q^2)\cdots(1+q^n),$$

and since we know that $M(n)$ is rank unimodal we obtain immediately:

Corollary 4.2. *The polynomial* $(1+q)(1+q^2)\cdots(1+q^n)$ *has unimodal coefficients.*

This result is very easy to state but no simple proof of it is known. Other proofs use algebraic techniques similar to those we used here, or analytic techniques, including computer aided estimates [20]. To date there is no proof of the rank unimodality of $M(n)$ analogous to O'Hara's proof of the unimodality of the q-binomial coefficients. It would be interesting to find a conceptual, combinatorial proof.

We now turn to a nice number theoretic application of the fact that the poset $M(n)$ has the Sperner property.

4.1.3. *A conjecture of Erdös and Moser, and more*

Consider an n-element set $S\subset\mathbb{R}$ of real numbers, and a real number α. Let $f(S,\alpha)=|\{T\subseteq S\colon \sum_{i\in T} i=\alpha\}|$, that is, $f(S,\alpha)$ denotes the number of subsets of S the sum of whose elements is α. For example, if $S=\{1,2,3,4,5\}$ and $\alpha=7$, the subsets $\{3,4\}$, $\{2,5\}$, $\{1,2,4\}$ are the only subsets of S whose elements add up to 7, and so $f(\{1,2,3,4,5\},7)=3$. We also take the sum of the elements of the empty set to be zero. We ask the following question: given n, how large can $f(S,\alpha)$ be?

In other words, for given n, what is $t(n) := \max\{f(S,\alpha): S \subset \mathbb{R},\ |S| = n,\ \alpha \in \mathbb{R}\}$?

It does not take much experimentation to observe that in order to maximize $f(S,\alpha)$, the elements of S must be "nicely" and "uniformly" situated in $\mathbb{R}$. By contrast, if the elements of S are chosen at random from $\mathbb{R}$ and are linearly independent over the rationals, then no two subsets of S have equal sums of elements. There are two very plausible conjectures:

Conjecture 4.3. *If $S \subset \mathbb{R}^+$ and $|S| = n$, then*

$$f(S,\alpha) \leq f\left(\{1,2,\ldots,n\},\ \left[\frac{1}{2}\binom{n+1}{2}\right]\right).$$

If we allow negative real numbers, we can achieve a larger number of subsets with equal sum:

Conjecture 4.4 (Erdös and Moser). *If $S \subset \mathbb{R}$ and $S = 2n+1$, then*

$$f(S,\alpha) \leq f(\{-n,-n+1,\ldots,n\},0).$$

A similar conjecture exists for sets S of even cardinality.

The following lemma, proved before the development of the algebraic machinery, shows that these conjectures are related to the lattice $M(n)$.

Lemma 4.5 (Lindström [16]). *If the poset $M(n)$ is Sperner, then Conjecture 4.3 holds.*

Proof. Suppose S is a set of positive real numbers, $S = \{a_1, a_2, \ldots, a_n\}$, and that the elements of S are arranged increasingly, $a_1 < a_2 < \cdots < a_n$.

We claim that if two subsets of S have equal sums of their elements, then the sets of subscripts of the elements in the two subsets are incomparable in $M(n)$. Suppose otherwise, i.e.,

$$\sum_{k=1}^{r} a_{i_k} = \sum_{k=1}^{s} a_{j_k}$$

and, say, $r \leq s$, $i_1 \leq j_1$, $i_2 \leq j_2, \ldots, i_r \leq j_r$. Since the elements of S were indexed in increasing order, this implies that $a_{i_1} \leq a_{j_1}$, $a_{i_2} \leq a_{j_2}, \ldots, a_{i_r} \leq a_{j_r}$. On the other hand, the sums of these elements must be equal, so we must have $r = s$, and $a_{i_k} = a_{j_k}$ for all $1 \leq k \leq r$.

The claim is now proved, and so any family of subsets of S which have the same sum of elements corresponds to an antichain in $M(n)$ of the same size. Thus, $f(S,\alpha)$ cannot exceed the size of the largest antichain in $M(n)$. □

We have just mentioned that it follows from properties of the spin representation of $\mathrm{so}(2n+1,\mathbb{C})$ that the poset $M(n)$ is rank symmetric (this can be seen easily by complementation), rank unimodal and Sperner. Therefore the largest antichain in $M(n)$ has size equal to the size of the middle rank, which is $f(\{1,2,\ldots,n\},[\frac{1}{2}\binom{n+1}{2}])$; hence Conjecture 4.3 is true.

We can also prove Conjecture 4.4. We must look at the poset $M(n)\times M(n)^*$, the product of $M(n)$ and its dual. The elements of $M(n)$ will represent subsets of positive reals and those of $M(n)^*$ will represent subsets of negative reals. In the product poset, an element will correspond to the union of a subset of positives and a subset of negatives. Through an argument similar to that used for $M(n)$, it can be shown that $M(n)\times M(n)^*$ is Sperner, as well as rank symmetric and rank unimodal, and Conjecture 4.4 follows. This is basically the only known proof for Conjecture 4.4 (though it is possible by elementary reasoning [22] to deduce the Sperner property of $M(n)\times M(n)^*$ from that of $M(n)$). However, this is a case when the references to Lie algebras can be completely removed from the proof [24]. We consider the poset $M(n)$ or $M(n)\times M(n)^*$, and the details can be worked out so that the operators U and D are defined explicitly and the relation $UD-DU=H$ is verified by purely combinatorial arguments.

The final algebraic machine we will see in connection with Lie algebras is the hard Lefschetz theorem, a theorem which lies in the realm of algebraic geometry. It is a way of obtaining representations of $\mathrm{sl}(2,\mathbb{C})$, but this time we do not know how to make them explicit and find a combinatorial proof even by hindsight.

4.2. The hard Lefschetz theorem

Start out with a smooth, irreducible, complex projective variety X of complex dimension d. Again, we will not need full grasp of the technical points. Roughly speaking, *complex* means we will work over the field of complex numbers; a *projective variety* is the set of solutions of a system of homogeneous polynomial equations, so the solutions lie in a projective space (i.e., if a nonzero solution is multiplied by a nonzero complex number, then it remains a solution); *smooth* can be thought of as meaning that as a topological space inside the ambient projective space, the variety has no singularities — it is a topological manifold; *irreducible* in this context means connected.

As a topological space, every complex projective variety, not necessarily smooth or irreducible, has associated with it its (singular) cohomology ring. Let us try to describe some properties of this ring. We will take coefficients in $\mathbb{C}$, since complex coefficients will turn out to be the most natural choice for the hard Lefschetz theorem. Then the cohomology ring $H^*(X,\mathbb{C})$, abbreviated $H^*(X)$, is a vector space over the complex numbers, and it is the direct sum, in a natural way, of finite dimensional subspaces $H^i(X)$, $0\le i\le 2d$. The indices i run from 0 to $2d$ as a consequence of the fact that the variety X has complex dimension d, and hence real dimension $2d$. Thus,

$$H^*(X,\mathbb{C})=H^*(X)=H^0(X)\oplus H^1(X)\oplus\cdots\oplus H^{2d}(X).$$

There is also a multiplication on $H^*(X)$, giving it the structure of a graded ring, i.e., $H^i(X)H^j(X)\subseteq H^{i+j}(X)$. The ith cohomology group of X (over $\mathbb{C}$) is $H^i(X)$.

Now, we intersect our projective variety X with a generic hyperplane $\mathscr{H}$ which lies in the ambient projective space. The intersection $\mathscr{H}\cap X$ is a subvariety of X, and in algebraic geometry there is a standard construction which associates to a closed subvariety an element of the cohomology ring. Since our complex hyperplane has real codimension 2, we obtain some element called the "class of a hyperplane section", $[\mathscr{H}\cap X]=\omega\in H^2(X)$ in the second cohomology group.

We consider two linear transformations from $H^*(X)$ to itself. The first is given by multiplication by the class of hyperplane section, and we denote it by ω as well:

$$\omega : H^*(X)\to H^*(X),$$

$$x\mapsto \omega x,$$

so, because $H^*(X)$ is a graded ring and ω has degree 2, we have $\omega(H^i(X))\subseteq H^{i+2}(X)$.

A second transformation we want to consider is

$$\gamma : H^*(X)\to H^*(X),$$

$$\gamma(x)=(i-d)x, \quad \text{if } x\in H^i(X),$$

so each $H^i(X)$ is an eigenspace for γ with eigenvalue $i-d$.

We now come to the hard Lefschetz theorem, a very deep result, which also has an interesting history associated with the names of Lefschetz, Hodge, Chern, and others. Lefschetz stated the theorem, although not in terms of $\mathrm{sl}(2,\mathbb{C})$, but his proof had some gaps. Hodge gave the first correct proof using his theory of harmonic integrals, and later Chern gave a proof involving $\mathrm{sl}(2,\mathbb{C})$. Finally, Deligne fixed the gaps in Lefschetz's original proof by resorting to incredibly powerful machinery.

Our statement of the theorem is not the original one, rather it is most convenient for our purposes.

Theorem 4.6 (The hard Lefschetz theorem). *There exists on $H^*(X)=H^*(X,\mathbb{C})$ a (unitary) scalar product such that if we let ω^* be the adjoint of ω with respect to this scalar product, then*

$$\mathrm{span}_{\mathbb{C}}\{\omega,\gamma,\omega^*\}\simeq \mathrm{sl}(2,\mathbb{C}).$$

The scalar product mentioned in the theorem is one which is natural from the point of view of differential geometry. Another way of phrasing the theorem is that there is a natural representation $\mathrm{sl}(2,\mathbb{C})\to \mathrm{gl}(H^*(X,\mathbb{C}))$, that is, the elements of $\mathrm{sl}(2,\mathbb{C})$ act as linear transformations on $H^*(X,\mathbb{C})$. In particular, it follows as before that $\omega^*\omega-\omega\omega^*=\gamma$, and the same arguments used before yield:

Corollary 4.7. *For each i,* $\dim H^i(X)=\dim H^{2d-i}(X)$.

(This symmetry property also follows from Poincaré duality, which applies because X is smooth, and hence it is an orientable manifold.)

Also, $\omega: H^i(X) \to H^{i+2}(X)$ *is one-to-one if* $i<d$ *and onto if* $i \geq d$.

Note that in this situation we do not necessarily have a poset, we have only a representation of $\mathrm{sl}(2,\mathbb{C})$. We can try to find the right basis for the cohomology so that ω acts as an order raising operator. This is not always easy to do, but at least we have a purely numerical result which we will find useful in proving that combinatorially interesting sequences are unimodal.

Corollary 4.8. *Let* $\beta_i = \dim H^i(X)$, *the ith Betti number of the variety X. Then the sequences* $\beta_0, \beta_2, \beta_4, \ldots, \beta_{2d}$ *and* $\beta_1, \beta_3, \beta_5, \ldots, \beta_{2d-1}$ *are symmetric and unimodal.*

4.2.1. Applications

• *q-binomial coefficients.* A variety which is very popular among algebraic geometers is the Grassmann variety $X = G(n,k)$. This is obtained by turning into a projective variety, in a very natural way, the set of all k-dimensional subspaces of a complex n-dimensional vector space. The computation of the cohomology and Betti numbers goes back to Ehresmann [9] who showed that $\beta_{2i-1} = 0$ while

$$\sum_i \beta_{2i} q^i = \begin{bmatrix} n \\ k \end{bmatrix}_q .$$

Thus, by applying the preceding corollary, we derive again (though this is a harder proof) the unimodality of the q-binomial coefficients.

• *Permutation statistics and Hessenberg varieties.* Our next example is one where the application of the hard Lefschetz theorem gives the only known proof of a new result, due to DeMari and Shayman.

The combinatorial context is the following. Let us fix two positive integers n and p. Given a permutation $\pi \in S_n$, the symmetric group on n elements, i.e., a permutation $\pi = a_1 a_2 \ldots a_n$ of the integers $1, 2, \ldots, n$, we define

$$d_p(\pi) = |\{(i,j):\ i<j,\ a_i > a_j,\ j-i \leq p\}|.$$

The first two conditions define an *inversion* in π, and the third puts a bound on how far apart the two inverted numbers can be. So, $d_p(\pi)$ is the number of such inversions in π, and it constitutes a statistic on S_n. Two cases of this statistic ($p=1$ and $p=n-1$) have already been studied and are well-known examples of permutation statistics. When $p=1$, then d_1 counts only inversions between adjacent numbers, called *descents*. When $p=n-1$, then the condition $j-i \leq p$ is not a restriction in any way, and d_{n-1} counts all inversions of the permutation.

Look at how many permutations in S_n have a prescribed number of inversions between numbers no farther apart than p positions, and let

$$A(p,n,k) := |\{\pi \in S_n:\ d_p(\pi) = k\}|.$$

We wish to study the distribution of the statistic d_p and determine properties of the sequence $\{A(p,n,k)\}_k$. For example, when $p=n-1$ and the statistic is just the number of inversions, it is easy to see that $\sum_k A(n-1,n,k)q^k=(1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1})$, and that the sequence $\{A(n-1,n,k)\}_k$ is unimodal. Also, when $p=1$, the numbers $A(1,n,k)$ are called *Eulerian numbers*, and they too are well known to form a unimodal sequence.

DeMari and Shayman showed that not only for the extreme values of p, but for the intermediate values as well, the sequence $\{A(p,n,k)\}_k$ is unimodal. They did so, by considering a certain variety, the "Hessenberg variety", which occurs in the theory of Hessenberg forms of matrices. Their motivation was from the point of view of numerical analysis and computer science, where it is of interest to work with matrices in an efficient way. They proved:

Theorem 4.9 (DeMari and Shayman [8]). *The Hessenberg variety $X=X_{n,p}$ is a smooth irreducible complex projective variety such that $\beta_{2k+1}(X)=0$ and $\beta_{2k}(X)=A(p,n,k)$, for all k.*

Now we can apply the hard Lefschetz theorem to this variety (specifically, the last corollary of it that we stated), and get:

Corollary 4.10. *For fixed p and n, the sequence $A(p,n,0), A(p,n,1), \ldots$ is symmetric and unimodal.*

While the symmetry can be proved easily directly, no other proof is known for the unimodality property. It would be interesting to get other information about the numbers $\{A(p,n,k)\}_k$, analogous to now classical results pertaining to inversions and descents. For instance, standard combinatorial questions involving recurrence relations, generating functions, etc., are not resolved for the numbers $A(p,n,k)$ when $1<p<n-1$.

Our last topic will turn out to be another application of the hard Lefschetz theorem, but first we will present a fair amount of background, which for the most part is independent of the previous parts of this paper.

• *Polytopes and f-vectors.*

•(1) *Background.* A simple example of a polytope is the 3-dimensional cube. It has 8 vertices, 12 edges, and 6 faces. Euler proved in 1752 that if a connected planar

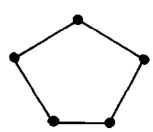

Fig. 10. Example of a 2-dimensional convex polytope.

graph (map of countries) has V vertices, E edges and F faces, then $V-E+F=2$. This is *Euler's formula* for the plane, and in fact it was known to Descartes earlier, but Descartes did not publish it. Geometers were interested in generalizing Euler's formula to higher dimensions; the first question is to what objects should the generalization apply? A very convenient object, though not the most general, is a *convex polytope*, that is, the convex hull of finitely many points in Euclidean space. Convex polytopes are homeomorphic to balls, and the dimension of a polytope is the dimension of the corresponding ball.

A *supporting hyperplane* is a hyperplane which intersects the polytope so that the entire polytope lies on one side of the hyperplane. The intersection of the polytope with a supporting hyperplane is a *face* of the polytope. It is easy to see that the faces themselves are convex polytopes too. A *facet* is a face of maximum dimension. For example, the convex hull of the five points in the plane represented in Fig. 10 has five vertices which are its 0-dimensional faces and five edges which are its 1-dimensional faces or facets. Its dimension is 2.

If we let f_i be the number of i-dimensional faces of a d-dimensional polytope P, we can form the *f-vector of P*, $f(P)=(f_0,f_1,\dots,f_{d-1})$. The f-vector of the 3-dimensional cube (which is a 3-dimensional polytope) is $(8,12,6)$; the f-vector of the polytope in Fig. 10 is $(5,5)$.

A generalization of Euler's formula was known to geometers of the nineteenth century, but they did not have a correct proof until Poincaré invented algebraic topology. This generalization is the so-called Euler–Poincaré formula, and Poincaré's 1893 proof itself had an error which he corrected in 1899. Only much later, in the 1950's, a nonalgebraic, purely geometric proof was found. For an interesting discussion of the history of the Euler–Poincaré formula, see [11, Section 8.6].

Theorem 4.11 (The Euler–Poincaré formula). *If $f=(f_0,f_1,\dots,f_{d-1})$ is the f-vector of a d-dimensional polytope, then*

$$f_0-f_1+f_2-\cdots+(-1)^{d-1}f_{d-1}=1+(-1)^{d-1}.$$

The right-hand side is the *Euler characteristic*, which has value 2 for planar graphs, and in the case of the 3-dimensional cube we can verify that $f_0-f_1+f_2=8-12+6=1+(-1)^2$.

Our goal is to obtain information related to the question: what properties must a vector satisfy in order to be the f-vector of a polytope? Are there other conditions, beside the linear relation in the Euler–Poincaré formula, that the f-vector satisfies? A complete characterization of the f-vectors of general polytopes seems very hard to give. However, very interesting results can be obtained if certain natural assumptions are made about the polytopes under consideration.

While for general polytopes it was proved that the Euler–Poincaré formula is the only linear relation satisfied by the f-vector, many more linear relations hold if we consider a *simplicial polytope*, that is, a polytope all of whose proper faces are simplices. For example, the octahedron and the icosahedron are simplicial polytopes, while the cube and the dodecahedron are not simplicial. The notion of simplicial polytope is dual to the concept of simple which appears in linear programming.

If P is a simplicial polytope, then much more can be said about its f-vector. For instance, $d \cdot f_{d-1} = 2 \cdot f_{d-2}$. Indeed, each $(d-2)$-dimensional face is contained in two faces of dimension $d-1$, and each $(d-1)$-dimensional face contains d faces of dimension $d-2$. Think for example of $d=3$, and triangles and edges.

A concept which is useful and more general than the simplicial polytopes, consists of *triangulations of spheres*.

First of all, let us consider the concept of an *abstract simplicial complex*. Consider a finite set V of vertices. A family Δ of subsets of V is a *simplicial complex* if it satisfies the following two conditions: (i) for every $x \in V$, we have $\{x\} \in \Delta$, and (ii) Δ is a hereditary system of sets, i.e., if $F \in \Delta$ and $G \subseteq F$, then $G \in \Delta$. The members F of Δ are the *faces* of the simplicial complex Δ, and the *dimension* of a face F is $|F|-1$, one unit less than its cardinality. Then we can talk about the f-vector of a simplicial complex, $f(\Delta) = (f_0, f_1, \ldots, f_{d-1})$, where as before, f_i is the number of i-dimensional faces of Δ, and the dimension of Δ is $\dim \Delta = d-1$, the maximum of the dimensions of its faces.

Every abstract simplicial complex Δ has a topological space associated with it, called its *geometric realization* and denoted $|\Delta|$. Without giving a precise definition, think of the vertices of $|\Delta|$ as points in a Euclidean space and each face of Δ as an actual simplex in Euclidean space. We must work in a Euclidean space of sufficiently high dimension, so that the simplices can fit together according to how the faces of Δ share vertices. For example, the pentagon in the plane (Fig. 10) is the geometric realization of $\Delta = \{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{e\}, \{a,b\}, \{b,c\}, \{c,d\}, \{d,e\}, \{a,e\}\}$. In this example, $|\Delta|$ is homeomorphic to the 1-dimensional sphere, S^1, and $f(\Delta) = (5,5)$.

An example of a triangulation of a sphere is obtained from any simplicial polytope. Say, P is a d-dimensional simplicial polytope. A natural simplicial complex associated with P is the *boundary complex* $\Delta(P)$, whose vertices are the vertices of P and whose faces are formed precisely by those vertices which are the vertices of faces in P. Of course, the boundary of a d-dimensional simplicial polytope is a $(d-1)$-dimensional sphere, so $\dim |\Delta(P)| = d-1$ and we have the homeomorphism $|\Delta(P)| \simeq S^{d-1}$. Thus, we have a whole class of triangulations of spheres obtained from simplicial polytopes via the boundary complex.

A natural question then is: is *every* triangulation of a sphere the boundary complex of some simplicial polytope? In the case of 2-dimensional spheres the answer is yes, and constitutes Steinitz's theorem: all 2-spheres are polytopal. However, for spheres of dimension 3 or higher, the answer is no; there exist nonpolytopal spheres. The first one was found by Grünbaum in 1965 [11], and now it is known [13] that there are *many more* nonpolytopal than polytopal spheres. Asymptotic formulae show that the nonpolytopal spheres completely dominate the polytopal ones.

We will look now at what can be said about the f-vector of triangulations of spheres, and what more can be said in the case of polytopal spheres. In fact, it will be better to look not at the f-vector itself, but at a related, equivalent vector. If Δ is a $(d-1)$-dimensional simplicial complex, or a triangulation of the $(d-1)$-dimensional sphere, or polytope, consider its f-vector and form the polynomial $\sum_{i=0}^{d} f_{i-1}(x-1)^{d-i}$, with $f_{-1}=1$. Expand it in powers of x and let h_i be the coefficient of x^{d-i}. Thus,

$$\sum_{i=0}^{d} h_i x^{d-i} = \sum_{i=0}^{d} f_{i-1}(x-1)^{d-i}.$$

Then $h(\Delta)=(h_0,h_1,\dots,h_d)$ is called the *h-vector* of the simplicial complex, or triangulation of the $(d-1)$-dimensional sphere, or polytope, Δ. The h-vector is of course equivalent to the f-vector, but it will turn out to be a more natural combinatorial object.

From the definition, it follows immediately that $h_0=1$, $h_1=f_0-d$, and $\sum_i h_i = f_{d-1}$. For example, for the 3-dimensional octahedron, which is a simplicial polytope, the h-vector is $(1,3,3,1)$. Notice that this h-vector is symmetric. In fact, this is true in general about the h-vector of triangulations of spheres, and we get additional linear relations satisfied in the case of triangulations of spheres. These relations are called the Dehn–Sommerville equations.

Theorem 4.12 (Dehn–Sommerville equations). *If Δ is a triangulation of the $(d-1)$-dimensional sphere, then $h_i=h_{d-i}$ for all $i\in\{0,1,\dots,d\}$.*

McMullen proved that these are the most general linear relations that hold for the h-vector, hence, f-vector, of triangulations of spheres. We can ask what other relations are satisfied, not necessarily linear relations. For example, one such question, motivated by the performance of the simplex algorithm in linear programming, is:

Given n, d, and i, what is the maximum number of i-dimensional faces that we can have in a triangulation of the $(d-1)$-dimensional sphere with n vertices?

There was a conjectured answer to this question, formulated by Motzkin with regard to simplicial polytopes and extended by Klee to triangulations of spheres.

Conjecture 4.13 (The upper bound conjecture (UBC) for spheres). *If $|\Delta|\simeq S^{d-1}$ and $f_0=n$, then*

$$f_i(\Delta)\le f_i(C(n,d)),$$

where $C(n,d)$ is the convex hull of any $n>d$ points on the "moment curve" $\{(\tau,\tau^2,\dots,\tau^d)\colon \tau\in\mathbb{R}\}$. It can be verified that this is indeed a simplicial polytope of dimension d with n vertices (the n points chosen on the moment curve), and that its combinatorial type is independent of the choice of the n points.

The "cyclic polytope" $C(n,d)$ itself defines a triangulation of the $(d-1)$-

dimensional sphere, and thus, the UBC for spheres says that if Δ is any triangulation of a $(d-1)$-dimensional sphere and has n vertices, then, for each i, it cannot have more i-dimensional faces than the cyclic polytope $C(n,d)$ has.

Why is the UBC plausible? It is not too hard to show that it is equivalent to the following statement: there exists a triangulation of the $(d-1)$-dimensional sphere with n vertices which maximizes *all* f_i, $1 \leq i \leq d-1$, simultaneously. This statement is indeed very plausible, because a larger number of i-dimensional faces helps have a larger number of j-dimensional faces. However, there is no known straightforward proof that there exists a triangulation which maximizes all numbers of faces simultaneously.

Significant progress was made on the UBC. In particular, McMullen proved three results:

(M1) The UBC holds for simplicial polytopes.

McMullen proved this special case of the UBC using the fact that the boundary complex of a (simplicial) polytope is shellable. Unfortunately, since there do exist nonshellable triangulations of spheres, McMullen's proof cannot settle the full conjecture. On the other hand, a byproduct of his proof is:

(M2) If P is a simplicial polytope, then $h_i \geq 0$ for each i.

One can also ask whether this is true for spheres.

(M3) If the h-vector satisfies the inequalities $h_i \leq \binom{n-d+i-1}{i}$, then the UBC holds.

Thus, McMullen proved a sufficient condition for the UBC for spheres.

In view of this evidence and partial results, the big question is whether the UBC holds indeed for triangulations of spheres. We will succeed in proving that it does using algebraic techniques, namely certain results from commutative algebra.

•(2) *Commutative algebra.* The main algebraic object on which we will concentrate is called a *standard graded algebra.* K will be a field and $R = R_0 \oplus R_1 \oplus \cdots$ a vector space over K which is the direct sum of subspaces indexed by the nonnegative integers. Furthermore, R is a commutative ring with unity, $R_0 = K$ and R is generated by R_1 as a K-algebra. We assume that $\dim_K R_1 < \infty$, which says that R is finitely generated as an algebra. Finally, the statement that R is *graded* means that $R_i R_j \subseteq R_{i+j}$. Such R is called a standard graded algebra.

For example, $K[x_1, x_2, \ldots, x_n]$, the ring of polynomials in n variables with coefficients in the field K, in which we define the degree of each variable to be 1, is a standard graded algebra over K. The ith graded piece consists of all homogeneous polynomials of degree i, together with 0.

The assumption $\dim_K R_1 < \infty$ implies that $\dim_K R_i < \infty$ for all i, and we can investigate the growth of these vector space dimensions, the *Hilbert function* $H(R,i) := \dim_K R_i$. The study of Hilbert functions is a very interesting topic on the interface of commutative algebra and combinatorics. As is often the case, we study the function $H(R,i)$ by studying its generating function $F(R,\lambda) := \sum_{i \geq 0} H(R,i)\lambda^i$, called the *Hilbert series* of the graded algebra R.

Theorem 4.14 (Hilbert). *For every standard graded algebra defined as above, the Hilbert series is a rational function,*

$$F(R,\lambda)=\frac{h_0+h_1\lambda+\cdots+h_s\lambda^s}{(1-\lambda)^d}.$$

Let us assume that the Hilbert series expression is reduced to lowest terms, i.e., $\sum_{i=0}^{s} h_i \neq 0$. Then d, which controls the rate of growth of the Hilbert function $H(R,i)$, is the Krull dimension of R, denoted $\dim R$, a very important parameter of the ring R.

A key definition which we state here in combinatorial terms, and for which we assume that the field K is infinite, is that of a Cohen–Macaulay algebra. A standard graded algebra R of Krull dimension d is *Cohen–Macaulay* if there exist $\theta_1, \theta_2, \ldots, \theta_d \in R_1$ (called a *homogeneous system of parameters*) such that the Hilbert series of the quotient $R/(\theta_1, \theta_2, \ldots, \theta_d)$, which inherits the grading from R, satisfies

$$F(R/(\theta_1,\theta_2,\ldots,\theta_d),\lambda)=(1-\lambda)^d F(R,\lambda)=h_0+h_1\lambda+\cdots+h_s\lambda^s.$$

In particular, $\dim_K(R/(\theta_1,\theta_2,\ldots,\theta_d))=\sum h_i<\infty$, so the quotient is a finite dimensional vector space.

A necessary condition for R to be Cohen–Macaulay is that $h_i \geq 0$, because h_i is the value of the Hilbert function on the degree i subspace of the quotient $R/(\theta_1, \theta_2, \ldots, \theta_d)$.

As an aside, here is how to reconcile the above definition of Cohen–Macaulay with the usual algebraic definition [18]. If we mod out successively by the θ_i's when R is Cohen–Macaulay, each time the Hilbert series is multiplied by $1-\lambda$. It is easy to see that the effect on the Hilbert series is multiplication by $1-\lambda$ if and only if each θ_i is a nonzero divisor at the time when we mod out by it. This is the (usual) algebraic definition of the Cohen–Macaulay property: there exist $\theta_1, \theta_2, \ldots, \theta_d$, homogeneous of positive degree, such that each θ_i is a nonzero divisor in $R/(\theta_1, \theta_2, \ldots, \theta_{i-1})$.

•(3) *The upper bound conjecture.* Now we can return to simplicial complexes and tie in this new piece of algebra. Let Δ be a simplicial complex on a vertex set $V=\{x_1, x_2, \ldots, x_n\}$ and of dimension $d-1$. We associate with it its *face ring*, $K[\Delta]:=K[x_1, x_2, \ldots, x_n]/I_\Delta$, where we think of the vertices as independent variables and I_Δ is the ideal generated by the monomials $x_{i_1}x_{i_2}\cdots x_{i_j}$ such that $\{x_{i_1}, x_{i_2}, \ldots, x_{i_j}\} \notin \Delta$. This means that after we mod out, the only monomials left are those supported by faces of Δ. The multiplication of monomials is carried out as usual except that the monomials whose variables do not form a face of Δ are identified with zero. For example, the simplicial complex with facets ab, bc, cd, ad has the face ring $K[\Delta]=K[a,b,c,d]/(ac,bd)$; I_Δ is generated by ac and bd since I_Δ is an ideal and $\{a,c\}$ and $\{b,d\}$ are the minimal sets of vertices which do not form faces.

Let $\deg x_i = 1$ for each vertex x_i. This turns the face ring $K[\Delta]$ into a graded algebra, and the monomials supported by faces of Δ form a vector space basis. We wish to compute the Hilbert series of $K[\Delta]$. By keeping track of the contribution from each face, it is not hard to prove that:

Proposition 4.15. *If $K[\Delta]$ is the face ring of a $(d-1)$-dimensional simplicial complex Δ, then its Hilbert series is*

$$F(K[\Delta], \lambda) = \frac{h_0 + h_1\lambda + \cdots + h_d\lambda^d}{(1-\lambda)^d},$$

where the coefficients h_i form the h-vector of the simplicial complex.

Observe that while in general the degree of the numerator may be arbitrarily large compared to that of the denominator, in the case of the face ring of a simplicial complex the degree of the numerator cannot exceed the degree of the denominator. Also, the Krull dimension of the face ring is the dimension of the simplicial complex augmented by one unit. Thus, this proposition relates parameters of the simplicial complex with parameters of the face ring, and in particular, it brings the h-vector into the picture, relating it to the algebra.

An immediate consequence of our previous discussion is:

Corollary 4.16. *Let Δ be a triangulation of the $(d-1)$-dimensional sphere. If its face ring $K[\Delta]$ is Cohen–Macaulay, then the UBC holds for Δ.*

Proof. Recall the definition of a Cohen–Macaulay ring and the preceding proposition. If $K[\Delta]$ is Cohen–Macaulay, then $K[\Delta]/(\theta_1, \theta_2, \ldots, \theta_d)$ is generated by $h_1 = n-d$ elements of degree 1, since the original number of generators, before modding out, was n (the number of vertices of Δ), and we have modded out by d elements of degree 1. Since the Hilbert series of $K[\Delta]/(\theta_1, \theta_2, \ldots, \theta_d)$ has coefficients h_i, there are h_i linearly independent elements in the ith graded piece of the quotient, that is, there are h_i linearly independent monomials of degree i in $n-d$ variables. But it is an undergraduate level exercise to show that the total number of monomials of degree i in $n-d$ variables is $\binom{n-d+i-1}{i}$. Hence, for each i we have $h_i \leq \binom{n-d+i-1}{i}$, which is McMullen's sufficient condition (M3) for the UBC. □

In order to appreciate the power of this corollary, we must determine *when* the face ring is Cohen–Macaulay. In 1976, independently of these combinatorial developments, Gerald Reisner, a student of Melvin Hochster, proved a necessary and sufficient condition for the face ring $K[\Delta]$ to be Cohen–Macaulay. This condition is in terms of the homology groups of various subcomplexes of the simplicial complex Δ and is rather difficult to prove. It turns out that any triangulation of a sphere satisfies this condition, and thus we have:

Theorem 4.17 (Reisner, 1976, [26]). *If Δ is a triangulation of a sphere, then $K[\Delta]$ is Cohen–Macaulay.*

Assuming Reisner's theorem, and thus leaving out the hardest part of the proof, we have established the UBC for spheres:

Corollary 4.18. *The UBC holds for triangulations of spheres.*

•(4) *The g-conjecture.* Having obtained a proof of the upper bound conjecture, we can ask for more detailed information about the f-vectors of simplicial polytopes. We can be ambitious enough to ask for the complete characterization of the f-vectors of simplicial polytopes or triangulations of spheres. The "g-conjecture" given below (its name comes from the use of the notation g_i for the coefficients of a certain generating function) is essentially due to McMullen, who stated it in 1971 in a numerical form without making the connection with the Hilbert function.

Conjecture 4.19 (The g-conjecture). *A vector $(h_0, h_1, \ldots, h_d) \in \mathbb{Z}^{d+1}$ is the h-vector of some d-dimensional simplicial polytope if and only if the Dehn–Sommerville equations hold (i.e., $h_i = h_{d-i}$) and there exists a standard graded algebra $R = R_0 \oplus R_1 \oplus \cdots$ whose Hilbert function satisfies $H(R,i) = h_i - h_{i-1}$, for each $i \leq [d/2]$.*

Note that since $H(R,i)$ is the dimension of a vector space, it is nonnegative, so the g-conjecture says in particular that the h-vector of a simplicial polytope must be a unimodal sequence. The unimodality of the h-vector was conjectured earlier by McMullen and Walkup under the name of "the generalized lower bound conjecture" (GLBC) because, in turn, it implies the "lower bound conjecture" (LBC). The LBC concerns the minimum possible number of faces of each dimension in a triangulation of a sphere on n vertices. The LBC was proved by Barnette [3], while the GLBC remained open as did the g-conjecture. It is interesting that McMullen made the g-conjecture based on numerical evidence, and there was no proof for either the necessity or the sufficiency of the condition in the conjecture.

The g-conjecture is now known to be true. We give an outline of the results and methods used in its proof.

The proof of the sufficiency requires that given a vector h which satisfies the conditions above, we construct a polytope having h as its h-vector. This was done by Billera and Lee in 1979 [4] using a very ingenious inductive argument which we will not describe here.

The proof of the necessity relies on the use of some further algebraic machinery, namely the theory of *toric varieties*, which was created by Demazure in 1970, and was elaborated upon by Mumford et al., 1973. Let us see briefly what this entails.

Suppose we have a vector h which is the h-vector of a simplicial polytope P. The idea is to associate with P a certain algebraic variety which is a special case of a so-called toric variety. However, this cannot be done with an arbitrary polytope, and we will have to put some conditions on P. First, we may assume that P lies in the d-dimensional space, $\mathbb{R}^d$, and (through a translation) that the origin is an interior point of the polytope, $0 \in P - \partial P$. We may also assume that the vertices of P have rational coordinates; this can be achieved by very small perturbations of the vertices, without changing the combinatorial type of the polytope, in particular, without changing the h-vector. It is essential that P is a simplicial polytope, since there exist nonsimplicial polytopes for which this assumption cannot be made.

Given a simplicial polytope P satisfying the above assumptions, we associate with it a certain d-dimensional irreducible complex projective variety $X = X(P)$. The variety $X(P)$ may not be smooth, so the situation we have here is not quite the same as in our earlier discussion of the hard Lefschetz theorem. Fortunately, there are many theorems which pertain to such varieties as we have now. One of them, proved by Danilov in 1978, deals with the computation of the cohomology ring, say, over the complex field. In general, the cohomology is in degrees 0 through $2d$, but for toric varieties it is zero in odd degrees. Incidentally, this is the condition needed to ensure that the cohomology ring is commutative.

Theorem 4.20 (Danilov, 1978 [7]). *With the notation above, the cohomology ring*

$$H^*(X(P), \mathbb{C}) = H^0(X) \oplus H^2(X) \oplus \cdots \oplus H^{2d}(X)$$

is isomorphic to the quotient

$$\mathbb{C}[\Delta(P)]/(\theta_1, \ldots, \theta_d)$$

of the face ring of the boundary complex of the polytope by a certain homogeneous system of parameters $\theta_1, \ldots, \theta_d$ of degree 1. (The isomorphism between $\mathbb{C}[\Delta(P)]/(\theta_1, \ldots, \theta_d)$ and $H^*(X)$ doubles degree, so θ_i, regarded as an element of $H^*(X)$ lies in $H^2(X)$.)

Recall that Reisner's theorem tells us that $\mathbb{C}[\Delta]$ is Cohen–Macaulay. It follows from the definition of a Cohen–Macaulay ring together with the computation of $F([\mathbb{C}[\Delta], \lambda)$ that $\dim_{\mathbb{C}} H^{2i}(X) = h_i$. Also, $H^*(X)$ is a standard graded algebra because the quotient $\mathbb{C}[\Delta(P)]/(\theta_1, \ldots, \theta_d)$ is standard, being generated by elements in $H^2(X)$. This is not true in general; for example, the cohomology ring of the Grassmann variety is not standard (though it is commutative).

Using Danilov's theorem we have obtained the above ring $H^*(X)$ whose Hilbert function agrees with the h_i's, but our task is to find a ring R whose Hilbert function satisfies $H(R, i) = h_i - h_{i-1}$. For this we need one final theorem, which is considerably more difficult than the hard Lefschetz theorem for smooth varieties.

Theorem 4.21 (M. Saito [27]). *If P is a simplicial polytope, then the corresponding toric variety $X(P)$ satisfies the hard Lefschetz theorem.*

Recalling the hard Lefschetz theorem, this implies further that there exists $\omega \in H^2(X(P))$ such that multiplication by ω gives a one-to-one map $\omega : H^{2(i-1)}(X(P)) \to H^{2i}(X(P))$ for each i, $1 \le i \le [d/2]$. Now, the ring we are seeking will be

$$R = H^*(X(P))/(\omega).$$

Since ω raises the degree by 2 units, we have

$$R = H^0 \oplus H^2/\omega H^0 \oplus H^4/\omega H^2 \oplus \cdots.$$

Moreover, since ω is one-to-one when $1 \le i \le [d/2]$, for such i we get that the dimension of the ith graded piece of R is

$$\dim(H^{2i}/\omega H^{2(i-1)}) = \dim H^{2i} - \dim \omega H^{2(i-1)} = \dim H^{2i} - \dim H^{2(i-1)}.$$

Finally, by Danilov's theorem, we have $\dim H^{2i}/\omega H^{2(i-1)} = h_i - h_{i-1}$, and the g-conjecture is proved.

•(5) *Finite group actions revisited.* We return now to finite group actions, this time in the context of polytopes. As a quick survey of the main result in this direction we will consider only the most interesting case, that of the group of order 2.

Assume we have a triangulation of the $(d-1)$-dimensional sphere, $|\Delta| = S^{d-1}$, with a *free involution* σ acting on it. This means that σ is an automorphism of Δ, denoted $\sigma \in \mathrm{Aut}(\Delta)$, such that (i) $\sigma^2 = \mathrm{id}$ (σ is an involution) and (ii) for each vertex x, $\{x, \sigma(x)\} \notin \Delta$; in particular, since each vertex is a face, no vertex is fixed by σ. Thus, σ is fixed point free on the vertices and in fact does not fix any face. For example, if P is a centrally symmetric simplicial d-dimensional polytope, we may take $\Delta = \Delta(P)$, and the automorphism σ given by reflection through the center: $\sigma(x)$ is the point antipodal to x.

What more can be said about f-vectors of centrally symmetric simplicial polytopes or spheres, in addition to what we already know? One result is:

Theorem 4.22. *If Δ is a triangulation of a $(d-1)$-dimensional sphere with a free involution σ, then $h_i \ge \binom{d}{i}$. In particular, the number of facets is $f_{d-1} = \sum_i h_i \ge 2^d$.*

For the case of convex polytopes, this theorem was proved by Bárány and Lovász [2] using the Borsuk–Ulam theorem. Their proof, however, does not extend to spheres. They also stated this theorem in its dual form, that is, any $(d-1)$-dimensional centrally symmetric *simple* polytope has at least 2^d *vertices*. This lower bound is achieved by the d-cube. In the simplicial case, the dual of the cube (the octahedron for the 3-cube) achieves the bound for the number of facets.

Proof (sketch). The involution gives an action of the group $\{\mathrm{id}, \sigma\}$ on $\mathbb{C}[\Delta]$. Because σ is a free involution, there exists a linear homogeneous system of parameters $\theta_1, \theta_2, \ldots, \theta_d$ with the additional property that for each i we have $\sigma(\theta_i) = -\theta_i$. In particular, the ideal generated by these θ_i's is fixed by σ, and therefore the group acts on the quotient $\mathbb{C}[\Delta]/(\theta_1, \theta_2, \dots, \theta_d)$. Moreover, since the quotient is graded,

$$\mathbb{C}[\Delta]/(\theta_1, \theta_2, \dots, \theta_d) \simeq A_0 \oplus A_1 \oplus \cdots \oplus A_d,$$

and the group acts on each of the subspaces A_i. Since $|\Delta|$ is a sphere, $\mathbb{C}[\Delta]$ is Cohen–Macaulay and $\dim A_i = h_i = h_i(\Delta)$. It is not too difficult to compute the dimension of the subspace of A_i which is fixed pointwise under the group action. The dimension of this invariant subspace turns out to be $\frac{1}{2}(h_i + \binom{d}{i})$, and cannot exceed the dimension h_i of A_i. Thus, $\frac{1}{2}(h_i + \binom{d}{i}) \le h_i$, proving the theorem. □

Just as in the case of general simplicial polytopes rather than triangulations of spheres, we can say more if we assume that the centrally symmetric triangulation is polytopal: $\Delta = \Delta(P)$. Recall that we have associated a toric variety $X(P)$ with the polytope P. If the polytope is centrally symmetric, then the group $\mathbb{Z}/2\mathbb{Z}$ acts on $X(P)$, hence, induces an action on the cohomology ring $H^*(X(P))$, and its action can be shown (using general considerations from algebraic geometry) to commute with the multiplication by the element $\omega \in H^2(X(P))$ of the hard Lefschetz theorem. This is entirely analogous to the poset situation where the raising operator commutes with the group action.

All this leads to the following additional condition for the h-vector of a simplicial centrally symmetric polytope: for $i \le [d/2]$, not only must $h_i - h_{i-1}$ be nonnegative, as for all simplicial polytopes, but

$$h_i - h_{i-1} \ge \binom{d}{i} - \binom{d}{i-1}.$$

This fact was conjectured by Björner and it implies an earlier conjecture of Bárány and Lovász. Recall that they had proved that given a simplicial centrally symmetric d-polytope P, the smallest possible number of facets (maximum dimension faces) is 2^d. They also had a conjectured value for the smallest possible number of i-dimensional faces for *any* number $2n$ of vertices, i.e., for the quantity $\min\{f_i: P$ simplicial centrally symmetric polytope, $\dim P = d,\ f_0 = 2n\}$. The conjecture of Bárány and Lovász is implied by the (now proved) conjecture of Björner.

5. Open questions

We close with a few interesting open problems in this area.

Maybe the most central open problem in this area is whether the g-conjecture, or just the unimodality of the h-vector, $h_0 \le h_2 \le \cdots \le h_{[d/2]}$, still holds for triangulations of spheres. Recall that the proof which we have presented of the g-conjecture for simplicial polytopes depended on the association of toric varieties with simplicial polytopes. This cannot be done for spheres in general. Although it is possible to associate varieties with a certain class of spheres which includes the simplicial polytopes, these varieties are not projective, so the proof based on the hard Lefschetz theorem does not apply, and we do not get new results.

Another question that can be asked is what can be said about nonsimplicial polytopes. Here are three interesting conjectures in this direction.

Conjecture 5.1 (Kalai). *If P is any centrally symmetric d-polytope (not necessarily simplicial), then*

$$1+f_0+\cdots+f_{d-1}\geq 3^d,$$

i.e., the total number of faces, including the empty face, is at least 3^d.

For example, the cube achieves this bound. Unfortunately, there are many more polytopes which achieve this bound as well, and they have different values for the f_i's. This seems to be the cause for the difficulty of this conjecture.

Conjecture 5.2 (Kupitz). *Let P be a d-polytope with no triangular 2-face. Then its number of vertices must satisfy*

$$f_0\geq 2^d.$$

The d-cube achieves this lower bound.[1]

Finally, a very intriguing "Ramsey-type" conjecture of Kalai, for convex polytopes.

Conjecture 5.3 (Kalai). *Given a positive integer $k\geq 1$, there exists an integer $d>k$ such that every d-polytope has a k-face which is either a simplex or is combinatorially equivalent to a cube.*

Very recently the case $k=2$ of this conjecture was proved by Kalai [14]. The resulting theorem solves a longstanding open problem in the theory of convex polytopes.

Theorem 5.4. *Every 5-dimensional polytope has a 2-dimensional face which is either a triangle or a quadrilateral.*

The value $d=5$ corresponding to $k=2$ is best possible since in three dimensions, the dodecahedron has only pentagonal 2-dimensional faces and in four dimensions there is a regular polytope whose 3-dimensional faces are all dodecahedra, so all its 2-dimensional faces are pentagons as well. Thus, 5 dimensions is the minimum for d.

For some results on nonsimplicial polytopes related to the intersection homology of toric varieties, see [29].

[1] This conjecture has recently been proved by G. Blind and R. Blind.

References

[1] M. Aigner, Combinatorial Theory (Springer, New York, 1979).

[2] I. Bárány and L. Lovász, Borsuk's theorem and the number of facets of centrally symmetric polytopes, Acta Math. Acad. Sci. Hungar. 40 (1982) 323-329.

[3] D. Barnette, A proof of the lower bound conjecture for convex polytopes, Pacific J. Math. 46 (1973) 349-354.

[4] L. Billera and C. Lee, A proof of the sufficiency of McMullen's conditions for f-vectors of simplicial convex polytopes, J. Combin. Theory Ser. A 31 (1981) 237-255.

[5] G. Birkhoff, Lattice Theory (Amer. Math. Soc., Providence, RI, 3rd ed., 1967).

[6] L.M. Butler, A unimodality result in the enumeration of subgroups of a finite abelian group, Proc. Amer. Math. Soc. 101 (1987) 771-775.

[7] V.I. Danilov, The geometry of toric varieties, Russian Math. Surveys 33 (1978) 97-154.

[8] F. DeMari and M. Shayman, Generalized Eulerian numbers and the topology of the Hessenberg variety of a matrix, Acta Appl. Math. 12 (1988) 213-235.

[9] C. Ehresmann, Sur la topologie de certains espaces homogènes, Ann. of Math. 35 (1934) 396-443.

[10] I.M. Gelfand and M.L. Zetlin, Finite dimensional representations of the group of unimodular matrices, Dokl. Akad. Nauk. USSR 71 (1950) 825-828 (in Russian).

[11] B. Grünbaum, Convex Polytopes (Interscience, London, 1967).

[12] J. Humphreys, Introduction to Lie Algebras and Representation Theory (Springer, New York, 1972).

[13] G. Kalai, Many triangulated spheres, Discrete Comput. Geom. 3 (1988) 1-14.

[14] G. Kalai, On low-dimensional faces that high-dimensional polytopes must have, Preprint.

[15] W.M. Kantor, On incidence matrices of finite projective and affine spaces, Math. Z. 124 (1972) 315-318.

[16] B. Lindström, Conjecture on a theorem similar to Sperner's, in: R. Guy, H. Hanani, M. Sauer and J. Schönheim, eds., Combinatorial Structures and Their Applications (Gordon and Breach, New York, 1970) 241.

[17] D. Livingstone and A. Wagner, Transitivity of finite permutation groups on unordered sets, Math. Z. 90 (1965) 393-403.

[18] H. Matsumura, Commutative Algebra (Benjamin/Cummings, Reading, MA, 2nd ed., 1980).

[19] P. McMullen, The maximum numbers of faces of a convex polytope, Mathematika 17 (1970) 179-184.

[20] A. Odlyzko and B. Richmond, On the unimodality of some partition polynomials, European J. Combin. 3 (1982) 69-84.

[21] K. O'Hara, Unimodality of Gaussian coefficients: a constructive proof, J. Combin. Theory Ser. A 53 (1990) 29-52.

[22] G.W. Peck, Erdös conjecture on sums of distinct numbers, Stud. Appl. Math. 63 (1980) 87-92.

[23] M. Pouzet and I.G. Rosenberg, Sperner properties for groups and relations, European J. Combin. 7 (1986) 349-370.

[24] R. Proctor, Representations of $sl(2,\mathbb{C})$ on posets and the Sperner property, SIAM J. Algebraic Discrete Methods 3 (1982) 275-280.

[25] F. Regonati, Whitney numbers (of the II kind) of finite modular lattices, in preparation.

[26] G. Reisner, Cohen-Macaulay quotients of polynomial rings, Adv. in Math. 21 (1976) 30-49.

[27] M. Saito, Hodge structure via filtered D-modules, Asterisque 130 (1985) 342-351.

[28] R.P. Stanley, Enumerative Combinatorics, Vol. 1 (Wadsworth and Brooks/Cole, Monterey, CA, 1986).

[29] R.P. Stanley, Generalized h-vectors, intersection cohomology of toric varieties, and related results, in: Commutative Algebra, Advanced Studies in Pure Mathematics 11 (North-Holland, Amsterdam, 1987) 187-213.

[30] J.J. Sylvester, Proof of the hitherto undemonstrated fundamental theorem of invariants, in: Collected Mathematics Papers, Vol. 3 (Chelsea, New York, 1973) 117–126.

[31] D. Zeilberger, Kathy O'Hara's constructive proof of the unimodality of the Gaussian polynomials, Amer. Math. Monthly 96 (1989) 590–602.