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On the Number of Faces of Centrally-Symmetric Simplicial Polytopes

Richard P. Stanley*

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA

Abstract. I. Bárány and L. Lovász [Acta Math. Acad. Sci. Hung. **40**, 323–329 (1982)] showed that a d-dimensional centrally-symmetric simplicial polytope $\mathscr{P}$ has at least 2^d facets, and conjectured a lower bound for the number f_i of i-dimensional faces of $\mathscr{P}$ in terms of d and the number $f_0 = 2n$ of vertices. Define integers $h_0, \ldots, h_d$ by $\sum_{i=0}^{d} f_{i-1}(x-1)^{d-i} = \sum_{i=0}^{d} h_i x^{d-i}$. A. Björner conjectured (unpublished) that $h_i \geq \binom{d}{i}$ (which generalizes the result of Bárány-Lovász since $f_{d-1} = \sum h_i$), and more strongly that $h_i - h_{i-1} \geq \binom{d}{i} - \binom{d}{i-1}$, $1 \leq i \leq \lfloor d/2 \rfloor$, which is easily seen to imply the conjecture of Bárány-Lovász. In this paper the conjectures of Björner are proved.

1. Introduction

Let $\mathscr{P}$ be a simplicial d-polytope, i.e., a d-dimensional simplicial convex polytope. Let f_i denote the number of i-dimensional faces of $\mathscr{P}$, where we set $f_{-1} = 1$. Define the *h-vector* $h(\mathscr{P}) = (h_0, h_1, \ldots, h_d)$ of $\mathscr{P}$ by the formula

$$\sum_{i=0}^{d} f_{i-1}(x-1)^{d-i} = \sum_{i=0}^{d} h_i x^{d-i}. \tag{1}$$

Suppose now that $\mathscr{P}$ is also centrally-symmetric (about the origin), i.e., $\mathscr{P}$ is embedded in Euclidean space so that if $v \in \mathscr{P}$ then $-v \in \mathscr{P}$. Bárány and Lovász [1] showed that $\mathscr{P}$ then has at least 2^d facets (or $(d-1)$-faces), equality being achieved by the d-dimensional cross-polytope (the dual to the d-cube). They also conjectured that if $\mathscr{P}$ has $2n$ vertices (i.e., $f_0 = 2n$), then

$$f_i \geq 2^{i+1}\binom{d}{i+1} + 2(n-d)\binom{d}{i}, \quad 0 \leq i \leq d-2, \tag{2}$$

$$f_{d-1} \geq 2^d + 2(n-d)(d-1). \tag{3}$$

(Actually, Bárány and Lovász deal with *simple* polytopes and state all their results and conjectures in dual form to ours.) The inequalities (2) and (3) are best possible,

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since they can be achieved by taking the d-cross-polytope and applying $n-d$ successive pairs of stellar subdivisions of antipodal facets.

In terms of the h-vector, the inequality $f_{d-1} \geq 2^d$ of Bárány-Lovász takes the form

$$h_0 + h_1 + \cdots + h_d \geq 2^d.$$

Moreover, for any simplicial d-polytope we have $h_i = h_{d-i}$ (the Dehn-Sommerville equations) and $1 = h_0 \leq h_1 \leq \cdots \leq h_{[d/2]}$ (the Generalized Lower-Bound Conjecture), as surveyed in [10]. For the d-cross-polytope we have $h_i = \binom{d}{i}$. These considerations led A. Björner to conjecture (unpublished) that for any centrally-symmetric simplicial d-polytope $\mathscr{P}$ with h-vector $(h_0, h_1, \ldots, h_d)$, we have

$$h_i \geq \binom{d}{i}, \quad 0 \leq i \leq d, \tag{4}$$

and more strongly (since $h_0 = 1$),

$$h_i - h_{i-1} \geq \binom{d}{i} - \binom{d}{i-1}, \quad 1 \leq i \leq [d/2]. \tag{5}$$

It is easily seen (see Corollary 4.2) that the inequalities (5) imply (2) and (3).

In this paper we will prove the conjectures (4) and (5) of Björner. We will establish (4) for a much broader class of objects than centrally-symmetric simplicial polytopes, but for (5) we are unable to weaken these hypotheses. Briefly, the idea behind the proofs is as follows. If Δ is a Cohen-Macaulay simplicial complex, then the theory of Cohen-Macaulay rings shows that $h_i \geq 0$ by interpreting h_i as the dimension of a certain vector space A_i. When in fact Δ is the boundary complex of a simplicial polytope, the theory of toric varieties allows us to construct injective linear transformations $A_{i-1} \to A_i$, $1 \leq i \leq [d/2]$. Hence here we get $h_0 \leq h_1 \leq \cdots \leq h_{[d/2]}$. When we have a group G of order 2 acting on Δ in a suitable way (which for the boundary complex of a centrally-symmetric polytope is induced by the map $v \to -v$ on $\mathscr{P}$) then G acts on the vector spaces A_i, and by decomposing this action into isotypic components (with respect to the two inequivalent irreducible representations of G), we can improve the inequalities $h_i \geq 0$ and $h_0 \leq h_1 \leq \cdots \leq h_{[d/2]}$ to (4) and (5), respectively.

2. Algebraic Background

We now review some algebraic concepts associated with simplicial complexes. See, e.g., [2] or [10] for more details. Let Δ be an abstract simplicial complex on the vertex set $V = \{x_1, \ldots, x_n\}$. Let K be a field, and let I_Δ be the ideal of the polynomial ring $K[x_1, \ldots, x_n]$ generated by all square-free monomials $x_{i_1} \cdots x_{i_r}$ for which $\{x_{i_1}, \ldots, x_{i_r}\} \notin \Delta$. Let $K[\Delta] = K[x_1, \ldots, x_n]/I_\Delta$, called the *face ring* (or Stanley-Reisner ring) of Δ. Let $K[\Delta]_i$ denote the space of all homogeneous polynomials of degree i in $K[\Delta]$, so $K[\Delta]$ has the structure

$$K[\Delta] = K[\Delta]_0 \oplus K[\Delta]_1 \oplus \cdots$$

of a graded K-algebra. If $\dim \Delta = d - 1$ (i.e., the largest face $F \in \Delta$ has d vertices), then d is the maximum number of algebraically independent (over K) elements of $K[\Delta]$ (or of $K[\Delta]_1$). A set $\theta_1, \ldots, \theta_d \in K[\Delta]_1$ is called a *homogeneous system of parameters* (h.s.o.p.) of degree one if $\dim_K K[\Delta]/(\theta_1, \ldots, \theta_d) < \infty$. (This implies that $\theta_1, \ldots, \theta_d$ are algebraically independent.) An h.s.o.p. of degree one always exists if K is infinite, which for convenience we will assume henceforth.

We say that $K[\Delta]$ is *Cohen-Macaulay* (or that Δ is Cohen-Macaulay over K) if for some (equivalently, every) h.s.o.p. $\theta_1, \ldots, \theta_d$ of degree one, $K[\Delta]$ is a finitely-generated *free* module over the polynomial subring $K[\theta_1, \ldots, \theta_d]$. (Equivalently, θ_i is a non-zero-divisor modulo $(\theta_1, \ldots, \theta_{i-1})$, $1 \le i \le d$.) Thus

$$K[\Delta] = \coprod_{1}^{t} \eta_i \cdot K[\theta_1, \ldots, \theta_d], \quad \text{(vector-space direct sum)} \tag{6}$$

where each η_i is a non-zero-divisor on $K[\theta_1, \ldots, \theta_d]$. We can choose each η_i to be homogeneous, and conversely a set $\eta_1, \ldots, \eta_t$ of homogeneous elements of $K[\Delta]$ satisfies (6) if and only if they form a K-basis of the quotient ring

$$A = K[\Delta]/(\theta_1, \ldots, \theta_d).$$

The ring A inherits a grading $A = A_0 \oplus A_1 \oplus \cdots$, and a simple counting argument shows that

$$\dim A_i = h_i(\Delta),$$

where Δ has $f_i = f_i(\Delta)$ i-dimensional faces (so $f_0 = n$) and $h_i = h_i(\Delta)$ is defined by (1).

Let us recall the fundamental theorem of G. Reisner (see [8, Thm. 5] or [9, p. 70]) characterizing Cohen-Macaulay complexes. Given any face $F \in \Delta$, define the *link* of F by

$$\operatorname{lk} F = \{G \in \Delta \colon F \cup G \in \Delta \text{ and } F \cap G = \varnothing\}.$$

In particular, $\operatorname{lk} \varnothing = \Delta$.

Theorem 2.1. *Let Δ be a (finite) simplicial complex. Then Δ is Cohen-Macaulay over K if and only if for all $F \in \Delta$,*

$$\tilde{H}_i(\operatorname{lk} F; K) = 0 \quad \text{if} \quad i < \dim(\operatorname{lk} F),$$

where $\tilde{H}_i(\operatorname{lk} F; K)$ denotes reduced simplicial homology over K. □

In particular, all triangulations of spheres are Cohen-Macaulay. More generally (e.g. [8, Thm. 5]), the question of whether Δ is Cohen-Macaulay depends only on the geometric realization $|\Delta|$ of Δ.

We also need a characterization of h.s.o.p.'s of degree one in $K[\Delta]$. If

$$y = \sum_{x \in V} \alpha_x \cdot x \in K[\Delta]_1$$

where $\alpha_x \in K$, then define the *restriction* $y|_F$ of y to the face $F \in \Delta$ by

$$y|_F = \sum_{x \in F} \alpha_x \cdot x.$$

Lemma 2.2 (see, e.g., [2, p. 66]). *For any $(d-1)$-dimensional simplicial complex Δ, a set $\theta_1, \ldots, \theta_d \in K[\Delta]_1$ is an h.s.o.p. if and only if for all $F \in \Delta$, the vector space spanned by $\theta_1|_F, \ldots, \theta_d|_F$ has dimension equal to $|F|$.* □

3. Group Actions

Now let G be a group of automorphisms of the simplicial complex Δ. Thus each $\sigma \in G$ is a bijection $\sigma\colon \Delta \to \Delta$ such that if $F \subset F' \in \Delta$, then $\sigma(F) \subset \sigma(F')$. In particular, σ permutes the vertex set V, and σ is completely determined by its action on V. We say that G acts *freely* on Δ if for every $\sigma \neq 1$ in G and every vertex $x \in V$, we have that $x \neq \sigma(x)$ and that $\{x, \sigma(x)\}$ is not an edge of Δ. Equivalently, for every $x \in V$ the open stars of the elements of the orbit Gx are pairwise disjoint.

We come to the first of our two main results.

Theorem 3.1. *Let Δ be a $(d-1)$-dimensional Cohen-Macaulay simplicial complex, and suppose that a group G of order 2 acts freely on Δ. Then*

$$h_i(\Delta) \geq \binom{d}{i}, \quad 0 \leq i \leq d.$$

In particular,

$$f_{d-1} = h_0 + \cdots + h_d \geq 2^d.$$

Proof. Let $G = \{1, \sigma\}$, and let K be a field of characteristic $\neq 2$. If W is any K-vector space on which G acts, then define

$$W^+ = \{w \in W\colon \sigma(w) = w\}$$
$$W^- = \{w \in W\colon \sigma(w) = -w\}.$$

It is clear that $W = W^+ \oplus W^-$. (In terms of representation theory, W^+ and W^- are the isotypic components corresponding to the trivial and non-trivial irreducible representations of G, respectively. However, G is such a "simple" group that there is no need here to invoke explicitly the representation theory of finite groups.)

The action of G on Δ induces an action on the face ring $K[\Delta]$. Let x^α be a monomial in $K[\Delta]$ of degree $i > 0$. Since G acts freely on Δ, the K-span of the G-orbit of x^α has a basis consisting of $x^\alpha + \sigma(x^\alpha) \in K[\Delta]_i^+$ and $x^\alpha - \sigma(x^\alpha) \in K[\Delta]_i^-$. Hence

$$\dim K[\Delta]_i^+ = \dim K[\Delta]_i^- = \frac{1}{2}\dim K[\Delta]_i, \quad i \geq 1. \tag{7}$$

Assume now K is infinite and (as above) $\operatorname{char} K \neq 2$. We claim there exists an h.s.o.p. $\theta_1, \ldots, \theta_d \in K[\Delta]_1^-$ of $K[\Delta]$. To see this, choose $V' \subset V$ to consist of exactly one element from each G-orbit of V. Since K is infinite, there exist functions $f_1, \ldots,$ $f_d\colon V' \to K$ such that the restrictions of $f_1, \ldots, f_d$ to any d-element subset of V' are linearly independent. Extend $f_1, \ldots, f_d$ to all of V by defining $f_i(\sigma(x)) = -f_i(x)$ for

$x \in V'$. Define

$$\theta_i = \sum_{x \in V} f_i(x)x.$$

Clearly $\theta_i \in K[\Delta]_1^-$. Since x and $\sigma(x)$ are not both vertices of any face of Δ, it follows from Lemma 2.2 that $\theta_1, \ldots, \theta_d$ form an h.s.o.p., as desired.

Let $A = K[\Delta]/(\theta_1, \ldots, \theta_d)$, with $\theta_1, \ldots, \theta_d$ as above. Since $\sigma(\theta_i) = -\theta_i$, it follows that G acts on the ideal $(\theta_1, \ldots, \theta_d)$, and therefore on the graded algebra $A = A_0 \oplus \cdots \oplus A_d$. We want to compute $\dim A_i^+$ and $\dim A_i^-$. Let q and t be indeterminates, and set $t^2 = 1$. If $V = V_0 \oplus V_1 \oplus \cdots$ is any graded vector space, with $\dim V_i < \infty$, on which G acts, set

$$F(V, q) = \sum_{i \geq 0} (\dim V_i) q^i$$

and

$$F(V, q, t) = \sum_{i \geq 0} [(\dim V_i^+) + (\dim V_i^-)t] q^i.$$

Since (see [10, eqn. (5)])

$$F(K[\Delta], q) = (1 - q)^{-d} \sum_{i=0}^{d} h_i(\Delta) q^i,$$

it follows from (7) that (writing $h_i = h_i(\Delta)$)

$$F(K[\Delta], q, t) = 1 + \frac{1}{2}\left[\frac{\sum_{i=0}^{d} h_i q^i}{(1 - q)^d} - 1\right](1 + t). \tag{8}$$

(We have $F(K[\Delta], 0, t) = 1$ since G fixes the empty face $\varnothing$.)

Now since $\sigma(\theta_i) = -\theta_i$ and since the decomposition $K[\Delta] = K[\Delta]^+ \oplus K[\Delta]^-$ defines a G-grading of $K[\Delta]$ (i.e., $K[\Delta]^+ \cdot K[\Delta]^+ \subseteq K[\Delta]^+$, etc.), the ideal (θ_1) satisfies (since θ_1 is a non-zero-divisor of degree one)

$$F((\theta_1), q, t) = qt \cdot F(K[\Delta], q, t).$$

Hence

$$F(K[\Delta]/(\theta_1), q, t) = (1 - qt) F(K[\Delta], q, t).$$

Each time we divide out by another θ_i we pick up another factor of $1 - qt$, so

$$\begin{aligned} F(A, q, t) &= (1 - qt)^d F(K[\Delta], q, t) \\ &= \frac{(1 - qt)^d}{2}\left[1 - t + (1 + t)(1 - q)^{-d} \sum_{i=0}^{d} h_i q^i\right], \end{aligned} \tag{9}$$

by (8). Since $(1 + t) = t(1 + t)$, it follows that $g(t)(1 + t) = g(1)(1 + t)$ for any function g. Similarly $g(t)(1 - t) = g(-1)(1 - t)$. In particular,

$$(1 - qt)^d(1 - t) = (1 + q)^d(1 - t),$$

$$(1 - qt)^d(1 + t) = (1 - q)^d(1 + t).$$

Hence

$$F(A,q,t) = \frac{1}{2}\left[(1+q)^d(1-t) + (1+t)\sum_{i=0}^{d} h_i q^i\right]$$

It follows that

$$\begin{aligned} h_i^+ &= \dim A_i^+ = \frac{1}{2}\left(h_i + \binom{d}{i}\right), \\ h_i^- &= \dim A_i^- = \frac{1}{2}\left(h_i - \binom{d}{i}\right). \end{aligned} \tag{10}$$

Since $\dim A_i^- \geq 0$, the proof follows. □

Note. Rather than choosing each $\theta_i \in K[\Delta]_1^-$, we could choose ℓ of the θ_i's to belong to $K[\Delta]_1^+$ and $d-\ell$ to belong to $K[\Delta]_1^-$. We could then compute h_i^+ and h_i^- for this choice of θ_i's and hope that some additional information about the h_i's will arise. If we choose $\ell = d$ then we obtain $h_i \geq \binom{d}{i}$ as in the case of $\ell = 0$ (except that for $\ell = d$ we need to use that both $h_i^+ \geq 0$ and $h_i^- \geq 0$, rather than just $h_i^- \geq 0$, as in the proof of Theorem 2.3 below.) If, however, we choose $0 < \ell < d$, then it can be checked that we obtain inequalities weaker than $h_i \geq \binom{d}{i}$. Thus the choice $\ell = 0$ leads to the strongest possible result, and we will see in the next section why it is more "natural" than the choice $\ell = d$.

Before turning to the case of centrally-symmetric polytopes, let us briefly consider extending Theorem 3.1 to other groups G. We only deal with the case where G is abelian; for nonabelian G we need to consider delicate properties of irreducible representations of G and their tensor products.

Theorem 3.2. *Let Δ be a $(d-1)$-dimensional Cohen-Macaulay simplicial complex, and suppose that an abelian group G of order $g > 1$ acts freely on Δ. Then*

$$h_i(\Delta) \geq \binom{d}{i}, \quad i \text{ even},$$

$$h_i(\Delta) \geq (g-1)\binom{d}{i}, \quad i \text{ odd}.$$

In particular,

$$f_{d-1} = h_0 + \cdots + h_d \geq g \cdot 2^{d-1}.$$

Proof. Let $\hat{G}$ denote the group of all homomorphisms $\chi: G \to \mathbb{C}^* = \mathbb{C} - \{0\}$ (so $\hat{G}$ and G are isomorphic as abstract groups). Then $\mathbb{C}[\Delta]$ has an $\mathbb{N} \times \hat{G}$ grading,

$$\mathbb{C}[\Delta] = \coprod_{i \geq 0} \coprod_{\chi \in \hat{G}} \mathbb{C}[\Delta]_i^\chi,$$

given by

$$\mathbb{C}[\Delta]_i^\chi = \{f \in \mathbb{C}[\Delta]_i : w \cdot f = \chi(w) f \text{ for all } w \in G\}.$$

For any $(\mathbb{N} \times \hat{G})$-graded vector space $V = \coprod V_i^\chi$ with each $\dim V_i^\chi < \infty$, define the Hilbert series

$$F(V, G; q) = \sum_{i \geq 0} \sum_{\chi \in \hat{G}} (\dim V_i^\chi) \chi q^i,$$

an element of the ring $(\mathbb{Z}\hat{G}) \otimes \mathbb{Z}[[q]]$, where $\mathbb{Z}\hat{G}$ is the ring of virtual characters of $\hat{G}$ (formal $\mathbb{Z}$-linear combinations of elements of $\hat{G}$).

Since G acts freely on Δ, the $\mathbb{C}$-span of the orbit of any monomial $x^\alpha \in \mathbb{C}[\Delta]$ of positive degree affords the regular representation of G (since the only transitive faithful permutation representation of a finite abelian group is the regular representation). It follows that

$$F(\mathbb{C}[\Delta], G; q) = 1 + \frac{1}{g}\left[\frac{\sum_0^d h_i q^i}{(1-q)^d} - 1\right]\left(\sum_{\chi \in \hat{G}} \chi\right), \tag{11}$$

where $g = |G|$.

Let ι denote the trivial character of G. The hypothesis that G acts freely on Δ implies, as in the proof of Theorem 3.1, that there is an h.s.o.p. $\theta_1, \ldots, \theta_d \in \mathbb{C}[\Delta]_1^\iota$. Let $A = \mathbb{C}[\Delta]/(\theta_1, \ldots, \theta_d)$. We obtain as in (9) that

$$F(A, G; q) = (1 - q\iota)^d F(\mathbb{C}[\Delta], G; q). \tag{12}$$

Substituting (11) into (12) and using the "symmetrizing" property of $\sum \chi$, we get

$$F(A, G; q) = (1 - q\iota)^d + \frac{1}{g}\left[\sum_0^d h_i q^i - (1-q)^d\right](\sum \chi). \tag{13}$$

Suppose i is even. Let $\iota \neq \chi \in \hat{G}$. The coefficient of $q^i \chi$ in the right-hand side of (13) is $\left(h_i - \binom{d}{i}\right)/g$, so $h_i \geq \binom{d}{i}$.

Suppose i is odd. The coefficient of $q^i \iota$ in the right-hand side of (13) is

$$-\binom{d}{i} + \frac{1}{g}\left(h_i + \binom{d}{i}\right) = \frac{1}{g}\left(h_i - (g-1)\binom{d}{i}\right).$$

Hence $h_i \geq (g-1)\binom{d}{i}$, and the proof is complete. □

As was the case for Theorem 3.1, one can check that choosing $\theta_i \in \mathbb{C}[\Delta]_1^{\chi_i}$ for arbitrary $\chi_1, \ldots, \chi_d \in \hat{G}$ does not lead to a stronger result.

The inequality $h_i \geq \binom{d}{i}$ in Theorem 3.1 is best possible, since the boundary complex Δ of the d-dimensional cross-polytope admits a free $(\mathbb{Z}/2\mathbb{Z})$-action and satisfies $h_i = \binom{d}{i}$ for $0 \leq i \leq d$. However, Theorem 3.2 is not sharp for $g > 2$. For instance, it is impossible for *any* Cohen-Macaulay simplicial complex to satisfy $h_2 = \binom{d}{2}$ and $h_3 = (g-1)\binom{d}{3}$ whenever $(g-2)(d-2) > 3$. We also have the following simple congruence condition.

Proposition 3.3. *Let Δ be any finite $(d-1)$-dimensional simplicial complex, and suppose that an abelian group G of order g acts freely on Δ. (In fact, we need only to assume that if $1 \neq \sigma \in G$ and $\varnothing \neq F \in \Delta$, then $\sigma(F) \neq F$.) Then*

$$h_i(\Delta) \equiv (-1)^i \binom{d}{i} (\operatorname{mod} g).$$

In particular, the reduced Euler characteristic

$$\tilde{\chi}(\Delta) := -f_{-1} + f_0 - \cdots + (-1)^{d-1} f_{d-1} = (-1)^{d-1} h_d(\Delta)$$

satisfies

$$\tilde{\chi}(\Delta) \equiv -1 \,(\operatorname{mod} g).$$

Proof. Since G is abelian, the orbit of any nonempty face F of Δ contains exactly g elements. Hence $f_i \equiv 0 \,(\operatorname{mod} g)$, $i > 0$. The proof follows from (1) (using that $f_{-1} = 1$). □

Consider once again the situation of Theorem 3.1. It is natural to ask what further information about the h-vector $(h_0, h_1, \ldots, h_d)$ of Δ can be obtained from the decomposition

$$A = A^+ \oplus A^- = \left(\coprod_0^d A_i^+ \right) \oplus \left(\coprod_0^d A_i^- \right).$$

In the case of arbitrary Cohen-Macaulay Δ (i.e., no group action), the ring structure on A leads to a complete characterization of the h-vector of Cohen-Macaulay simplicial complexes (see, e.g., [9, Thm. 2.2, p. 65]). In the present situation we don't see how to obtain such strong results. It is easy to see that A^+ is a graded algebra generated by elements of degrees one and two (whose number can be specified) and that A^- is a graded A^+-module with generators (as an A^+-module) in degree one. This observation leads to some information about the h-vector, but it seems far from definitive. For instance, we have the following result.

Proposition 3.4. *Let Δ be a finite $(d-1)$-dimensional Cohen-Macaulay simplicial complex admitting a free $(\mathbb{Z}/2\mathbb{Z})$-action. Suppose $h_i = \binom{d}{i}$ for some $i \geq 1$. Let $j \geq i$. If either j is even or $j - i$ is even, then $h_j = \binom{d}{j}$.*

Proof. Since $h_i = \binom{d}{i}$ we have $A_i^- = 0$ by (10). Suppose $j \geq i$ and $A_j^- \neq 0$. Since A^- is generated by A_1^- as an A^+-module and A^+ is generated by A_1^+ and A_2^+ as a K-algebra, there exist elements $t \in A_1^-$ and $u_1, \ldots, u_r \in A_1^+ \cup A_2^+$ such that

$$0 \neq u_1 \cdots u_r t \in A_j^-.$$

If j is even then some $u_s \in A_1^+$. Then some subproduct v of $u_1 \cdots u_r$ will have degree $i-1$. Hence $0 \neq vt \in A_i^-$, contradicting $h_i^- = 0$. Similarly if $j - i$ is even then again some subproduct of $u_1 \cdots u_r$ will have degree $i-1$, and we reach the same contradiction. □

Proposition 3.4 suggests the following conjecture.

Conjecture 3.5. Let Δ be a finite $(d-1)$-dimensional Cohen-Macaulay simplicial complex admitting a free $(\mathbb{Z}/2\mathbb{Z})$-action. Suppose $h_i = \binom{d}{i}$ for some $i \geq 1$. Then $h_j = \binom{d}{j}$ for all $j \geq i$. □

4. Centrally-Symmetric Simplicial Polytopes

Let $\mathscr{P}$ be a centrally-symmetric (about the origin) simplicial d-polytope. The boundary complex Δ of $\mathscr{P}$ is a $(d-1)$-dimensional Cohen-Macaulay (since the geometric realization $|\Delta|$ is a $(d-1)$-sphere) simplicial complex with a free $(\mathbb{Z}/2\mathbb{Z})$-action induced by the map $v \to -v$ on $\mathscr{P}$. Hence by Theorem 3.1, $h_i(\mathscr{P}) := h_i(\Delta) \geq \binom{d}{i}$. But in this situation we can say considerably more.

Theorem 4.1. *If $\mathscr{P}$ is a centrally-symmetric simplicial d-polytope, then*

$$h_i(\mathscr{P}) - h_{i-1}(\mathscr{P}) \geq \binom{d}{i} - \binom{d}{i-1}, \quad 1 \leq i \leq [d/2]. \tag{14}$$

Proof. We may assume that $\mathscr{P} \subset \mathbb{R}^d$. Moreover, since any sufficiently small perturbations of the vertices of a simplicial polytope do not affect its combinatorial type, we may assume that $\mathscr{P}$ is *rational*, i.e., the vertices of $\mathscr{P}$ have rational coordinates (with $\mathscr{P}$ still centrally-symmetric about the origin). We can now invoke the theory of toric varieties, as discussed, e.g., in [10]. Let $X = X(\mathscr{P})$ be the toric variety corresponding to $\mathscr{P}$ with cohomology ring (over $\mathbb{R}$, say)

$$H^*(X) = H^*(X; \mathbb{R}) = H^0(X) \oplus H^2(X) \oplus \cdots \oplus H^{2d}(X).$$

Let Δ denote the boundary complex of $\mathscr{P}$. By a result of Danilov [3, Thm. 10.8],

$$\begin{aligned} H^*(X) &\cong \mathbb{R}[\Delta]/(\theta_1, \ldots, \theta_d) \\ &= A = A_0 \oplus \cdots \oplus A_d, \end{aligned} \tag{15}$$

for a certain h.s.o.p. $\theta_1, \ldots, \theta_d$ of $\mathbb{R}[\Delta]$ of degree one, the grading being such that $A_i \cong H^{2i}(X)$. The h.s.o.p. $\theta_1, \ldots, \theta_d$ is described as follows. Let $\varphi_1, \ldots, \varphi_d$ be any set of linearly independent linear functionals $\varphi_i \colon \mathbb{R}^d \to \mathbb{R}$. Then

$$\theta_i = \sum_{x \in V} \varphi_i(x) x,$$

where V is the set of vertices of $\mathscr{P}$ (or Δ). Since $\mathscr{P}$ is centrally-symmetric, any vertex $x \in V$ has an antipodal vertex $\bar{x} = -x$, and $\varphi_i(\bar{x}) = -\varphi_i(x)$ since φ_i is linear.

The group $G = \{1, \sigma\}$ of order two acts on $\mathscr{P}$ by $\sigma(v) = -v$ for $v \in \mathscr{P}$. This induces an action on $X(\mathscr{P})$ (as is easily seen from the definition of $X(\mathscr{P})$) and on $H^*(X)$ by the rule $\sigma(x) = \bar{x}$ for $x \in V$ (identifying $H^*(X)$ with A as in (15)). It follows that $\sigma(\theta_i) = -\theta_i$, so by (10) the vector space $A_i = H^{2i}(X)$ decomposes under G into

$$H^{2i}(X) = H^{2i}(X)^+ \oplus H^{2i}(X)^-,$$

where

$$\dim H^{2i}(X)^- = \frac{1}{2}\left(h_i - \binom{d}{i}\right). \tag{16}$$

We now use the fact [10, p. 219] that X satisfies the hard Lefschetz theorem. (J. Steenbrink has informed me that his original proof [11] of this fact is invalid, but that a correct proof was subsequently given by M. Saito [6] (see in particular [5]) based on the theory of perverse sheaves.) In particular, if $\omega \in H^2(X)$ is the class of a hyperplane section, then the map $\omega\colon H^{2(i-1)}(X) \to H^{2i}(X)$, given by multiplication by ω, is injective for $1 \leq i \leq [d/2]$.

We claim that the action of σ on $H^*(X)$ commutes with multiplication by ω, i.e., $\sigma(\omega y) = \omega(\sigma \cdot y)$ for all $y \in H^*(X)$. I am grateful to S. Kleiman for providing the following argument. It is clear from the definition [3, §6.9] of the embedding of X into projective space $\mathbb{P}$ that the action of σ on X extends to a linear transformation of $\mathbb{P}$. Hence if $H \subset \mathbb{P}$ is a hyperplane then so is $\sigma \cdot H$. Since any two hyperplane sections of X (with respect to a fixed embedding $X \subset \mathbb{P}$) represent the same cohomology class, we have $\sigma \cdot \omega = \omega$. Since σ acts on $H^*(X)$ by pullback, σ induces a ring homomorphism on $H^*(X)$. Hence $\sigma \cdot (\omega y) = (\sigma \cdot \omega)(\sigma \cdot y) = \omega(\sigma \cdot y)$, as desired.

It follows that the subspaces $H^*(X)^+$ and $H^*(X)^-$ are ω-invariant. Thus in particular ω sends $H^{2(i-1)}(X)^-$ to $H^{2i}(X)^-$, $1 \leq i \leq [d/2]$, and is of course still injective (being the restriction of the injective function $\omega\colon H^{2(i-1)}(X) \to H^{2i}(X)$). Therefore $\dim H^{2(i-1)}(X)^- \leq \dim H^{2i}(X)^-$, $1 \leq i \leq [d/2]$, so by (16),

$$\frac{1}{2}\left(h_{i-1} - \binom{d}{i-1}\right) \leq \frac{1}{2}\left(h_i - \binom{d}{i}\right), \quad 1 \leq i \leq [d/2].$$

This is equivalent to (14), completing the proof. □

Corollary 4.2. *Let $\mathscr{P}$ be as in the previous theorem, and suppose $\mathscr{P}$ has f_i i-faces, $0 \leq i \leq d-1$. Let $f_0 = 2n$. Then the f_i's satisfy* (2) *and* (3).

Proof. We have $h_1 = f_0 - d = 2n - d$. Hence by (14) and the Dehn-Sommerville equations,

$$h_i \geq 2(n-d) + \binom{d}{i}, \quad 1 \leq i \leq d-1.$$

Thus

$$f_i = \sum_{j=0}^{i+1} \binom{d-j}{i+1-j} h_j$$
$$\geq \begin{cases} \binom{d}{i+1} + \sum_{j=1}^{i+1} \binom{d-j}{i+1-j}\left(2(n-d) + \binom{d}{j}\right), & 0 \leq i \leq d-2 \\ 1 + \sum_{j=1}^{d-1}\left(2(n-d) + \binom{d}{j}\right) + 1, & i = d-1. \end{cases}$$

This is equivalent to (2) and (3) by simple binomial coefficient identities. □

As was the situation for Theorem 3.1, the ring $A = A^+ \oplus A^-$ together with the element $\omega \in A_1^+$ can be used to obtain some additional information about h-vectors of centrally-symmetric simplicial polytopes, but nothing nearly as definitive as McMullen's g-conjecture (see [10, p. 217]) for arbitrary simplicial polytopes. In particular, we don't know an analogue of the Upper Bound Conjecture for polytopes (or spheres). In other words, given a centrally-symmetric simplicial d-polytope $\mathscr{P}$ with $f_0 = 2n$ vertices, what is the largest possible value of f_i? Even a plausible conjecture is not known. The most obvious conjecture is that f_i is maximized by choosing $f_j = 2^{j+1}\binom{n}{j+1}$, $0 < j \le [d/2] - 1$ (equivalently, if $0 \le j \le [d/2] - 1$ then every set of $j + 1$ vertices, no two antipodal, of $\mathscr{P}$ forms the vertices of a j-face). However, a result [4, Thm. 23] of McMullen and Shephard shows this conjecture to be false for $n > d + 2$. For additional results along these lines, see [7].

Let us also remark that Conjecture 3.5 is valid for centrally-symmetric simplicial polytopes. In fact, we have the following result.

Proposition 4.3. *Let $\mathscr{P}$ be a centrally-symmetric simplicial d-polytope. Suppose that for some $1 \le i \le d - 1$ we have $h_i(\mathscr{P}) = \binom{d}{i}$. Then $h_j(\mathscr{P}) = \binom{d}{j}$ for all j, and $\mathscr{P}$ is affinely equivalent to a cross-polytope.*

Proof. Let $k_i = h_i(\mathscr{P}) - \binom{d}{i}$. By Theorem 4.1 the sequence $(k_0, k_1, \ldots, k_d)$ is non-negative (since $h_0 = 1$, or by Theorem 3.2) and unimodal. Moreover, $k_i = k_{d-i}$ by the Dehn-Sommerville equations. Thus if $k_i = 0$ for some $1 \le i \le d - 1$, then $k_1 = 0$. Hence $f_0(\mathscr{P}) = 2d$. Let F by any facet of $\mathscr{P}$ and $\bar{F}$ the antipodal facet. Since $F \cup \bar{F}$ contains $2d$ vertices, it follows that $\mathscr{P}$ is the convex hull of $F \cup \bar{F}$ and is therefore affinely equivalent to a cross-polytope. □

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