ADVANCES IN MATHEMATICS 77, 76–115 (1989)

Some Combinatorial Properties of Jack Symmetric Functions

RICHARD P. STANLEY*

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

1. INTRODUCTION

We use throughout this paper notation and terminology related to partitions and symmetric functions from [M1]. Let $\lambda = (\lambda_1, \lambda_2, ...)$ be a partition and $\lambda' = (\lambda'_1, \lambda'_2, ...)$ the conjugate partition to λ. The number λ'_1 of parts of λ is denoted $l(\lambda)$, called the *length* of λ. Set

$$b(\lambda) = \sum (i-1)\lambda_i = \sum \binom{\lambda'_i}{2}. \tag{1}$$

(Macdonald $[\mathrm{M}_1]$ uses $n(\lambda)$ for our $b(\lambda)$, but using $n(\lambda)$ here would lead to confusion with other uses of the letter n.)

We will identify a partition λ with its *diagram*

$$\lambda = \{(i, j): 1 \leqslant i \leqslant l(\lambda),\ 1 \leqslant j \leqslant \lambda_i\}.$$

If $\lambda_1 + \lambda_2 + \cdots = n$, then write $\lambda \vdash n$ or $|\lambda| = n$. If μ is another partition, then write $\mu \subseteq \lambda$ if $\mu_i \leqslant \lambda_i$ for all i (i.e., if the diagram of λ contains the diagram of μ). If $|\mu| = |\lambda|$ then write $\mu \overset{\mathrm{R}}{\leqslant} \lambda$ (reverse lexicographic order) if either $\mu = \lambda$ or the first nonvanishing difference $\lambda_i - \mu_i$ is positive. For instance (writing $\lambda_1 \lambda_2 \cdots \lambda_l$ for $(\lambda_1, \lambda_2, ..., \lambda_l)$), $5 \overset{\mathrm{R}}{>} 41 \overset{\mathrm{R}}{>} 32 \overset{\mathrm{R}}{>} 311 \overset{\mathrm{R}}{>} 221 \overset{\mathrm{R}}{>} 2111 \overset{\mathrm{R}}{>} 11111$. Finally write $\mu \leqslant \lambda$ if $|\mu| = |\lambda|$ and $\mu_1 + \mu_2 + \cdots + \mu_i \leqslant \lambda_1 + \lambda_2 + \cdots \lambda_i$ for all i. Macdonald $[\mathrm{M}_1$, p. 6] call the partial ordering $\leqslant$ the "natural ordering," but we will call it the *dominance* ordering

Let $x = (x_1, x_2, ...)$ be an infinite set of indeterminates. As in $[\mathrm{M}_1]$, we

* Partially supported by ONR Grant N00014-85-K-0546 and NSF Grant 8401376-DMS. Much of the research was performed when the author was a Sherman Fairchild Distinguished Scholar at Caltech.

0001-8708/89 $7.50

use the following notation for certain symmetric functions $a_\lambda = a_\lambda(x)$ indexed by partitions λ:

m_λ, monomial symmetric function;

e_λ, elementary symmetric function;

h_λ, complete homogeneous symmetric function;

p_λ, power-sum symmetric function;

s_λ, Schur function.

If λ has $m_i = m_i(\lambda)$ parts equal to i, then write

$$z_\lambda = (1^{m_1} 2^{m_2} \cdots) \, m_1! \, m_2! \cdots. \tag{2}$$

Let α be a parameter (indeterminate), and let $\mathbb{Q}(\alpha)$ denote the field of all rational functions of α with rational coefficients. Define a (bilinear) scalar product $\langle \, , \, \rangle$ on the vector space $\Lambda \otimes \mathbb{Q}(\alpha)$ of all symmetric functions of bounded degree over the field $\mathbb{Q}(\alpha)$ by the condition

$$\langle p_\lambda, p_\mu \rangle = \delta_{\lambda\mu} z_\lambda \alpha^{l(\lambda)}, \tag{3}$$

where $\delta_{\lambda\mu} = 0$ if $\lambda \neq \mu$ and $\delta_{\lambda\lambda} = 1$. When $\alpha = 1$, (3) coincides with the usual scalar product [M_1, p. 35] on symmetric functions. We also let $\Lambda^k \otimes \mathbb{Q}(\alpha)$ denote the vector space of all homogeneous symmetric functions of degree k over the field $\mathbb{Q}(\alpha)$. Thus

$$\Lambda \otimes \mathbb{Q}(\alpha) = \coprod_{k \geqslant 0} \Lambda^k \otimes \mathbb{Q}(\alpha) \qquad \text{(vector space direct sum).}$$

The following fundamental result is due to Macdonald [M_3, Chap. VI, (4.5)] and will not be proved here.

1.1. THEOREM. *There are unique symmetric functions $J_\lambda = J_\lambda(x; \alpha) \in \Lambda \otimes \mathbb{Q}(\alpha)$, where λ ranges over all partitions of all nonnegative integers, satisfying the following three conditions:*

(P1) (*orthogonality*) *$\langle J_\lambda, J_\mu \rangle = 0$ if $\lambda \neq \mu$, where the scalar product is given by* (3).

(P2) (*triangularity*) *Write*

$$J_\lambda = \sum_\mu v_{\lambda\mu}(\alpha) m_\mu. \tag{4}$$

Then $v_{\lambda\mu}(\alpha) = 0$ unless $\mu \leqslant \lambda$.

(P3) (*normalization*) *If $|\lambda| = n$, then the coefficient $v_{\lambda, 1^n}$ of $x_1 x_2 \cdots x_n$ in J_λ is equal to $n!$.*

Though we do not prove Theorem 1.1 here, let us point out where the difficulty lies. (See also the note after the proof of Theorem 3.1.) If the dominance order were a total order, then Theorem 1.1 would follow easily from Gram–Schmidt orthogonalization. Since dominance order is only a partial order, we must take some compatible total order (such as $\overset{R}{\leqslant}$) before applying Gram–Schmidt. Then Theorem 1.1 amounts essentially to saying that whatever total order we take compatible with $\leqslant$, we wind up with the same basis.

We call the symmetric functions J_λ *Jack symmetric functions* (for a reason explained after the proof of Proposition 1.2). If we set all but finitely many variables equal to 0 (say $x_{n+1} = x_{n+2} = \cdots = 0$) in J_λ, then we obtain a *polynomial* $J_\lambda(x_1, ..., x_n; \alpha)$ (with coefficients in $\mathbb{Q}(\alpha)$).

Two specializations of Jack symmetric functions are immediate from the definition. Given $x = (i, j) \in \lambda$, define $h(x) = \lambda_i + \lambda_j' - i - j + 1$, the *hook-length* of λ at x [M$_1$, p. 9]. Set

$$H_\lambda = \prod_{x \in \lambda} h(x),$$

the product of all hook-lengths of λ.

1.2. Proposition. (a) $J_\lambda(x; 1) = H_\lambda s_\lambda(x)$.

(b) $J_\lambda(x; 2) = Z_\lambda(x)$, *the zonal symmetric function indexed by* λ (*normalized as in* [Jam$_2$], *so the coefficient of* $x_1 \cdots x_n$ *is* $n!$).

Proof. It is well-known that Schur funtions s_λ satisfy (P1) (with $\alpha = 1$) and (P2) [M$_1$, Chap I, (4.8) and (6.5)]. Hence they must agree with $J_\lambda(x; 1)$ up to a scalar multiple. But the coefficient of $x_1 \cdots x_n$ in s_λ is $n!/H_\lambda$ [M$_1$, Ex. 2, p. 43], so (a) follows.

Similarly it is well-known that zonal polynomials Z_λ satisfy (P1) (with $\alpha = 2$), (P2), and (P3), so (b) follows. ∎

Two additional specializations of $J_\lambda(x; \alpha)$ are given in Proposition 7.6.

Jack symmetric functions were first defined by H. Jack [Jac] (using the total order $\overset{R}{\leqslant}$ in (P2) instead of $\leqslant$). He established Proposition 1.2(a) and conjectured Proposition 1.2(b), but did not pursue their properties much further. Foulkes [F, pp. 90–91] raised the problem of obtaining a combinatorial interpretation of Jack symmetric functions. I. G. Macdonald, in unpublished work, established some fundamental properties of Jack symmetric functions and made several conjectures concerning them. We have already appealed to one of Macdonald's results (Theorem 1.1), and in Section 3 we will consider further results of Macdonald which will play an essential role in our work. Subsequently Macdonald [M$_3$] has extended Jack symmetric functions to an even more general class of symmetric functions $P_\lambda(x; q, t)$ to which almost all the results and techniques concerning Jack symmetric functions can be carried over. (Macdonald [M_3] uses a normalization of Jack symmetric functions different from ours.) Before turning to Macdonald's results in Section 3, we will first establish some elementary properties of Jack symmetric functions in Section 2.

2. Elementary Properties

Properties (P2) and (P3) of Jack symmetric functions immediately yield

$$J_{1^n} = n!\, m_{1^n} = n!\, e_n. \tag{5}$$

We also seek an expression for J_n, which will be deduced from the following result. We henceforth use the notation

$$j_\lambda = \langle J_\lambda, J_\lambda \rangle.$$

2.1. Proposition. *Let $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$ be two sets of indeterminates. Then*

$$\sum_\lambda J_\lambda(x)\, J_\lambda(y)\, j_\lambda^{-1} = \prod_{i,j} (1 - x_i y_j)^{-1/\alpha}, \tag{6}$$

summed over all partitions λ.

Proof. The proof parallels the $\alpha = 1$ case in [M_1, (4.3), p. 33]. First one checks, just as in [M_1, (4.1), p. 33], that

$$\sum_\lambda z_\lambda^{-1} \alpha^{-l(\lambda)} p_\lambda(x)\, p_\lambda(y) = \prod_{i,j} (1 - x_i y_j)^{-1/\alpha}.$$

It then follows, just as in [M_1, (4.6), p. 34], that if (u_λ) and (v_λ) are bases for the space $\Lambda^n \otimes \mathbb{Q}(\alpha)$ of homogeneous symmetric functions of degree n over $\mathbb{Q}(\alpha)$, then the following conditions are equivalent:

(a) $\langle u_\lambda, v_\mu \rangle = \delta_{\lambda\mu}$ for all λ, μ;

(b) $\sum_\lambda u_\lambda(x)\, v_\lambda(y) = \prod_{i,j} (1 - x_i y_j)^{-1/\alpha}$.

Then (6) follows from (P1) after setting $u_\lambda = J_\lambda$ and $v_\lambda = J_\lambda / j_\lambda$. ∎

2.2. Proposition. *For any $n \geq 0$, the Jack symmetric function J_n (short for $J_{(n)}$, where (n) denotes the partition $(n, 0, 0, \ldots)$) has the following expansions:*

(a) $J_n = \sum_{\lambda \vdash n} \binom{n}{\lambda_1, \lambda_2, \ldots}(\prod_{i \geqslant 1} P_{\lambda_i}(\alpha)) m_\lambda$,

where

$$\binom{n}{\lambda_1, \lambda_2, \ldots} = \frac{n!}{\lambda_1! \, \lambda_2! \cdots} \qquad \textit{(multinomial coefficient)},$$

$$P_k(\alpha) = 1 \cdot (1+\alpha)(1+2\alpha) \cdots (1+(k-1)\alpha).$$

(b) $J_n = \sum_{\lambda \vdash n} \alpha^{n-l(\lambda)} n! \, z_\lambda^{-1} p_\lambda$.

(c) $J_n = \sum_{\lambda \vdash n} (-1)^{n-l(\lambda)} \alpha^{n-l(\lambda)} n! \, P_{l(\lambda)}(\alpha)(m_1(\lambda)! \, m_2(\lambda)! \cdots)^{-1} e_\lambda$,

where λ has $m_i(\lambda)$ parts equal to i.

(d) $J_n = \sum_{\lambda \vdash n} \alpha^{n-l(\lambda)} n! \, P_{l(\lambda)}(-\alpha)(m_1(\lambda)! \, m_2(\lambda)! \cdots)^{-1} h_\lambda$.

(e) $J_n = \sum_{\lambda \vdash n} f_\lambda [\prod_{(i,j) \in \lambda} (1+(j-i)\alpha)] s_\lambda$,

where $f_\lambda = n!/H_\lambda$, the number of standard tableaux [M_1, p. 5] *of shape λ.*

Proof. (a) Set $y_1 = t$, $y_2 = y_3 = \cdots = 0$ in (6). By property (P2) of Jack symmetric functions, we have for $\lambda \vdash n$,

$$J_\lambda(t, 0, \ldots, 0) = \begin{cases} v_n(\alpha) t^n, & \lambda = (n), \\ 0, & \lambda \neq (n), \end{cases}$$

where $v_n(\alpha) \in \mathbb{Q}(\alpha)$. Hence writing

$$w_n(\alpha) = v_n(\alpha) j_n^{-1}, \tag{7}$$

we get from (6) that

$$\begin{aligned} \sum_{n \geqslant 0} J_n(x) \, w_n(\alpha) t^n &= \prod_i (1 - x_i t)^{-1/\alpha} \\ &= \prod_i \sum_{j \geqslant 0} \binom{-1/\alpha}{j} (-1)^j x_i^j t^j \\ &= \sum_n \frac{t^n}{\alpha^n n!} \sum_{\lambda \vdash n} \binom{n}{\lambda_1, \lambda_2, \ldots} \left(\prod_{i \geqslant 1} P_{\lambda_i}(\alpha) \right) m_\lambda. \end{aligned} \tag{8}$$

This determines J_n up to normalization, and property (P3) then yields (a).

Note. We now get from (8) that $w_n(\alpha) = 1/\alpha^n n!$. By definition, $v_n(\alpha)$ is the coefficient of x_1^n (or m_n) in J_n, which by (a) is given by

$$v_n(\alpha) = P_n(\alpha) = (1+\alpha)(1+2\alpha) \cdots (1+(n-1)\alpha).$$

Thus by (7), we get

$$j_n = \langle J_n, J_n \rangle = \alpha^n n!\,(1+\alpha)(1+2\alpha)\cdots(1+(n-1)\alpha).$$

In Theorem 5.8 we will determine j_λ for any λ.

(b) We have shown above that

$$\sum_{n \geqslant 0} J_n(x)\, t^n/\alpha^n n! = \prod_i (1 - x_i t)^{-1/\alpha}. \tag{9}$$

But

$$\begin{aligned}
\prod_i (1 - x_i t)^{-1/\alpha} &= \exp \log \prod_i (1 - x_i t)^{-1/\alpha} \\
&= \exp \sum_i \sum_{j \geqslant 1} \frac{1}{\alpha} \frac{x_i^j t^j}{j} \\
&= \exp \frac{1}{\alpha} \sum_{j \geqslant 1} \frac{p_j(x)\, t^j}{j} \\
&= \prod_{j \geqslant 1} \exp \frac{p_j(x)\, t^j}{\alpha j} \\
&= \prod_{j \geqslant 1} \sum_{m_j \geqslant 0} \frac{(p_j(x)\, t^j)^{m_j}}{\alpha^{m_j} j^{m_j} m_j!} \\
&= \sum_\lambda \alpha^{-l(\lambda)} z_\lambda^{-1} p_\lambda(x)\, t^{|\lambda|},
\end{aligned}$$

as in [M_1, p. 17]. Comparing with (9) yields (b).

(c) We have

$$\begin{aligned}
\prod_i (1 - x_i t)^{-1/\alpha} &= \left(\sum_{j \geqslant 0} (-1)^j e_j(x)\, t^j \right)^{-1/\alpha} \\
&= \sum_{r \geqslant 0} \binom{-1/\alpha}{r} \left(\sum_{j \geqslant 1} (-1)^j e_j(x)\, t^j \right)^r \\
&= \sum_\lambda (-1)^{|\lambda| - l(\lambda)} \alpha^{-l(\lambda)} P_{l(\lambda)}(\alpha)(m_1(\lambda)!\, m_2(\lambda)! \cdots)^{-1} e_\lambda t^{|\lambda|}.
\end{aligned}$$

Comparing with (9) yields (c).

(d) Analogous to (c), using

$$\prod_i (1 - x_i t)^{-1/\alpha} = \left(\sum_{j \geqslant 0} h_j(x)\, t^j \right)^{1/\alpha}.$$

(e) A well-known identity in the theory of symmetric functions [L, Sect. 7.2; M_1, Ex.1, p. 36] asserts that

$$\prod_i (1-x_i t)^{-\beta} = \sum_\lambda \binom{\beta}{\lambda'} s_\lambda(x) t^{|\lambda|},$$

where (in the notation of [M_1, Ex. 4, pp. 28–29])

$$\binom{\beta}{\lambda'} = \prod_{(i,j)\in\lambda} \frac{\beta+j-i}{h(i,j)}.$$

Putting $\beta = 1/\alpha$ and comparing with (9) yields (e). ∎

The next result gives a further interesting consequence of Proposition 2.1.

2.3. Proposition. *We have*

$$J_1^n = (x_1 + x_2 + \cdots)^n = \alpha^n n! \sum_{\lambda \vdash n} J_\lambda j_\lambda^{-1}.$$

(*The value of* j_λ^{-1} *is given by Theorem* 5.8.)

Proof. Take the coefficient of $y_1 y_2 \cdots y_n$ on both sides of (6). On the left-hand side we obtain (by property (P3))

$$n! \sum_\lambda J_\lambda j_\lambda^{-1}.$$

On the right-hand side we get $\alpha^{-n} J_1^n$, and the proof follows. ∎

The previous proposition may be generalized as follows. Given a partition μ, define

$$\mathscr{J}_\mu = \mathscr{J}_\mu(x;\alpha) = J_{\mu_1} J_{\mu_2} \cdots.$$

Also set $\mu! = \mu_1!\, \mu_2! \cdots$, and

$$\mathscr{J}_\mu = \sum_\lambda q_{\mu\lambda}(\alpha) J_\lambda. \tag{10}$$

2.4. Proposition. *We have*

$$v_{\lambda\mu}(\alpha)\, \alpha^{|\mu|} \mu! = q_{\mu\lambda}(\alpha)\, j_\lambda,$$

where $v_{\lambda\mu}(\alpha)$ *is given by* (4).

Proof. From (9) we have

$$\prod_{i,j}(1-x_i y_j)^{-1/\alpha}=\prod_j\left[\sum_{n\geqslant 0}J_n(x)\,y_j^n/\alpha^n n!\right]$$
$$=\sum_\lambda \mathscr{J}_\lambda(x)\,m_\lambda(y)/\alpha^{|\lambda|}\lambda!,$$

as in [M_1, Chap. I, proof of (4.2)]. Hence by the proof of Proposition 2.1, we have

$$\langle \mathscr{J}_\lambda, m_\mu\rangle=\delta_{\lambda\mu}\alpha^{|\mu|}\mu!.$$

Thus from (4) we get

$$\langle J_\lambda, \mathscr{J}_\mu\rangle=v_{\lambda\mu}(\alpha)\,\alpha^{|\mu|}\mu!,$$

while from (P1) and (10),

$$\langle J_\lambda, \mathscr{J}_\mu\rangle=q_{\mu\lambda}(\alpha)\,j_\lambda,$$

and the proof follows. ∎

Proposition 2.4 reduces to Proposition 2.3 when $\mu=(1^n)$, since by (P3) we have $v_{\lambda,1^n}(\alpha)=n!$. The case when $\alpha=1$ is due to Kostka (see, e.g., [M_1, Chap. I, (6.7)(vii)]).

It is clear from properties (P1) and (P2) that the J_λ's with $\lambda\vdash n$ form a basis for the space $\Lambda^n\otimes\mathbb{Q}(\alpha)$ of homogeneous symmetric functions of degree n with coefficients in $\mathbb{Q}(\alpha)$. The next result considers the dependencies which result when we set all but r variables equal to 0.

2.5. Proposition. *Let $r\geqslant 0$. The Jack polynomials $J_\lambda(x_1, ..., x_r)$ vanish for $r<l(\lambda)$ and are linearly independent otherwise.*

Proof. Suppose $r<l(\lambda)$. Let $|\mu|=|\lambda|$ and $\mu\leqslant\lambda$. Then $m_\mu(x_1, ..., x_r)=0$, so by property (P2) we have $J_\lambda(x_1, ..., x_r)=0$. If on the other hand $\lambda^1, \lambda^2, ...$ are distinct partitions all of length $\leqslant r$, then the monomial symmetric functions $m_{\lambda^i}(x_1, ..., x_r)$ are linearly independent. Hence by (P1) and (P2) the $J_{\lambda^i}(x_1, ..., x_r)$'s are linearly independent. ∎

3. Duality

The results in this section are due to I. G. Macdonald, and I am grateful to him for communicating them to me. They will play an essential role in the derivation of our later results.

A. James [Jam_3] used the fact that zonal polynomials are spherical functions for the pair $(GL_n(\mathbb{R}), O_n(\mathbb{R}))$ to deduce that they are eigenfunctions of a certain partial differential operator, the *Laplace–Beltrami operator*. We first give a formal generalization of this result for Jack polynomials. Fix a positive integer n. Define the operator $D(\alpha)\colon \Lambda\otimes\mathbb{Q}(\alpha)\to\Lambda\otimes\mathbb{Q}(\alpha)$ by

$$D(\alpha)=\frac{\alpha}{2}\sum_{1}^{n} x_i^2\frac{\partial^2}{\partial x_i^2}+\sum_{i\neq j}\frac{x_i^2}{x_i-x_j}\frac{\partial}{\partial x_i}. \tag{11}$$

It is easy to see that if $f\in\Lambda\otimes\mathbb{Q}(\alpha)$, then $D(\alpha)f\in\Lambda\otimes\mathbb{Q}(\alpha)$. Moreover, if f is homogeneous of degree n, then so is $D(\alpha)f$. When $\alpha=2$, (11) reduces to the Laplace–Beltrami operator of James. For general α, there no longer seems to be a group-theoretic interpretation of (11).

3.1. THEOREM. *The Jack polynomials $J_\lambda(x_1, ..., x_n; \alpha)$, where $l(\lambda)\leqslant n$, are eigenfunctions of $D(\alpha)$. The eigenvalue $e_\lambda(\alpha)$ corresponding to J_λ is given by*

$$e_\lambda(\alpha)=\alpha b(\lambda')-b(\lambda)+(n-1)\,|\lambda|.$$

There are no further eigenfunctions linearly independent from the J_λ's.

Sketch of proof. One shows by direct computation that

$$\begin{aligned}D(\alpha)\,p_\lambda=\frac{1}{2}p_\lambda\Bigg[&\sum_k(\alpha\lambda_k(\lambda_k-1)+\lambda_k(2n-\lambda_k-1)\\&+\alpha\sum_{r\neq s}\frac{\lambda_r\lambda_s p_{\lambda_r+\lambda_s}}{p_{\lambda_r}p_{\lambda_s}}+\sum_k\frac{\lambda_k}{p_{\lambda_k}}\sum_{j=1}^{\lambda_k-1}p_jp_{\lambda_k-j}\Bigg].\end{aligned}$$

From this it is easy to deduce that $D(\alpha)$ is *self-adjoint*, i.e.,

$$\langle D(\alpha)f, g\rangle=\langle f, D(\alpha)g\rangle$$

for all $f, g\in\Lambda\otimes\mathbb{Q}$, by checking the cases $f=p_\lambda$, $g=p_\mu$. It follows that $D(\alpha)$ has a set of orthogonal eigenfunctions which form a basis of $\Lambda\otimes\mathbb{Q}(\alpha)$. One also checks that for any partition λ,

$$D(\alpha)m_\lambda=\sum_{\mu\leqslant\lambda}b_{\lambda\mu}(\alpha)m_\mu \tag{12}$$

for certain scalars $b_{\lambda\mu}(\alpha)\in\mathbb{Q}(\alpha)$, where $b_{\lambda\lambda}(\alpha)=e_\lambda(\alpha)\neq 0$. It then follows from Theorem 1.1 (using only $\mu\overset{R}{\leqslant}\lambda$ in (P2) instead of $\mu\leqslant\lambda$) that $D(\alpha)\,J_\lambda(x;\alpha)=e_\lambda(\alpha)\,J_\lambda(x;\alpha)$. ∎

Note. It follows easily from (12) that for any partition λ there exists an eigenfunction $K_\lambda(x;\alpha)$ of $D(\alpha)$ of the form

$$K_\lambda(x;\alpha)=m_\lambda+\sum_{\mu<\lambda} h_{\lambda\mu}(\alpha)m_\mu.$$

If all the eigenvalues of $D(\alpha)$ were distinct, it would follow using $\mu \overset{R}{\leqslant} \lambda$ in (P2) instead of $\mu\leqslant\lambda$ that $K_\lambda(x;\alpha)$ is a scalar multiple of $J_\lambda(x;\alpha)$. Hence Theorem 1.1 would be proved. Unfortunately the eigenvalues of $D(\alpha)$ are not distinct, i.e., we can have $\lambda\neq\mu$ but $e_\lambda(\alpha)=e_\mu(\alpha)$ (as well as $|\lambda|=|\mu|$). Macdonald [M_3, Chap. VI] circumvents this difficulty in two ways. For the first way, he defines more general symmetric functions $P_\lambda(x;q,t)$ (mentioned in Section 1) and a more general operator $D_1^{(q,t)}$ such that the $P_\lambda(x;q,t)$'s are eigenfunctions of $D_1^{(q,t)}$ with *distinct* eigenvalues. For the second way, he defines a class $D_r^{(\alpha)}$ of operators (where $D_1^{(\alpha)}=D(\alpha)$) which separates the eigenfunctions $J_\lambda(x;\alpha)$. In what follows we can circumvent the problem of repeated eigenvalues with the following simple lemma, whose proof is omitted. (We do not see, however, how to use this lemma to prove Theorem 1.1. Our primary purpose for including Theorem 3.1 here is its use in the proof of Proposition 5.1.)

3.2. LEMMA. *If $|\lambda|=|\mu|$ and $e_\lambda(\alpha)=e_\mu(\alpha)$, then λ and μ are incomparable in dominance order.*

To state a second fundamental result of Macdonald, define for $0\neq\beta\in\mathbb{Q}(\alpha)$ a $\mathbb{Q}(\alpha)$-algebra automorphism $\omega_\beta\colon\Lambda\otimes\mathbb{Q}(\alpha)\to\Lambda\otimes\mathbb{Q}(\alpha)$ by the condition $\omega_\beta p_r=\beta p_r$, $r\geqslant 1$. Thus the usual involution $\omega\colon\Lambda\to\Lambda$ defined in [M_1] is given by $\omega f=(-1)^n\,\omega_{-1}f$, where f is homogeneous of degree n.

3.3. THEOREM. *Let*

$$\hat{J}_\lambda(x;\alpha)=(-1)^{|\lambda|}\,\omega_{-1/\alpha}J_{\lambda'}(x;1/\alpha). \tag{13}$$

Then $\hat{J}_\lambda(x;\alpha)=\alpha^{-|\lambda|}J_\lambda(x;\alpha)$.

Sketch of proof. One shows that $\omega_{-\alpha}J_\lambda(\alpha)$ is an eigenfunction of $D(1/\alpha)$ corresponding to the eigenvalue $e(\lambda',1/\alpha)$. It follows that $D(\alpha)\hat{J}_\lambda(x;\alpha)=e(\lambda',1/\alpha)\hat{J}_\lambda(x;\alpha)$. Moreover, by (11) we have that $D(\alpha)\hat{J}_\lambda(x;\alpha)$ is a linear combination of monomial symmetric functions m_μ with $\mu\leqslant\lambda$. Hence by Theorem 3.1 and Lemma 3.2, $\hat{J}_\lambda(\alpha)=u_\lambda(\alpha)J_\lambda(\alpha)$ for some $u_\lambda(\alpha)$.

If we expand J_λ in terms of the power-sums p_μ, then by property (P3) we have

$$J_\lambda(x;\alpha)=p_1^n+\text{other terms}.$$

Similarly,

$$\begin{aligned}\omega_{-1/\alpha} J_{\lambda'}(x; 1/\alpha) &= \omega_{-1/\alpha}(p_1^n + \cdots) \\ &= (-1)^n \alpha^{-n} p_1^n + \text{other terms}.\end{aligned}$$

Comparing with (13) completes the proof. ∎

Since $\omega_{-1/\alpha}$, as well as the map sending $f \in \Lambda^k \otimes \mathbb{Q}(\alpha)$ to $(-1/\alpha)^k f$, is a $\mathbb{Q}$-algebra automorphism, we immediately obtain the following useful corollary of Theorem 3.3.

3.4. COROLLARY. *The $\mathbb{Q}(\alpha)$-linear map $\Lambda \otimes \mathbb{Q}(\alpha) \to \Lambda \otimes \mathbb{Q}(\alpha)$ defined by*

$$J_\lambda(x; \alpha) \mapsto J_{\lambda'}(x; 1/\alpha)$$

is an algebra automorphism of $\Lambda \otimes \mathbb{Q}(\alpha)$. In particular,

$$\langle J_\mu J_\nu, J_\lambda \rangle \neq 0 \Leftrightarrow \langle J_{\mu'} J_{\nu'}, J_{\lambda'} \rangle \neq 0.$$

An alternative statement of Theorem 3.3 is the following.

3.5. COROLLARY. *Let*

$$J_\lambda = \sum_\mu c_{\lambda\mu}(\alpha)\, p_\mu. \tag{14}$$

Then

$$J_{\lambda'} = \sum_\mu (-\alpha)^{|\lambda| - l(\mu)} c_{\lambda\mu}(1/\alpha)\, p_\mu. \quad ∎$$

We conclude this section with a further consequence of Theorem 3.3.

3.6. PROPOSITION. *Using the notation* (4), *we have*

$$j_\lambda := \langle J_\lambda, J_\lambda \rangle = \alpha^{|\lambda|} v_{\lambda\lambda}(\alpha)\, v_{\lambda'\lambda'}(1/\alpha).$$

Proof. Let $\langle\ ,\ \rangle_1$ denote the ordinary scalar product on symmetric functions (the case $\alpha = 1$ of (3)), and let ω denote the usual involution (satisfying $\omega(p_\lambda) = (-1)^{|\lambda| - l(\lambda)} p_\lambda$) [$M_1$, Chap. I, (2.13)]. Note that

$$\begin{aligned}\langle p_\lambda, \omega_{-1/\alpha}(p_\mu) \rangle &= \langle p_\lambda, (-1/\alpha)^{l(\mu)} p_\mu \rangle \\ &= (-1)^{|\mu|} \langle p_\lambda, \omega p_\mu \rangle_1.\end{aligned}$$

Hence for any $f, g \in \Lambda \otimes \mathbb{Q}(\alpha)$, we have

$$\langle f, \omega_{-1/\alpha}(g) \rangle = \langle f, \omega(g) \rangle_1.$$

Let $f = J_\lambda(x; \alpha)$ and $g = J_{\mu'}(x; 1/\alpha)$. Then (13) yields

$$\langle J_\lambda, \hat{J}_\lambda \rangle = \langle J_\lambda(x; \alpha), \omega J_{\lambda'}(x; 1/\alpha) \rangle_1.$$

By property (P2) of J_λ and of Schur functions s_μ, when we expand J_λ in terms of Schur functions (using also the fact that the coefficient of m_μ is s_μ is 1) we obtain

$$J_\lambda = v_{\lambda\lambda}(\alpha) s_\lambda + \text{lower order terms (in dominance order)}.$$

Since $\omega s_\mu = s_{\mu'}$ [M_1, Chap. I, (3.8)] and conjugation is an anti-automorphism of dominance order [M_1, Chap. I, (1.11)], we also have

$$\omega J_{\lambda'}(x; 1/\alpha) = v_{\lambda'\lambda'}(1/\alpha) s_\lambda + \text{higher order terms}.$$

Since $\langle s_\lambda, s_\mu \rangle_1 = \delta_{\lambda\mu}$ [M_1, Chap. I, (4.8)], it follows that

$$\langle J_\lambda, \hat{J}_\lambda \rangle = v_{\lambda\lambda}(\alpha)\, v_{\lambda'\lambda'}(1/\alpha). \tag{15}$$

On the other hand, by Theorem 3.3 we have

$$\langle J_\lambda, \hat{J}_\lambda \rangle = \alpha^{-|\lambda|} \langle J_\lambda, J_\lambda \rangle = \alpha^{-|\lambda|} j_\lambda. \tag{16}$$

Comparing (15) and (16) completes the proof. ∎

4. Skew Jack Symmetric Functions

The skew Schur functions $s_{\lambda/\mu}$, besides being of intrinsic interest in themselves, play a basic role in developing properties of the ordinary Schur functions s_λ [M_1, Sect. 5]. Similarly we need to develop a theory of skew Jack symmetric functions. If λ and μ are any partitions, then define the skew Jack symmetric function $J_{\lambda/\mu} \in \Lambda \otimes \mathbb{Q}(\alpha)$ by the rule

$$\langle J_{\lambda/\mu}, J_\nu \rangle = \langle J_\lambda, J_\mu J_\nu \rangle, \tag{17}$$

for all partitions ν. Hence the linear transformation $\phi_\mu \colon \Lambda \otimes \mathbb{Q}(\alpha) \to \Lambda \otimes \mathbb{Q}(\alpha)$ defined by $\phi_\mu(J_\lambda) = J_{\lambda/\mu}$ is adjoint (with respect to the scalar product (3)) to multiplication by J_μ.

Define rational functions $g^\lambda_{\mu\nu}(\alpha)$ by

$$g^\lambda_{\mu\nu} = g^\lambda_{\mu\nu}(\alpha) = \langle J_\mu J_\nu, J_\lambda \rangle.$$

Equivalently,

$$J_\mu J_\nu = \sum j_\lambda^{-1} g^\lambda_{\mu\nu}(\alpha)\, J_\lambda. \tag{18}$$

It then follows from the definition (17) of $J_{\lambda/\mu}$ that

$$J_{\lambda/\mu}=\sum_{\nu} j_\nu^{-1} g_{\mu\nu}^{\lambda}(\alpha)\, J_\nu. \tag{19}$$

Clearly $g_{\mu\nu}^{\lambda}(\alpha)=0$ unless $|\lambda|=|\mu|+|\nu|$. The *Littlewood–Richardson coefficients* $c_{\mu\nu}^{\lambda}$ are defined by

$$s_\mu s_\nu=\sum_{\lambda} c_{\mu\nu}^{\lambda} s_\lambda.$$

Since $J_\lambda(x;1)=H_\lambda s_\lambda$, we have

$$g_{\mu\nu}^{\lambda}(1)=H_\mu H_\nu H_\lambda^{-1} c_{\mu\nu}^{\lambda}.$$

The Littlewood–Richardson rule [M_1, Sect. I.9] gives a combinatorial interpretation of $c_{\mu\nu}^{\lambda}$, and one of our main concerns here will be to generalize it as far as possible to $g_{\mu\nu}^{\lambda}(\alpha)$ (see Theorem 6.1 and Proposition 8.6).

If $\lambda=(\lambda_1, \lambda_1, ...)$ and $\mu=(\mu_1, \mu_2, ...)$ are partitions, then define $\lambda+\mu$ by $(\lambda+\mu)_i=\lambda_i+\mu_i$, and define $\lambda\cup\mu$ to be the partition whose parts are the parts of λ and the parts of μ, rearranged in decreasing order [M_1, p. 5]. It follows that $(\lambda+\mu)'=\lambda'\cup\mu'$ and $(\lambda\cup\mu)'=\lambda'+\mu'$.

4.1. PROPOSITION. *If* $g_{\mu\nu}^{\lambda}(\alpha)\neq 0$ *then*

$$\mu\cup\nu\leqslant\lambda\leqslant\mu+\nu.$$

Proof. Write

$$m_\mu m_\nu=\sum_{\lambda} t_{\mu\nu}^{\lambda} m_\lambda.$$

It is easily seen that if $t_{\mu\nu}^{\lambda}\neq 0$ then $\lambda\leqslant\mu+\nu$. Hence by property (P2) of Jack symmetric functions, if $g_{\mu\nu}^{\lambda}(\alpha)\neq 0$ then $\lambda\leqslant\mu+\nu$. Now apply Corollary 3.4 to get

$$g_{\mu\nu}^{\lambda}(\alpha)\neq 0 \Leftrightarrow g_{\mu'\nu'}^{\lambda'}(\alpha)\neq 0.$$

But if $g_{\mu'\nu'}^{\lambda'}(\alpha)\neq 0$ then $\lambda'\leqslant\mu'+\nu'$, which is equivalent to $\mu\cup\nu\leqslant\lambda$. ∎

Later (Corollary 6.4) we will obtain a stronger result than Proposition 4.1, but for the present Proposition 4.1 will suffice.

Now let $x=(x_1, x_2, ...)$ and $y=(y_1, y_2, ...)$ be two sets of variables.

4.2. PROPOSITION. *We have*

$$J_{\lambda/\mu}(x, y;\alpha)=\sum J_{\nu/\mu}(x;\alpha)\, J_{\lambda/\nu}(y;\alpha)\, j_\nu^{-1}. \tag{20}$$

In particular (*taking* $\mu=\varnothing$),

$$J_\lambda(x, y; \alpha)=\sum_\nu J_\nu(x; \alpha)\, J_{\lambda/\nu}(y; \alpha)\, j_\nu^{-1}.$$

Proof. The proof parallels exactly the corresponding argument [M$_1$, (5.10), p. 41] for Schur functions. Namely, let $z=(z_1, z_2, ...)$ be a third set of variables. Extend the scalar product (3) to functions symmetric in variables x and z separately by the rule $\langle J_\lambda(x)\, J_\mu(z),\ J_\sigma(x)\, J_\tau(z)\rangle = \langle J_\lambda, J_\sigma\rangle\langle J_\mu, J_\tau\rangle$. Now for fixed μ,

$$\sum_\lambda J_{\lambda/\mu}(x)\, J_\lambda(z)\, j_\lambda^{-1}=\sum_\nu J_\mu(z)\, J_\nu(x)\, J_\nu(z)\, j_\nu^{-1},$$

since the scalar product of the left-hand side with $J_\nu(x)\, J_\lambda(z)$ is $\langle J_{\lambda/\mu}, J_\nu\rangle$, while that of the right-hand side is $\langle J_\mu J_\nu, J_\lambda\rangle$. Hence by Proposition 2.1,

$$\begin{aligned}\sum_{\lambda,\mu} J_{\lambda/\mu}(x)\, J_\lambda(z)\, J_\mu(y)\, j_\mu^{-1} j_\lambda^{-1} &= \sum_{\mu,\nu} J_\mu(y)\, J_\mu(z)\, J_\nu(x)\, J_\nu(z)\, j_\mu^{-1} j_\nu^{-1}\\ &=\prod_{i,k}(1-x_i z_k)^{-1/\alpha}\cdot\prod_{j,k}(1-y_j z_k)^{-1/\alpha}\\ &=\sum_\lambda J_\lambda(x, y)\, J_\lambda(z)\, j_\lambda^{-1}.\end{aligned}$$

Comparing coefficients of $J_\lambda(z)$ yields

$$J_\lambda(x, y)=\sum_\mu J_{\lambda/\mu}(x)\, J_\mu(y)\, j_\mu^{-1}. \tag{21}$$

Substituting (x, y) for x and z for y gives

$$\begin{aligned}\sum_\mu J_{\lambda/\mu}(x, y)\, J_\mu(z)\, j_\mu^{-1} &= J_\lambda(x, y, z)\\ &=\sum_\nu J_{\lambda/\nu}(x)\, J_\nu(y, z)\, j_\nu^{-1}\\ &=\sum_{\mu,\nu} J_{\lambda/\nu}(x)\, J_{\nu/\mu}(y)\, J_\mu(z)\, j_\nu^{-1} j_\mu^{-1},\end{aligned}$$

again by (21). Taking the coefficients of $J_\mu(z)$ at each end of this chain of equalities yields (20). ∎

5. Further Properties of Jack Symmetric Functions

In this section we derive a number of properties of Jack symmetric functions, some of interest in their own right and some of which will be used in the proofs of the main results in the next section.

5.1. PROPOSITION. *Let* $l(\lambda)=n$, *and write* $\lambda-I=\lambda-I_n=(\lambda_1-1, \lambda_2-1, ..., \lambda_n-1)$. *Then there exists a rational function* $c_\lambda(\alpha)\in\mathbb{Q}(\alpha)$ *for which*

$$J_\lambda(x_1, ..., x_n)=c_\lambda(\alpha)\, x_1\cdots x_n J_{\lambda-I}(x_1, ..., x_n). \tag{22}$$

Proof. Write $L=x_1\cdots x_n J_{\lambda-I}(x_1, ..., x_n)$. We have

$$\begin{aligned} D(\alpha)L &= \left[\frac{\alpha}{2}\sum_{i=1}^{n} x_i^2\frac{\partial^2}{\partial x_i^2}+\sum_{i\neq j}\frac{x_i^2}{x_i-x_j}\frac{\partial}{\partial x_i}\right]L \\ &= x_1\cdots x_n D(\alpha)\, J_{\lambda-I}(x_1, ..., x_n) \\ &\quad +\alpha x_1\cdots x_n\sum_{i=1}^{n} x_i\frac{\partial}{\partial x_i}J_{\lambda-I}(x_1, ..., x_n) \\ &\quad + x_1\cdots x_n\left(\sum_{i\neq j}\frac{x_i}{x_i-x_j}\right)J_{\lambda-I}(x_1, ..., x_n). \end{aligned}$$

Now for any homogeneous function $f(x_1, ..., x_n)$ of degree m we have

$$\sum_{i=1}^{n} x_i\frac{\partial}{\partial x_i}f=mf.$$

Moreover, since

$$\frac{x_i}{x_i-x_j}+\frac{x_j}{x_j-x_i}=1,$$

we have

$$\sum_{i\neq j}\frac{x_i}{x_i-x_j}=\binom{n}{2}.$$

Hence by Theorem 3.1 and the fact the fact that $\lambda_1'=n$, we have

$$\begin{aligned} D(\alpha)L &= \left[\alpha b((\lambda-I)')-b(\lambda-I)+(n-1)\,|\lambda-I|+\alpha\,|\lambda-I|+\binom{n}{2}\right]L \\ &= \left[\alpha\sum_{i\geqslant 2}(i-2)\lambda_i'-\sum_{i\geqslant 2}\binom{\lambda_i'}{2}+(n-1)\,|\lambda|\right. \\ &\qquad \left.-n(n-1)+\alpha\,|\lambda|-\alpha n+\binom{n}{2}\right]L \\ &= \left[\alpha\sum_{i\geqslant 1}(i-1)\,\lambda_i'-\sum_{i\geqslant 1}\binom{\lambda_i'}{2}+(n-1)\,|\lambda|\right]L \\ &= e_\lambda(\alpha)L. \end{aligned}$$

It follows from Theorem 3.1 that L is a linear combination of $J_\mu(x_1, ..., x_n)$'s for which $e_\mu(\alpha) = e_\lambda(\alpha)$ and $|\lambda| = |\mu|$. Now by (P2) every monomial symmetric function m_ν appearing in L satisfies $\nu \leqslant \lambda$. It follows that

$$L = \sum_{\mu \leqslant \lambda} c_{\lambda\mu}(\alpha)\, J_\mu(x_1, ..., x_n)$$

for certain $c_{\lambda\mu}(\alpha) \in \mathbb{Q}(\alpha)$. Since $c_{\lambda\mu}(\alpha) = 0$ unless $e_\lambda(\alpha) = e_\mu(\alpha)$, it follows from Lemma 3.2 that L is a multiple $c_\lambda(\alpha)$ of $J_\lambda(x_1, ..., x_n)$. ∎

Given a partition λ, let $m = \lambda_1$. By property (P2) we have

$$J_\lambda(x) = x_1^m f_\lambda(x') + g_\lambda(x),$$

where $f_\lambda(x')$ is a symmetric function in the variables $x' = (x_2, x_3, ...)$, and where no monomial appearing in $g_\lambda(x)$ is divisible by x_1^m. Write

$$f_\lambda(x') = [x_1^m]\, J_\lambda(x),$$

the *coefficient of* x_1^m *in* $J_\lambda(x)$.

5.2. Proposition. *Let* $m = \lambda_1$ *as above, and write* $\lambda^- = (\lambda_2, \lambda_3, ...)$. *Then there exists a rational function* $d_\lambda(\alpha)$ *for which*

$$[x_1^m]\, J_\lambda(x) = d_\lambda(\alpha)\, J_{\lambda^-}(x'). \tag{23}$$

Proof. By Proposition 4.2,

$$J_\lambda(x) = J_\lambda(x_1, x') = \sum_\mu J_\mu(x_1)\, J_{\lambda/\mu}(x')\, j_\mu^{-1}. \tag{24}$$

By property (P2) we have $J_\mu(x_1) \neq 0$ if and only if μ consists of a single part r, in which case $J_\mu(x_1) = J_r(x_1) = v_{r,r}(\alpha) x_1^r$. It follows from (24) that

$$[x_1^m]\, J_\lambda(x) = v_{m,m}(\alpha)\, J_{\lambda/m}(x')\, j_m^{-1}.$$

Thus, writing

$$J_{\lambda/m}(x') = \sum_\mu d_{\lambda\mu}(\alpha)\, J_\mu(x'),$$

we have

$$d_{\lambda\mu}(\alpha) \neq 0 \Leftrightarrow \langle J_\lambda, J_m J_\mu \rangle \neq 0. \tag{25}$$

For any μ with $m \geqslant \mu_1$, consider the product $e_m J_{\mu'}$. When we restrict to the variables $x_1, ..., x_m$, we obtain by Proposition 5.1,

$$x_1 \cdots x_m J_{\mu'}(x_1, ..., x_m) = c_{\mu' + I}(\alpha)^{-1} J_{\mu' + I}(x_1, ..., x_m),$$

where $\mu' + I = (\mu'_1 + 1, ..., \mu'_m + 1)$. Hence by Proposition 2.5,

$$m \geqslant \mu_1 \text{ and } \langle e_m J_{\mu'}, J_{\lambda'} \rangle \neq 0 \Rightarrow \lambda' = \mu' + I \text{ or } l(\lambda') > m. \tag{26}$$

Assuming $m \geqslant \mu_1$, the condition $\lambda' = \mu' + I$ is equivalent to $\lambda = \mu \cup (m)$ (i.e., $\mu = \lambda^-$). Moreover $l(\lambda') = \lambda_1 = m$, so $l(\lambda') > m$ is impossible. Thus dualizing (26) (i.e., applying Corollary 3.4), we obtain

$$m \geqslant \mu_1 \text{ and } \langle J_m J_\mu, J_\lambda \rangle \neq 0 \Rightarrow \mu = \lambda \ . \tag{27}$$

But if $\langle J_m J_\mu, J_\lambda \rangle \neq 0$ then by Proposition 4.1 we have $\mu \cup m \leqslant \lambda$ so $\mu_1 \leqslant \lambda_1 = m$. Thus the condition $m \geqslant \mu_1$ is superfluous in (27), so by (25) the proof is complete. ∎

If $\mu \subseteq \lambda$ then [M$_1$, p. 4] the skew shape λ/μ (regarded as a difference $\lambda - \mu$ of diagrams) is called a *horizontal strip* if no two distinct points of λ/μ lie in the same column. Call a horizontal strip λ/μ an *n-strip* if $|\lambda/\mu| = n$. The next result, well-known for Schur functions [M$_1$, (5.16), p. 42], will be quantitatively improved in the next section (Theorem 6.1).

5.3. PROPOSITION. *$\langle J_\mu J_n, J_\lambda \rangle \neq 0$ if and only if $\mu \subseteq \lambda$ and λ/μ is a horizontal n-strip.*

Proof. The "if" part follows from the corresponding result mentioned above for Schur functions (the case $\alpha = 1$, except for irrelevant scalar factors). The "only if" part is proved by induction on $|\mu|$, the proof being clear for $|\mu| = 0$.

Put $\nu = (n)$ in (18) to get

$$J_\mu J_n = \sum_\lambda j_\lambda^{-1} g_{\mu n}^\lambda J_\lambda. \tag{28}$$

Since $\langle J_\mu J_n, J_\lambda \rangle \neq 0 \Rightarrow \mu \cup (n) \leqslant \lambda$ by Proposition 4.1, we may assume $l(\lambda) \geqslant l(\mu)$ in (28).

Case 1. $l(\lambda) = l(\mu) = l$. Restrict (28) to the variables $x_1, ..., x_l$. By Propositions 2.5 and 5.1, we obtain

$$\begin{aligned} c_\mu x_1 \cdots x_l J_{\mu - I}(x_1, ..., x_l)\, J_n(x_1, ..., x_l) \\ = \sum_{l(\lambda) = l} j_\lambda^{-1} g_{\mu n}^\lambda c_\lambda x_1 \cdots x_l J_{\lambda - I}(x_1, ..., x_l), \end{aligned} \tag{29}$$

where $I = I_l$. By induction, if $g^{\lambda}_{\mu n} \neq 0$ then $(\mu - I) \subseteq (\lambda - I)$ and $(\lambda - I)/(\mu - I)$ is a horizontal n-strip. But then $\mu \subseteq \lambda$ and λ/μ is also a horizontal n-strip.

Case 2. $l(\lambda) > l(\mu) = l$. Apply Corollary 3.4 to (28) to get

$$J_{\mu'} J_{1^n} = \sum_{\lambda} (1/\bar{j}_\lambda)\, \bar{g}^{\lambda}_{\mu\nu} J_{\lambda'}, \tag{30}$$

where we use the notation $\bar{f}(\alpha) = f(\alpha^{-1})$. The largest power of x_1 dividing any monomial appearing in $J_{\mu'} J_{1^n}$ is $x_1^{\mu_1' + 1} = x_1^{l+1}$, so this is also the largest power of x_1 dividing any $J_{\lambda'}$ appearing in (30). (By property (P2), these largest powers cannot "cancel out" of the right-hand side of (30).) Hence if $g^{\lambda}_{\mu n} \neq 0$ then $\lambda_1' \leqslant \mu_1' + 1$. Thus either $l(\lambda) = l$ (which was Case 1) or $l(\lambda) = l + 1$.

Now apply $[x_1^{l+1}]$ to (30). If $l(\lambda) = l$ then $[x_1^{l+1}] J_{\lambda'} = 0$. Thus by Proposition 5.2, we obtain

$$d_{\mu'} J_{(\mu')^-} J_{1^{n-1}} = \sum_{l(\lambda) = l+1} (1/\bar{j}_\lambda)\, \bar{g}^{\lambda}_{\mu n} d_{\lambda'} J_{(\lambda')^-}. \tag{31}$$

Now apply Corollary 3.4 to get

$$\bar{d}_{\mu'} J_{\mu - I_l} J_{n-1} = \sum_{l(\lambda) = l+1} j_\lambda^{-1} g^{\lambda}_{\mu n} \bar{d}_{\lambda'} J_{\lambda - I_{l+1}}. \tag{32}$$

By induction $(\mu - I_l) \subseteq (\lambda - I_{l+1})$ and $(\lambda - I_{l+1})/(\mu - I_l)$ is a horizontal $(n-1)$-strip. But then $\mu \subseteq \lambda$ and λ/μ is a horizontal n-strip, and the proof is complete. ∎

We are now ready to prove a series of explicit formulas involving Jack symmetric functions. The first of these formulas is well-known in the case of Schur functions [M$_1$, Ex. 4, p. 28] and (in an equivalent form) zonal polynomials [C; T, p. 50, Thm. 2]. It was independently conjectured for Jack symmetric functions by I. G. Macdonald and this writer. We write $J_\lambda(1^n)$ for the Jack symmetric function $J_\lambda(x; \alpha)$ evaluated at

$$x_1 = x_2 = \cdots = x_n = 1, \qquad x_{n+1} = x_{n+2} = \cdots = 0.$$

5.4. THEOREM. *We have*

$$J_\lambda(1^n) = \prod_{(i,j) \in \lambda} (n - (i-1) + \alpha(j-1)).$$

Proof. Induction on $|\lambda|$ and for fixed $|\lambda|$ on reverse dominance order. The initial conditions consist of the case $\lambda = (1^r)$, and here we have $J_{1^r}(1^n) = r!\, e_r(1^n) = r! \binom{n}{r} = n(n-1) \cdots (n-r+1)$, as desired.

It is easily seen that

$$m_\lambda(1^n) = \binom{n}{l(\lambda)} \binom{l(\lambda)}{m_1(\lambda), m_2(\lambda), ...}. \tag{33}$$

It follows from property (P3) that $J_\lambda(1^n)$ is a polynomial function of n of degree $|\lambda|$ and leading coefficient 1.

Now from Proposition 4.2 (with $\mu = \varnothing$) we have

$$J_\lambda(1^n) = \sum_\nu J_\nu(1^{n-1}) J_{\lambda/\nu}(1) j_\nu^{-1}. \tag{34}$$

Now $J_{\lambda/\nu}(1) \neq 0$ if and only if $\langle J_{\lambda/\nu}, J_m \rangle \neq 0$, where $m = |\lambda| - |\nu|$. Hence by Proposition 5.3, $\nu \subseteq \lambda$ and λ/ν is a horizontal strip. This means that ν (regarded as a diagram) must contain every element of λ which is not the bottom element of a column. In symbols,

$$J_{\lambda/\nu}(1) \neq 0 \Rightarrow \{(i, j) \in \lambda : i \neq \lambda_j'\} \subseteq \nu.$$

Hence by induction, every term on the right-hand side of (34) is divisible (as a polynomial in n) by

$$\prod_{\substack{(i,j) \in \lambda \\ i \neq \lambda_j'}} (n - 1 - (i-1) + \alpha(j-1)) = \prod_{\substack{(i,j) \in \lambda \\ i > 1}} (n - (i-1) + \alpha(j-1)). \tag{35}$$

Thus the polynomial $J_\lambda(1^n)$ is divisible by (35).

Now let $\mu = \lambda^- = (\lambda_2, \lambda_3, ...)$, let $m = \lambda_1$, and consider as in (18) the product

$$\begin{aligned} J_\mu(1^n) J_m(1^n) &= \sum_\nu j_\nu^{-1} g_{\mu m}^\nu(\alpha) J_\nu(1^n) \\ &= j_\lambda^{-1} g_{\mu m}^\lambda(\alpha) J_\lambda(1^n) + \sum_{\nu < \lambda} j_\nu^{-1} g_{\mu m}^\nu(\alpha) J_\nu(1^n). \end{aligned} \tag{36}$$

By properties (P1) and (P2) the coefficient $v_{\lambda\lambda}(\alpha) \neq 0$ in (4), and it follows again from (P1) and (P2) that $g_{\mu m}^\lambda(\alpha) \neq 0$. By Proposition 4.1 or by Proposition 5.3, if $g_{\mu m}^\nu(\alpha) \neq 0$ then $\nu_1 \geqslant m$. Hence by induction, if $\nu < \lambda$ and $g_{\mu m}^\nu(\alpha) \neq 0$, then $J_\nu(1^n)$ is divisible by

$$\prod_{j=1}^m (n + \alpha(j-1)). \tag{37}$$

Now by (9),

$$\sum_{m \geq 0} J_m(1^n)\, t^m/\alpha^m m! = (1-t)^{-n/\alpha}$$

$$= \sum_{m \geq 0} \binom{-n/\alpha}{m} (-1)^m t^m,$$

from which it follows that

$$J_m(1^n) = \prod_{j=1}^{m} (n + \alpha(j-1)).$$

Hence the left-hand side of (36) is also divisible by (37), so $J_\lambda(1^n)$ is also.

The polynomials (35) and (37) are relatively prime (since the zeros $(i-1)-\alpha(j-1)$, $i>1$, and $-\alpha(j-1)$ are distinct), and we have shown that $J_\lambda(1^n)$ is divisible by both (35) and (37). Hence $J_\lambda(1^n)$ is divisible by their product. But their product has degree $|\lambda| = \deg J_\lambda(1^n)$, so $J_\lambda(1^n)$ is a multiple of their product. Since $J_\lambda(1^n)$ is monic, it is equal exactly to the product of (35) and (37), and the proof follows by induction. ∎

Note. Even if one is interested in Theorem 5.4 only for a particular value of α (such as the zonal polynomial case $\alpha = 2$), it is necessary to work with general α since otherwise one cannot conclude that the polynomials (35) and (37) are relatively prime. Thus the more general viewpoint of Jack polynomials can lead to simpler and more elementary proofs than might be otherwise possible.

If $(i, j) \in \lambda$, then recall from Section 1 that the quantity $h_\lambda(i, j) = h(i, j) = \lambda_i + \lambda'_j - i - j + 1$ is called the *hook-length* at (i, j). We define two "α-refinements" of $h_\lambda(i, j)$ as follows:

$$h^*_\lambda(i, j) = h^*(i, j) = \lambda'_j - i + \alpha(\lambda_i - j + 1)$$

$$h_*^\lambda(i, j) = h_*(i, j) = \lambda'_j - i + 1 + \alpha(\lambda_i - j).$$

We call $h^*(i, j)$ the *upper hook-length* and $h_*(i, j)$ the *lower hook-length* at (i, j). Regard the diagram of λ as consisting of juxtaposed unit squares; e.g., Fig. 1 shows the diagram of $\lambda = (4, 3, 1, 1)$. The set $A_\lambda(x)$ of squares directly to the right of $x = (i, j) \in \lambda$ is the *arm* of x, of size $a_\lambda(x) = \# A_\lambda(x) = \lambda_i - j$. Similarly the set $L_\lambda(x)$ of squares directly below $x \in \lambda$ is the *leg* of x, of size $l_\lambda(x) = \# L_\lambda(x) = \lambda'_j - i$. Thus

$$h^*_\lambda(x) = l_\lambda(x) + \alpha(a_\lambda(x) + 1)$$

$$h_*^\lambda(x) = l_\lambda(x) + 1 + \alpha a_\lambda(x).$$

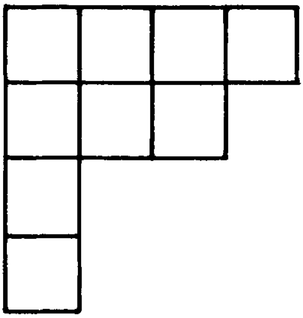

FIGURE 1

It is as if we are unable to decide whether the square x belongs to the arm or to the leg, so it is necessary to consider both possibilities. Figure 2 show the upper and lower hook-lengths of the partition (4, 3, 1, 1).

5.5 PROPOSITION. *Let $l(\lambda)=n$. Then*

$$c_\lambda(\alpha)=\prod_{i=1}^{n} h_*^\lambda(i,1),$$

where $c_\lambda(\alpha)$ is defined by Proposition 5.1.

Proof. Put $x_1=\cdots=x_n=1$ in (22). By Theorem 5.4,

$$\begin{aligned} c_\lambda(\alpha) &= \frac{\prod_{(i,j)\in\lambda}(n-(i-1)+\alpha(j-1))}{\prod_{(i,j)\in\lambda-l}(n-(i-1)+\alpha(j-1))} \\ &= \prod_{\substack{(i,j)\in\lambda \\ j=\lambda_i}} (n-(i-1)+\alpha(j-1)) \\ &= \prod_{i=1}^{n} h_*^\lambda(i,1). \quad \blacksquare \end{aligned}$$

$3+4\alpha$ $4+3\alpha$	$1+3\alpha$ $2+2\alpha$	$1+2\alpha$ $2+\alpha$	α 1
$2+3\alpha$ $3+2\alpha$	2α $1+\alpha$	α 1	
$1+\alpha$ 2			
α 1			

FIGURE 2

5.6. THEOREM. *The coefficient $v_{\lambda\lambda}(\lambda)$ of m_λ in J_λ (see (4)) is given by*

$$v_{\lambda\lambda}(\alpha)=\prod_{s\in\lambda} h_*^\lambda(s).$$

Proof. Induction on $|\lambda|$, the case $|\lambda|=0$ being trivial. Let $n=l(\lambda)$ and $I=I_n=(1, 1, ..., 1)$(n times), and set $x_{n+1}=x_{n+2}=\cdots=0$ in (4). We get by Proposition 5.1,

$$c_\lambda(\alpha)\, x_1\cdots x_n J_{\lambda-I}(x_1, ..., x_n)=\sum_{\mu\leqslant\lambda} v_{\lambda\mu}(\alpha)\, m_\lambda(x_1, ..., x_n).$$

If $\mu\leqslant\lambda$ then $m_\mu(x_1, ..., x_n)=x_1\cdots x_n m_{\mu-I}(x_1, ..., x_n)$ (which equals 0 unless $l(\mu)=l(\lambda)$), so

$$c_\lambda(\alpha)\, J_{\lambda-I}(x_1, ..., x_n)=\sum_{\mu\leqslant\lambda} v_{\lambda\mu}(\alpha)\, m_{\mu-I}(x_1, ..., x_n).$$

Thus in particular,

$$v_{\lambda\lambda}(\alpha)=v_{\lambda-I,\lambda-I}(\alpha)\, c_\lambda(\alpha),$$

and the proof follows using Proposition 5.5 and induction. ∎

5.7. PROPOSITION. *Let $\lambda_1=m$. Then*

$$d_\lambda(\alpha)=\prod_{j=1}^{m} h_*^\lambda(1, j),$$

where $d_\lambda(\alpha)$ is defined by Proposition 5.2.

Proof. By definition of $d_\lambda(\alpha)$ and $v_{\lambda\mu}(\alpha)$ it is clear that

$$v_{\lambda\lambda}(\alpha)=d_\lambda(\alpha)\, v_{\mu\mu}(\alpha),$$

where $\mu=\lambda^{-}=(\lambda_2, \lambda_3, ...)$. The proof follows from Theorem 5.6. ∎

5.8. THEOREM. *We have*

$$j_\lambda=\langle J_\lambda, J_\lambda\rangle=\prod_{s\in\lambda} h_*^\lambda(s)\, h_\lambda^*(s).$$

Proof. Immediate from Proposition 3.6 and Theorem 5.6. ∎

6. Pieri's Rule and the Evaluation of $v_{\lambda\mu}(\alpha)$

Pieri's rule is the name given to the formula [M_1, (5.16), p. 42]

$$s_\mu s_n = \sum_\lambda s_\lambda,$$

where λ ranges over all partitions for which $\mu \subseteq \lambda$ and λ/μ is a horizontal n-strip. Equivalently,

$$\langle s_\mu s_n, s_\lambda \rangle_1 = \begin{cases} 1, & \text{if } \mu \subseteq \lambda \text{ and } \lambda/\mu \text{ is a horizontal } n\text{-strip} \\ 0, & \text{otherwise,} \end{cases}$$

where the scalar product is given by (3) with $\alpha = 1$.

In this section we extend this rule to Jack symmetric functions, and we use this extension to give a (rather messy) combinatorial interpretation of $v_{\lambda\mu}(\alpha)$ analogous to what is done in [M_1] for Schur functions. A result equivalent to Theorem 6.1 in the case $\alpha = 2$ (i.e., for zonal polynomials) is due to Kushner [K].

Our main result is the following:

6.1. Theorem. *Let $\mu \subseteq \lambda$, and let λ/μ be a horizontal n-strip. Then*

$$\langle J_\mu J_n, J_\lambda \rangle = \left(\prod_{s \in \mu} A_{\lambda\mu}(s)\right)\left(\prod_{s \in (n)} h_n^*(s)\right)\left(\prod_{s \in \lambda} B_{\lambda\mu}(s)\right), \tag{38}$$

where

$$A_{\lambda\mu}(s) = \begin{cases} h_*^\mu(s), & \text{if } \lambda/\mu \text{ does not contain a square in the same column as } s \\ h_\mu^*(s), & \text{otherwise} \end{cases}$$

$$B_{\lambda\mu}(s) = \begin{cases} h_\lambda^*(s), & \text{if } \lambda/\mu \text{ does not contain a square in the same column as } s \\ h_*^\lambda(s), & \text{otherwise.} \end{cases} \tag{39}$$

Before turning to the proof, let us make a few observations concerning the form of (38). Theorem 6.1 shows that $\langle J_\mu J_n, J_\lambda \rangle$ is obtained by choosing either $h^*(s)$ or $h_*(s)$ for every element s of μ, (n), and λ, and then multiplying these expressions together. Moreover, we choose $h^*(s)$ and $h_*(s)$ exactly $|\lambda|$ times each. Note also that the middle product in (38) has the explicit evaluation

$$\prod_{s \in (n)} h_n^*(s) = n!\,\alpha^n.$$

EXAMPLE. $\langle J_{32}J_4, J_{531}\rangle$ is the product of all the entries of the three diagrams in Fig. 3.

Proof of Theorem 6.1. Induction on $|\mu|$, the proof being clear for $|\mu|=0$. The argument closely parallels the proof of Proposition 5.3, but now we know the values of the quantities j_λ, $c_\lambda(\alpha)$, $d_\lambda(\alpha)$ which appear there.

As in (28), write

$$J_\mu J_n = \sum_\lambda j_\lambda^{-1} g^\lambda_{\mu n}(\alpha) J_\lambda. \tag{40}$$

As in the proof of Proposition 5.3 (or by Proposition 5.3 itself), we may assume $l(\lambda)=l(\mu)$ or $l(\lambda)=l(\mu)+1$ in (40).

Case 1. $l(\lambda)=l(\mu)=l$. Restrict (40) to the variables $x_1, ..., x_l$, and let $I=I_l$. By (29) we have

$$c_\mu J_{\mu-I}(x_1, ..., x_l)\, J_n(x_1, ..., x_l) = \sum_{l(\lambda)=l} j_\lambda^{-1} g^\lambda_{\mu n} c_\lambda J_{\lambda-I}(x_1, ..., x_l).$$

Hence

$$g^\lambda_{\mu n} = j_\lambda j^{-1}_{\lambda-I} c_\mu c_\lambda^{-1} g^{\lambda-I}_{\mu-I,n}. \tag{41}$$

We know all the factors on the right-hand side of (41) by Proposition 5.5, Theorem 5.8, and induction, so it is simply a matter of checking how they cancel. The last factor in (41) consists of all factors in (38) except those of the form $s=(i,1)\in\mu$ and $s=(i,1)\in\lambda$. Since λ/μ has an empty first column, we need to show that

$$\left[\prod_{(i,1)\in\mu} h^\mu_*(i,1)\right]\left[\prod_{(i,1)\in\lambda} h^*_\lambda(i,1)\right] = j_\lambda j^{-1}_{\lambda-I} c_\mu(\alpha)\, c_\lambda(\alpha)^{-1}.$$

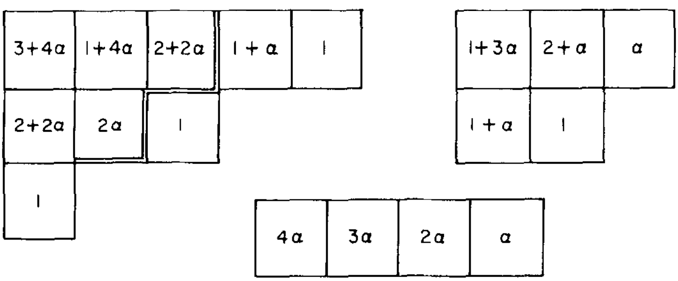

FIGURE 3

But it is immediate from Proposition 5.5 and Theorem 5.8 that

$$\prod_{(i,1)\in\mu} h_*^\mu(i,1) = c_\mu(\alpha),$$
$$\prod_{(i,1)\in\lambda} h_\lambda^*(i,1) = j_\lambda j_{\lambda-l}^{-1} c_\lambda(\alpha)^{-1},$$

so $g_{\mu n}^\lambda(\alpha)$ has the desired form.

Case 2. $l(\lambda) = 1 + l(\mu) = l + 1$. Now from (32),

$$\bar{d}_{\mu'} J_{\mu - 1_l} J_{n-1} = \sum_{l(\lambda)=l+1} j_\lambda^{-1} g_{\mu n}^\lambda \bar{d}_{\lambda'} J_{\lambda - 1_{l+1}}.$$

Hence

$$g_{\mu n}^\lambda = j_\lambda j_{\lambda - 1_{l+1}}^{-1} \bar{d}_{\mu'} (1/\bar{d}_{\lambda'}) \, g_{\mu - 1_l, n}^{\lambda - 1_{l+1}}. \tag{42}$$

Again the last factor in (42) consists of all factors in (38) except those of the form $s = (i, 1) \in \mu$ and $s = (i, 1) \in \lambda$. Since λ/μ has a nonempty first column, we need to show that

$$\left[\prod_{(i,1)\in\mu} h_\mu^*(i,1)\right]\left[\prod_{(i,1)\in\lambda} h_*^\lambda(i,1)\right] = j_\lambda j_{\lambda - 1_{l+1}}^{-1} \bar{d}_{\mu'} \bar{d}_{\lambda'}^{-1}.$$

But

$$\prod_{(i,1)\in\mu} h_\mu^*(i,1) = \alpha^l \bar{d}_{\mu'}$$
$$\prod_{(i,1)\in\lambda} h_*^\lambda(i,1) = \alpha^{-l} j_\lambda j_{\lambda - 1_{l+1}}^{-1} (1/\bar{d}_{\lambda'}),$$

and the proof follows. ▌

We next obtain a combinatorial interpretation of the coefficients $v_{\lambda\mu} = v_{\lambda\mu}(\alpha)$ of (4). More generally, set

$$J_{\lambda/\mu} = \sum_\nu v_{\lambda/\mu, \nu} J_\nu.$$

We will obtain a combinatorial interpretation of $v_{\lambda/\mu, \nu}$ analogous to [M_1, (5.12), p. 42]. Following [M_1, p. 4], define a *tableau* (or "column-strict plane partition") of *shape* λ/μ to be a sequence of partitions

$$\mu = \lambda^{(0)} \subseteq \lambda^{(1)} \subseteq \cdots \subseteq \lambda^{(r)} = \lambda,$$

such that each skew diagram $\theta^{(i)} = \lambda^{(i)} - \lambda^{(i-1)}$ $(1 \leqslant i \leqslant r)$ is a horizontal strip. We may represent T graphically by numbering each square of the skew diagram $\theta^{(i)}$ with the number i $(1 \leqslant i \leqslant r)$, and we often think of a tableau as a numbered skew diagram in this way. For instance, the tableau

$$(2) \subseteq (3, 1) \subseteq (5, 1, 1) \subseteq (5, 1, 1) \subseteq (6, 4, 1) \subseteq (6, 5, 2)$$

is represented by the diagram

$$\begin{array}{cccccc} & & 1 & 2 & 2 & 4 \\ 1 & 4 & 4 & 4 & 5 & \\ 2 & 5. & & & & \end{array}$$

If $a_i = |\theta^{(i)}|$, then the sequence $(a_1, ..., a_r)$ is called the *type* (or *weight*) of T. Define the monomial x^T associated with a tableau T to be

$$x^T = x_1^{a_1} \cdots x_r^{a_r}.$$

Thus the above tableau T has type $(2, 3, 0, 4, 2)$, and $x^T = x_1^2 x_2^3 x_4^4 x_5^2$.

The combinatorial interpretation of $s_{\lambda/\mu}$ takes the form [M_1, (5.12), p. 42]

$$s_{\lambda/\mu} = \sum_T x^T, \tag{43}$$

summed over all tableaux T of shape λ/μ. Thus

$$J_\lambda(x; 1) = H_\lambda \sum_T x^T.$$

We will give a combinatorial interpretation of $J_{\lambda/\mu}$ of the form

$$J_{\lambda/\mu} = \sum_T w(T)\, x^T,$$

summed over the same set as in (43), where $w(T) \in \mathbb{Q}(\alpha)$ is a certain weight associated with T.

6.2. Lemma. *Write $J_{\lambda/\mu}(t)$ for $J_{\lambda/\mu}(x)$ evaluated at $x_1 = t$, $x_2 = x_3 = \cdots = 0$. Then*

$$J_{\lambda/\mu}(t) = g_{\mu n}^{\lambda} t^n / \alpha^n n!,$$

where $n = |\lambda/\mu|$.

Proof. By property (P2) we have

$$J_\nu(t) = \begin{cases} v_{nn} t^n, & \text{if } \nu = (n) \\ 0, & \text{if } l(\nu) > 1. \end{cases}$$

Hence by (19),

$$J_{\lambda/\mu}(t) = j_n^{-1} v_{nn} g_{\mu n}^{\lambda}.$$

Since $j_n^{-1} v_{nn} = 1/\alpha^n n!$, the proof follows. ∎

6.3. THEOREM. *We have*

$$J_{\lambda/\mu} = \sum_T w(T) x^T,$$

summed over all tableaux

$$T\colon \mu = \lambda^{(0)} \subseteq \lambda^{(1)} \subseteq \cdots \subseteq \lambda^{(r)} = \lambda \tag{44}$$

of shape λ/μ, *where*

$$w(T) = \frac{j_\mu \prod_{i=1}^{r} \prod_{s \in \lambda^{(i)}} B_{\lambda^{(i)}, \lambda^{(i-1)}}(s)}{\prod_{i=1}^{r} \prod_{s \in \lambda^{(i-1)}} C_{\lambda^{(i)}, \lambda^{(i-1)}}(s)}. \tag{45}$$

Here $B_{\nu\sigma}(s)$ *is given by* (39), *while we define* $C_{\nu\sigma}(s)$ (*where* ν/σ *is a horizontal strip*) *by*

$$C_{\nu\sigma}(s) = \begin{cases} h_\sigma^*(s), & \text{if } \nu/\sigma \text{ does not contain} \\ & \text{a square in the same column as } s \\ h_*^\sigma(s), & \text{otherwise.} \end{cases}$$

EXAMPLE. Let T be the skew tableau

$$\begin{array}{cccccc} & & 1 & 1 & 2 & 4 \\ 2 & 2 & 3 & 4 & & \\ 3 & 4 & 4. & & & \end{array}$$

Then $w(T)$ is equal to the product of all entries appearing in Fig. 4, divided by the product of all entries appearing in Fig. 5. (Disregard for now the fact that some entries are circled.)

Proof of Theorem 6.3. By Proposition 4.2 and Lemma 6.2, we have

$$\begin{aligned} [x_1^{a_1}]\, J_{\lambda/\mu} &= [x_1^{a_1}] \sum_\nu J_{\nu/\mu}(x_1)\, J_{\lambda/\nu}(x_2, x_3, \ldots)\, j_\nu^{-1} \\ &= \sum g_{\mu, a_1}^{\nu} J_{\lambda/\nu}(x_2, x_3, \ldots)\, j_\nu^{-1}/\alpha^{a_1} a_1!. \end{aligned}$$

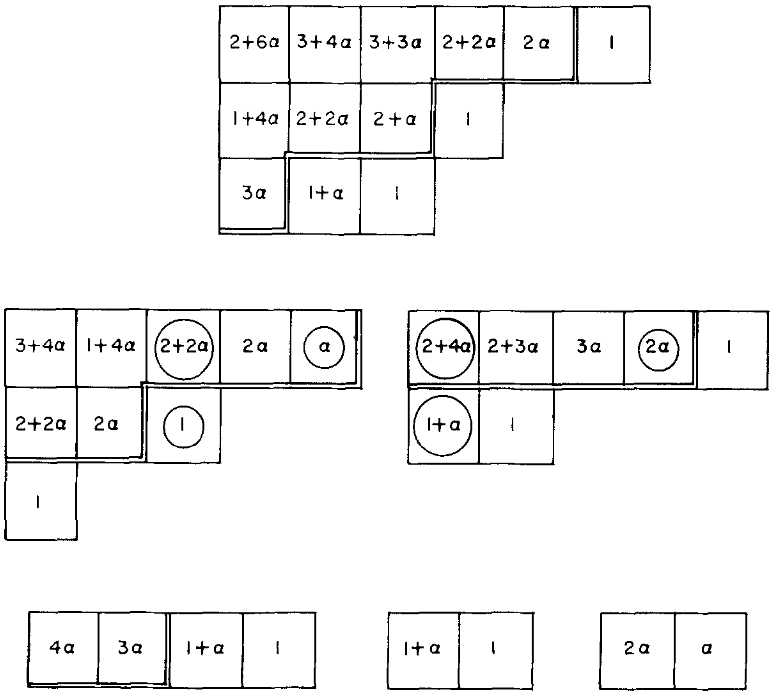

FIGURE 4

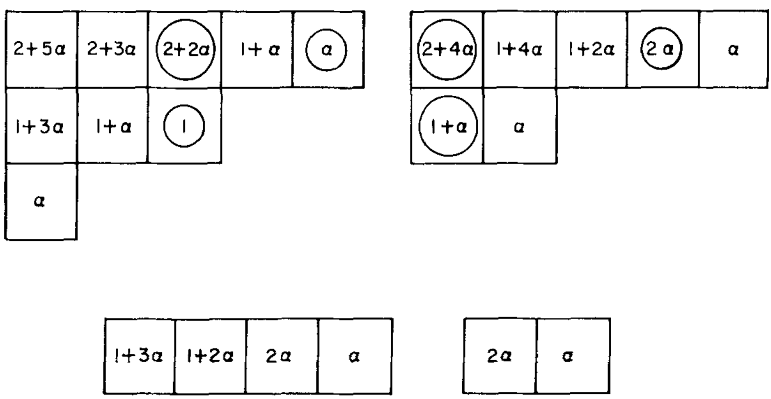

FIGURE 5

Now apply $[x_2^{a_2}]$ to this equation, and continue. If $a_1+a_2+\cdots+a_r=|\lambda/\mu|$, then we obtain

$$[x_1^{a_1}\cdots x_r^{a_r}]\,J_{\lambda/\mu}=\sum\frac{g^{\lambda^1}_{\lambda^0,a_1}\cdot g^{\lambda^2}_{\lambda^1,a_2}\cdots g^{\lambda^r}_{\lambda^{r-1},a_r}}{\alpha^{a_1}a_1!\,j_{\lambda^1}\alpha^{a_2}a_2!\,j_{\lambda^2}\cdots j_{\lambda^{r-1}}\alpha^{a_r}a_r!},$$

where the sum ranges over all tableaux $\mu=\lambda^0\subseteq\lambda^1\subseteq\cdots\subseteq\lambda^r=\lambda$ of type $(a_1,...,a_r)$. Now by Theorems 5.8 and 6.1, we have

$$\frac{g^{\lambda^i}_{\lambda^{i-1},a_i}}{\alpha^{a_i}a_i!\,j_{\lambda^{i-1}}}=\frac{\prod_{s\in\lambda^i}B_{\lambda^i,\lambda^{i-1}}(s)}{\prod_{s\in\lambda^{i-1}}C_{\lambda^i,\lambda^{i-1}}(s)},$$

and the proof follows. ∎

The formula (45) for $w(T)$ can be slightly simplified, since in certain circumstances we will have

$$B_{\lambda^i,\lambda^{i-1}}(s)=C_{\lambda^{i+1},\lambda^i}(s).$$

This will occur if s belongs to a column containing zero or two squares of $\lambda^{i+1}/\lambda^{i-1}$. We have circled in Figs. 4 and 5 those entries which are cancelled in this way. One can concoct certain additional cancellation rules which hold under rather specialized conditions, but in any event $w(T)$ will in general be a messy rational function of α.

An important corollary of Theorem 6.3 is the following strengthening of Propositions 4.1 and 5.3. An elegant proof based directly on Proposition 5.3 appears in [M_3, Chap. VI, (6.6)].

6.4. COROLLARY. *If $J_{\lambda/\mu}\neq 0$ then $\mu\subseteq\lambda$. Equivalently, if $\langle J_\lambda, J_\mu J_\nu\rangle\neq 0$ then $\mu\subseteq\lambda$ and $\nu\subseteq\lambda$.*

Proof. The tableau T of (44) will not exist unless $\mu\subseteq\lambda$. ∎

7. CONSEQUENCES

Many explicit formulas involving special Jack polynomials or special coefficients of Jack polynomials can be deduced from our previous results. Without attempting to be comprehensive, we give in this section a sampling of some of the more interesting results in this direction. We will express many of our formulas in terms of quantities like c_λ, d_λ, $v_{\lambda\lambda}$, and j_λ, which already have been explicitly evaluated.

7.1. PROPOSITION. *Let $\lambda=(\lambda_1,...,\lambda_l)$, with $\lambda_l>0$. Let $1\leqslant j\leqslant l$, and* *define* $\mu = (\lambda_1, ..., \lambda_j, 1, ..., 1)$, *where* $|\lambda| = |\mu|$ (*so* $l(\mu) = j + \lambda_{j+1} + \cdots + \lambda_l$). *Then*

$$v_{\lambda\mu} = d_{\lambda^1} d_{\lambda^2} \cdots d_{\lambda^j} (\lambda_{j+1} + \cdots + \lambda_l)!,$$

where $\lambda^i = (\lambda_i, \lambda_{i+1}, ..., \lambda_l)$.

Proof. We have

$$\begin{aligned} v_{\lambda\mu} &= [x_1^{\lambda_1} \cdots x_j^{\lambda_j} x_{j+1} \cdots x_{l(\mu)}] J_\lambda \\ &= d_{\lambda^1} [x_2^{\lambda_2} \cdots x_j^{\lambda_j} x_{j+1} \cdots x_{l(\mu)}] J_{\lambda^2} \qquad \text{(by (23))} \\ &= d_{\lambda^1} d_{\lambda^2} \cdots d_{\lambda^j} [x_{j+1} \cdots x_{l(\mu)}] J_{\lambda^{j+1}} \qquad \text{(by iterating (23))} \\ &= d_{\lambda^1} \cdots d_{\lambda^j} (\lambda_{j+1} + \cdots + \lambda_l)!, \end{aligned}$$

by property (P3). ∎

7.2. PROPOSITION. *The Jack symmetric function* $J_{2^i 1^j}$ *has the following expansion in terms of monomial symmetric functions,*

$$J_{2^i 1^j} = \sum_{r=0}^{i} (i)_r (\alpha + i + j)_r (2(i-r) + j)! \, m_{2^r 1^{2(i-r)+j}}, \tag{46}$$

where $(a)_b = a(a-1) \cdots (a-b+1)$ (*with* $(a)_0 = 1$, *even for* $a = 0$).

Proof. By property (P2) the only monomial symmetric functions appearing in $J_{2^i 1^j}$ are $m_{2^r 1^{2(i-r)+j}}$, $0 \leqslant r \leqslant i$. By Proposition 7.1,

$$v_{2^i 1^j, 2^r 1^{2(i-r)+j}} = d_{2^i 1^j} d_{2^{i-1} 1^j} \cdots d_{2^{i-r+1} 1^j} (2(i-r) + j)!.$$

From Proposition 5.7, one sees that

$$d_{2^a 1^b} = \begin{cases} a(a+b+\alpha), & a \neq 0 \\ b, & a = 0, \end{cases}$$

from which (46) follows. ∎

We conjecture that $J_{2^i 1^j}$ has the following expansion in terms of Schur functions:

$$J_{2^i 1^j} = \sum_{r=0}^{i} (i)_r (\alpha + i + j)_r (i - r - \alpha)_{i-r} (i + j - r)! \, s_{2^r 1^{2(i-r)+j}}. \tag{47}$$

It is easy to expand the Schur function $s_{2^r 1^{2(i-r)+j}}$ in terms of monomial symmetric functions. Substituting into (47) and comparing with (46) leaves a combinatorial identity to be verified, which should not be too difficult.

7.3. Proposition. *Let $|\lambda| = m$. Then*

$$v_{\lambda, 21^{m-2}}(\alpha) = (m-2)!\left[\binom{m}{2} - b(\lambda) + \alpha b(\lambda')\right],$$

where $b(\mu)$ is defined by (1).

Proof. Write $(n)_l = n(n-1)\cdots(n-l+1)$ as above, and set $v(\alpha) = v_{\lambda, 21^{m-2}}(\alpha)$. From (33) it follows that

$$\begin{aligned} J_\lambda(1^n; \alpha) &= \sum_\mu v_{\lambda\mu}(\alpha)(n)_{l(\mu)}/m_1(\mu)!\, m_2(\mu)! \\ &= m!\,(n)_m/m! + v(\alpha)(n)_{m-1}/(m-2)! + O(n^{m-2}) \\ &= n^m - \left(\binom{m}{2} - \frac{v(\alpha)}{(m-2)!}\right) n^{m-1} + O(n^{m-2}), \end{aligned} \tag{48}$$

since only the partition $\mu = (1^m)$ of m has m parts, and only $\mu = (21^{m-2})$ has $m-1$ parts. (Here $m_i(\mu)$ denotes the number of parts of μ equal to i.)

On the other hand, by Theorem 5.4 we have

$$\begin{aligned} J_\lambda(1^n; \alpha) &= n^m + \left(\sum_{(i,j)\in\lambda} (-(i-1) + \alpha(j-1))\right) n^{m-1} + O(n^{m-2}) \\ &= n^m - [b(\lambda) - \alpha b(\lambda')]n^{m-1} + O(n^{m-2}). \end{aligned} \tag{49}$$

Comparing coefficients of n^{m-1} in (48) and (49) completes the proof. ∎

A similar argument applies with power sum symmetric functions replacing monomial symmetric functions.

7.4. Proposition. *Let $|\lambda| = m$, and define $c_{\lambda\mu}(\alpha)$ by* (14). *Then*

$$c_{\lambda, 1^m}(\alpha) = 1$$
$$c_{\lambda, 21^{m-2}}(\alpha) = \alpha b(\lambda') - b(\lambda).$$

Proof. Since $p_\mu(1^m) = m^{l(\mu)}$, we get from (14), as in the proof of Proposition 7.3, that

$$n^m - [b(\lambda) - \alpha b(\lambda')]n^{m-1} + \cdots = c_{\lambda, 1^m}(\alpha)n^m + c_{\lambda, 21^{m-2}}(\alpha)n^{m-1} + \cdots.$$

Equate coefficients of n^m and n^{m-1} to complete the proof. ∎

When $\alpha = 1$, we have by [M_1, Chap. I, (7.5)] that

$$c_{\lambda\mu}(1) = H_\lambda z_\mu^{-1} \chi^\lambda(\mu), \tag{50}$$

where $\chi^\lambda(\mu)$ is the irreducible character χ^λ of the symmetric group S_m evaluated at a conjugacy class of type μ. There is a well-known combinatorial method due to Littlewood and Richardson (equivalent to the "Murnagham–Nakayama rule") [M_1, Chap. I, Ex. 7.5] for evaluating the characters $\chi^\lambda(\mu)$ of S_m. The basis for this rule is a formula for expanding $s_\lambda p_r$ in terms of Schur functions [M_1, Chap I, Ex. 3.11]. It is natural to ask whether these results extend to Jack symmetric functions. The answer seems to be negative; at any rate, it is false that if $\langle J_\mu p_r, J_\lambda \rangle \neq 0$, then $\lambda - \mu$ is a border strip (as is the case when $\alpha = 1$ by [M_1, Chap. I, Ex. 3.11]). For instance, $\langle J_1 p_2, J_{21} \rangle = 2\alpha^2(\alpha - 1)$.

It is also natural to ask whether the coefficients $c_{\lambda\mu}(\alpha)$ have a group-theoretic significance, as they do for $\alpha = 1$. For $\alpha = 2$, the theory of zonal polynomials [Jam_1, Thm. 4] provides an affirmative answer, and a similar result holds for $\alpha = \frac{1}{2}$[M_2]. In general, however, the question remains open. For a general conjecture involving $c_{\lambda\mu}(\alpha)$, see [H, p. 60].

Following [M_1, p. 3], write $(r \mid s)$ for the hook partition $(r + 1, 1^s)$. Our next result gives an explicit formula for $c_{\lambda\mu}(\alpha)$ when λ is the hook $(n - 1 \mid k)$.

7.5. Proposition. *Let*

$$J_\lambda = H_\lambda \sum_\mu z_\mu^{-1} \alpha^{n - l(\mu)} d_{\lambda\mu}(\alpha) p_\mu.$$

(Hence by (50) *we have* $d_{\lambda\mu}(1) = \chi^\lambda(\mu)$.) *Then*

$$\begin{aligned}\alpha^{k+1}(k+n)\, &d_{(n-1|k),\mu}(\alpha) \\ &= \sum_{j=0}^{k} (-1)^{k-j}\,(j + (n+k-j)\alpha) \sum_{\nu \vdash j} \left[\prod_i \binom{m_i(\mu)}{m_i(\nu)}\right] \varepsilon_\nu \alpha^{l(\nu)},\end{aligned} \tag{51}$$

where $\varepsilon_\nu = (-1)^{j - l(\nu)}$ *(and where* H_λ, z_μ, m_i *are as in Section* 1).

Proof. Letting $\mu = (1^k)$ in (38), we obtain

$$J_{1^k} J_n = \frac{1}{k + n\alpha}\,(n\alpha J_{(n-1|k)} + k J_{(n|k-1)}). \tag{52}$$

Now $H_{(n-1|k)} = (n+k)!/(n-1)!\,k!$ and $H_{(n|k-1)} = (n+k)!/n!\,(k-1)!$. Moreover, by Proposition 2.2(b) and Corollary 3.5 (or by [M_1, Chap. I, (2.14′)], we have

$$J_{1^k} = k!\,e_k = k! \sum_{\alpha \vdash k} \varepsilon_\alpha z_\alpha^{-1} p_\alpha. \tag{53}$$

Hence from (52), (53), and Proposition 2.2(b),

$$\left[\sum_{\beta\vdash k} \varepsilon_\beta z_\beta^{-1} p_\beta\right]\left[\sum_{\nu\vdash n} \alpha^{n-l(\nu)} z_\nu^{-1} p_\nu\right]$$

$$=\frac{k+n}{k+n\alpha}\left[\alpha \sum_{\mu\vdash n+k} d_{(n-1|k),\mu}\alpha^{n+k-l(\mu)} z_\mu^{-1} p_\mu\right.$$

$$\left.+\sum_{\rho\vdash n+k} d_{(n|k-1)}\alpha^{n+k-l(\rho)} z_\rho^{-1} p_\rho\right].$$

Equating coefficients of p_μ on both sides yields

$$\frac{k+n\alpha}{k+n}\left[\sum_{\substack{\beta\cup\gamma=\mu\\ \beta\vdash k\\ \gamma\vdash n}} \varepsilon_\beta z_\mu z_\beta^{-1} z_\gamma^{-1} \alpha^{l(\beta)-k}\right]=\alpha d_{(n-1|k),\mu}+d_{(n|k-1),\mu}. \tag{54}$$

Now from the definition (2) of z_λ we have

$$z_\mu z_\beta^{-1} z_\gamma^{-1}=\prod_i \binom{m_i(\mu)}{m_i(\beta)}. \tag{55}$$

Regarding (54) as a recurrence relation for $d_{(n-1|k),\mu}$ as a function of k with the initial condition $d_{(n+k|-1),\mu}=0$, we easily deduce (51) from (54) and (55). ∎

Note that when $\alpha=1$ in (51) the factor $k+n$ can be cancelled from both sides, and also that $(-1)^{k-j}\varepsilon_\nu=(-1)^{k-l(\nu)}$ (independent of j). Hence $d_{(n-1|k),\mu}(1)$ (i.e., $\chi^{(n-1|k)}(\mu)$) is a polynomial function $X_k(m_1, m_2, ..., m_k)$ of the variables $m_i(\mu)$, $1\leqslant i\leqslant k$. More generaly, for any partition λ of k there is a polynomial $X_\lambda(m_1, m_2, ..., m_k)$, called the *character polynomial*, such that $\chi^{n\cup\lambda}(\mu)=X_\lambda(m_1(\mu), ..., m_k(\mu))$ (see [S], and for a table see [K-T, pp. 288–312]). Proposition 7.5 shows that this result does not extend to Jack polynomials, since $d_{(n-1|k),\mu}(\alpha)$ is a function not only of $m_1(\mu), ..., m_k(\mu)$, but also of n. (This fact was already apparent for zonal polynomials from [D-L].) Some small values of $d_{(n-1|k),\mu}(\alpha)$ are given by

$$\alpha n d_{n,\mu}(\alpha)=n\alpha \qquad (\text{i.e., } d_{n,\mu}(\alpha)=1)$$

$$\alpha^2(n+1)\, d_{(n-1|1),\mu}(\alpha)=-(n+1)\alpha+\alpha(1+n\alpha)\, m_1(\mu)$$

$$\alpha^3(n+2)\, d_{(n-1|2),\mu}(\alpha)=(n+2)\alpha-\alpha(1+(n+1)\alpha)\, m_1(\mu)$$

$$+\alpha^2(2+n\alpha)\binom{m_1(\mu)}{2}-\alpha(2+n\alpha)\, m_2(\mu).$$

Other formulas for J_λ when λ is a hook appear in [M_3, Chap. VI, Ex. 10.5 and 10.6]. The second of these formulas was originally conjectured by Hanlon [H, Property 2].

We conclude this section with two specializations of $J_\lambda(x;\alpha)$ in addition to those of Proposition 1.2. This result was earlier proved by Macdonald [M_3, Chap. VI, (4.12)(v)–(vi)]. (The expression $J_\lambda(x;0)$ appearing in Proposition 7.6 makes no sense from the point of view of the definition of $J_\lambda(x;\alpha)$ provided by Theorem 1.1 (since the scalar product $\langle\ ,\ \rangle$ is degenerate when $\alpha=0$). However, since the coefficients $v_{\lambda\mu}(\alpha)$ of (4) are rational functions of α it makes sense to set $J_\lambda(x;0)=\sum v_{\lambda\mu}(0)m_\mu$. Conceivably some $v_{\lambda\mu}(\alpha)$ could become infinite when $\alpha=0$, but Proposition 7.6 shows in particular that this is not the case.)

7.6. PROPOSITION. *Let $|\lambda|=m$. Then we have*

$$J_\lambda(x;0)=\left(\prod_i \lambda_i'!\right)e_{\lambda'}(x), \tag{56}$$

$$\alpha^{m-l(\lambda)}J_\lambda(x;1/\alpha)\,|_{\alpha=0}=\left(\prod_{i=1}^{l(\lambda)}(\lambda_i-1)!\right)\left(\prod_{i\geqslant 1} m_i(\lambda)!\right)m_\lambda. \tag{57}$$

Sketch of proof. Let $D(\alpha)$ be the differential operator of (11). One checks that

$$D(0)\,e_{\lambda'}(x)=((n-1)\,|\lambda|-b(\lambda))\,e_{\lambda'}(x).$$

Hence by Theorem 3.1 we have that $J_\lambda(x;0)$ is a $\mathbb{Q}$-linear combination of $e_{\mu'}(x)$'s, where $|\mu|=|\lambda|$ and $b(\mu)=b(\lambda)$. It is easy to see (a slight refinement of Lemma 3.1) that if $|\mu|=|\lambda|$ and $b(\mu)=b(\lambda)$, then λ and μ are incomparable in dominance order. Moreover, the monomial m_μ occurs in $e_{\mu'}$, and if m_ν occurs then $\nu\leqslant\mu$. From this and (P2) we conclude that $J_\lambda(x;0)$ is a $\mathbb{Q}$-multiple of $e_{\lambda'}(x)$. The factor $\prod(\lambda_i'!)$ can be obtained most easily by comparing coefficients of $x_1x_2\cdots x_m$, where $|\lambda|=m$. This proves (56).

To show (57), write $w(T,\alpha)$ for the expression in (45). It is not difficult to verify that the degree of $w(T,\alpha)$ as a rational function of α (i.e., the degree of the numerator minus the degree of the denominator) is given by

$$\deg w(T,\alpha)=|\mu|-l(\mu)+|\lambda|-(\text{the number of entries } i \text{ of } T \text{ such that if this entry } i \text{ occurs in column } j, \text{ then no } i \text{ occurs in column } j+1).$$

(Note also that $|\mu|-l(\mu)+|\lambda|=\deg j_\mu+|\lambda/\mu|$.) It follows that if $\mu=\varnothing$ in (44), then

$$\alpha^{m-l(\lambda)}w(T,1/\alpha)\,|_{\alpha=0}=0$$

unless the type of T is a permutation of the parts of λ. Hence the left-hand side of (57) is a $\mathbb{Q}$-multiple $f_\lambda(\alpha)$ of m_λ. The factor $f_\lambda(\alpha)$ can be obtained directly from (45), or more easily by noting that

$$f_\lambda(\alpha)=\alpha^{m-l(\lambda)}v_{\lambda\lambda}(1/\alpha)\,|_{\alpha=0}$$

and applying Theorem 5.6. ▮

8. Open Problems

Many conjectures are suggested by our previous results and by empirical evidence. We collect the most attractive ones in this section. The first conjecture is due to I. G. Macdonald.

8.1. Conjecture. *Let*

$$\tilde{v}_{\lambda\mu}(\alpha)=v_{\lambda\mu}(\alpha)\Big/\prod_{i\geqslant 1}m_i(\lambda)!,$$

where $v_{\lambda\mu}(\alpha)$ *is given by* (4). *Then* $\tilde{v}_{\lambda\mu}(\alpha)$ *is a polynomial with nonnegative integer coefficients.* ▮

It is not even known whether $\tilde{v}_{\lambda\mu}(\alpha)$ (or $v_{\lambda\mu}(\alpha)$) is a polynomial. Some special cases of Conjecture 8.1 follow from Theorem 5.6 and Propositions 7.1–7.3.

The following conjecture is a consequence of Conjecture 8.1.

8.2. Conjecture. *The coefficients* $c_{\lambda\mu}(\alpha)$ *of* (14) *are polynomials with integer coefficients.* ▮

Proposition 7.5 shows that Conjecture 8.2 is true when λ is a hook. It should not be difficult to verify Conjecture 8.1 when λ is a hook using Proposition 7.5.

We have one additional conjecture analogous to the previous two.

8.3. Conjecture. *For fixed* λ, μ, *and* ν, *the quantity* $\langle J_\mu J_\nu, J_\lambda\rangle$ *is a polynomial in* α *with nonnegative integer coefficients.* ▮

It is not even known whether $\langle J_\mu J_\nu, J_\lambda\rangle$ is a polynomial in α. Of course Theorem 6.1 implies Conjecture 8.3 when $\nu=(n)$. The following conjecture is a consequence of Conjecture 8.3.

8.4. CONJECTURE. *We have*

$$\langle J_\mu J_\nu, J_\lambda \rangle \neq 0 \Leftrightarrow \langle s_\mu s_\nu, s_\lambda \rangle_1 \neq 0,$$

where $\langle \ , \ \rangle_1$ *denotes the case* $\alpha = 1$ *of* (3). ▮

A. Garsia has shown (private communication) that if $\langle J_\mu J_\nu, J_\lambda \rangle_{\alpha=2} \neq 0$, then $\langle s_{2\mu} s_{2\nu}, s_{2\lambda} \rangle_1 \neq 0$.

In view of Theorem 6.1 and the Littlewood–Richardson rule for Schur functions, it is natural to ask for a combinatorial interpretation of $\langle J_\mu J_\nu, J_\lambda \rangle$ in general. We have only been able to find a conjecture for this value in the following special case.

8.5. CONJECTURE. *If* $\langle s_\mu s_\nu, s_\lambda \rangle_1 = 1$, *then*

$$\langle J_\mu J_\nu, J_\lambda \rangle = \left(\prod_{s \in \mu} \tilde{h}_\mu(s) \right) \left(\prod_{s \in \nu} \tilde{h}_\nu(s) \right) \left(\prod_{s \in \lambda} \tilde{h}_\lambda(s) \right), \tag{58}$$

where $\tilde{h}_\sigma(s) = h^*_\sigma(s)$ *or* $h^\sigma_*(s)$. *Moreover, one chooses* $h^*_\sigma(s)$ *and* $h^\sigma_*(s)$ *exactly* $|\lambda|$ *times each in* (58). ▮

We do not have an explicit conjecture as to when to choose $h^*_\sigma(s)$ and $h^\sigma_*(s)$. One problem with coming up with an explicit conjecture from empirical data is that the possible choices for $\tilde{h}_\sigma(s)$ are not unique. For instance, we have (computed by P. Hanlon)

$$\langle J_{211} J_{211}, J_{22211} \rangle = 384\alpha^5 (1+\alpha)^3 (4+\alpha)(5+\alpha).$$

Figure 6 shows the possible choices for $\tilde{h}_\sigma(s)$ consistent with Conjecture 8.5, for each square of 211, 211, and 22211. Of the 16 different squares, the choice for $\tilde{h}_\sigma(s)$ is uniquely determined for 10 of them. However, there are

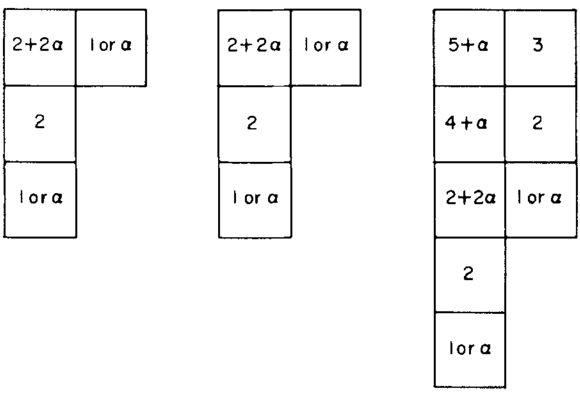

FIGURE 6

six squares s for which $h_\sigma^*(s)=\alpha$ and $h_*^\sigma(s)=1$. We must choose $\tilde{h}_\sigma(s)=\alpha$ for exactly five of these squares, but from the data alone there is no "correct" choice of these five squares.

We can prove Conjecture 8.5 in a special case generalizing Theorem 6.1. To state this result, we use the description of the Littlewood–Richardson rule in [M_1, Chap. I, (9.2)]; viz., $c_{\mu\nu}^\lambda := \langle s_\mu s_\nu, s_\lambda \rangle_1$ is equal to the number of tableaux T of shape $\lambda-\mu$ and weight ν such that $w(T)$ is a lattice permutation. Let us call such a tableau T an $L-R$ *tableaux* (of shape $\lambda-\mu$ and weight ν).

8.6. PROPOSITION. *Suppose that $c_{\mu\nu}^\lambda=1$, and that in addition the unique $L-R$ tableaux T of shape $\lambda-\mu$ and weight ν has the property that every column C of T consists of the integers $1, 2, ..., n_C$ (for some $n_C \geqslant 0$ depending on C). (Equivalently, for each i the number of columns of $\lambda-\mu$ of length i is equal to $\nu_i-\nu_{i+1}$.) Then Eq. (57) holds for the following values of $\tilde{h}_\sigma(s)$:*

(a) *Let $s=(i,j)\in\lambda$. Let r_i be the largest entry of T in row i of $\lambda-\mu$, and let c_j be the largest entry of T in column j of $\lambda-\mu$. Set $r_i=0$ (respectively, $c_j=0$) if row i (respectively, column j) of $\lambda-\mu$ is empty. Then*

$$\tilde{h}_\lambda(s)=\begin{cases} h_*^\lambda(s), & \text{if } r_i \leqslant c_j \text{ and } c_j>0 \\ h_\lambda^*(s), & \text{if } r_i>c_j, \text{ or if } r_i=c_j=0. \end{cases}$$

(b) *Let $s=(i,j)\in\mu$. Then*

$$\tilde{h}_\mu(s)=\begin{cases} h_*^\mu(s), & \text{if } \tilde{h}_\lambda(i+c_j, j)=h_\lambda^*(i+c_j, j) \\ h_\mu^*(s), & \text{if } \tilde{h}_\lambda(i+c_j, j)=h_*^\lambda(i+c_j, j). \end{cases}$$

(c) *Let $s\in\nu$. Then*

$$\tilde{h}_\nu(s)=h_\nu^*(s).$$

Sketch of proof. Let $l=l(\nu)$. Consider the product

$$J_\mu J_{\nu_1} J_{\nu_2} \cdots J_{\nu_l}=J_\mu \mathscr{J}_\nu,$$

where $\mathscr{J}_\nu$ is defined preceding Proposition 2.4. Now from Proposition 2.4 and (P2) we have that if $\langle \mathscr{J}_\nu, J_\rho \rangle \neq 0$ then $\nu \leqslant \rho$. We also have from the hypothesis on μ, ν, and λ that if $\langle J_\mu J_\rho, J_\lambda \rangle \neq 0$ then $\rho \leqslant \nu$. Hence if $q_{\nu\nu}$ is defined by (10), then

$$\langle J_\mu \mathscr{J}_\nu, J_\lambda \rangle = q_{\nu\nu} \langle J_\mu J_\nu, J_\lambda \rangle. \tag{59}$$

The left-hand side of (59) can be computed as follows. By the hypothesis on μ, ν, and λ, there is only *one* way to adjoin to μ a horizontal strip of length ν_1, then of length ν_2, up to length ν_l, which yields the shape λ. Thus let $\lambda[i]$ be the unique partition satisfying

$$\langle J_\mu J_{\nu_1} \cdots J_{\nu_i}, J_{\lambda[i]} \rangle \neq 0,$$

and

$$\langle J_{\lambda[i]} J_{\nu_{i+1}} \cdots J_{\nu_l}, J_\lambda \rangle \neq 0.$$

Then it follows that

$$\langle J_\mu \mathscr{J}_\nu, J_\lambda \rangle = \frac{\langle J_{\lambda[0]} J_{\nu_1}, J_{\lambda[1]} \rangle \langle J_{\lambda[1]} J_{\nu_2}, J_{\lambda[2]} \rangle \cdots \langle J_{\lambda[l-1]} J_{\nu_l}, J_{\lambda[l]} \rangle}{j_{\lambda[1]} j_{\lambda[2]} \cdots j_{\lambda[l-1]}}. \quad (60)$$

Every factor appearing on the right-hand side of (60) can be computed using Theorems 5.8 and 6.1. Moreover, $q_{\nu\nu}$ from (59) can be computed by Proposition 2.4 and Theorem 5.8. Hence $\langle J_\mu J_\nu, J_\lambda \rangle$ is expressed as a quotient of two products. By keeping careful track of cancellations, the desired result follows. ∎

Example. Let $\lambda = (7, 6, 6, 6, 5, 4, 2, 1, 1)$, $\mu = (7, 5, 5, 3, 3, 2, 2)$, $\nu = (5, 4, 2)$. Figure 7 shows the unique $L-R$ tableau of shape $\lambda - \mu$ and weight ν. Figure 8 shows in each square s of λ and μ the letter U or L, depending on whether $\tilde{h}_\sigma(s)$ equals $h^*_\sigma(s)$ or $h^\sigma_*(s)$, respectively.

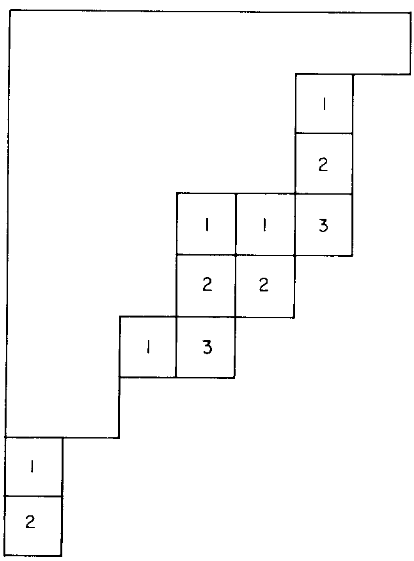

FIGURE 7

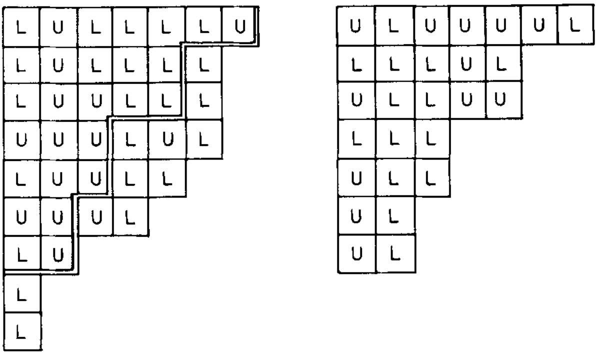

FIGURE 8

There are some additional small cases for which we can verify Conjecture 8.5. For instance, we can prove it whenever $\nu = (2, 1)$ (provided of course $c^{\lambda}_{\mu\nu} = 1$) using the formula

$$\langle J_\mu J_{21}, J_\lambda \rangle = \frac{j_{21}}{\langle J_2 J_1, J_{21} \rangle} \left[\langle J_\mu J_2 J_1, J_\lambda \rangle - \frac{\langle J_2 J_1, J_3 \rangle}{j_3} \langle J_\mu J_3, J_\lambda \rangle \right].$$

The details are rather messy.

We can try to extend Conjecture 8.5 to *any* $\langle J_\mu J_\nu, J_\lambda \rangle$. One possibility (consistent with the case $\alpha = 1$) is the following: $\langle J_\mu J_\nu, J_\lambda \rangle$ can be written as a sum of $c^{\lambda}_{\mu\nu}$ expressions of the form (58), each possibly multiplied by a power of α. Unfortunately this conjecture is *false* for the case $\mu = (2, 1)$, $\nu = (3, 1)$, $\lambda = (4, 2, 1)$. Here we have (computed by P. Hanlon)

$$\begin{aligned} \langle J_{21} J_{31}, J_{421} \rangle &= 8\alpha^5(9 + 97\alpha + 294\alpha^2 + 321\alpha^3 + 131\alpha^4 + 12\alpha^5) \\ &= 8\alpha^5 f(\alpha), \text{ say.} \end{aligned}$$

The polynomial $f(\alpha)$ has no rational zeros. Moreover, $c^{421}_{21,31} = 2$. One can check that any two expressions (58), say $f_1(\alpha)$ and $f_2(\alpha)$, have a common linear factor $L(\alpha) \neq \alpha$ or a common integer factor $L(\alpha)$ not dividing 8. Hence $\alpha^r f_1(\alpha) + \alpha^s f_2(\alpha)$ is divisible by $L(\alpha)$ and so cannot equal $8\alpha^5 f(\alpha)$.

Note added in proof. Equation (47) has been proved by K. Koike. The result of Garsia mentioned after Conjecture 8.4 appears in A. M. Garsia and N. Bergeron, Zonal polynomials and domino tableaux, preprint.

ACKNOWLEDGMENTS

I am indebted to Persi Diaconis for introducing me to zonal symmetric functions and suggesting them as objects for further study from a combinatorial viewpoint. I am also grateful to Ian Macdonald for explaining to me much of his work and for many valuable suggestions, and to Phil Hanlon for many helpful discussions and computations.

REFERENCES

[C] A. G. CONSTANTINE, Some noncentral distribution problems in multivariate analysis, *Ann. Math. Statist.* **34** (1963), 1270–1285.

[D-L] P. DIACONIS AND E. LANDER, Some formulas for zonal polynomials, in preparation.

[F] H. O. FOULKES, A survey of some combinatorial aspects of symmetric functions, *in* "Permutations," Gauthier-Villars, Paris, 1974.

[H] P. HANLON, Jack symmetric functions and some combinatorial properties of Young symmetrizers, *J. Combin. Theory Ser. A* **47** (1988), 37–70.

[Jac] H. JACK, A class of symmetric polynomials with a parameter, *Proc. Roy. Soc. Edinburgh Sect. A* **69** (1969–1970), 1–17.

[Jam_1] A. T. JAMES, Zonal polynomials of the real positive definite symmetric matrices, *Ann. of Math.* **74** (1961), 456–469.

[Jam_2] A. T. JAMES, Distributions of matrix variables and latent roots derived from normal samples, *Ann. Math. Statist.* **35** (1964), 475–501.

[Jam_3] A. T. JAMES, Calculation of zonal polynomial coefficients by use of the Laplace–Beltrami operator, *Ann. Math. Statist.* **39** (1968), 1711–1718.

[K] H. B. KUSHNER, The linearization of the product of two zonal polynomials, preprint.

[K-T] A. KERBER AND K.-J. THÜRLINGS, "Symmetrieklassen von Funktionen und ihre Abzählungstheorie," Teil II, Bayreuther Mathematische Schriften, Heft 15, 1983.

[L] D. E. LITTLEWOOD, "The Theory of Group Characters," 2nd ed., Oxford Univ. Press, Oxford, 1950.

[M_1] I. G. MACDONALD, "Symmetric Functions and Hall Polynomials," Oxford Univ. Press, Oxford, 1979.

[M_2] I. G. MACDONALD, Commuting differential equations and zonal spherical functions, *in* "Algebraic Groups, Utrecht 1986" (A. M. Cohen *et al.*, Eds.), Lecture Notes in Math., Vol. 1271, pp. 189–200, Springer-Verlag, Berlin/Heidelberg/New York, 1987.

[M_3] I. G. MACDONALD, [M_1, 2nd ed.], to appear.

[S] W. SPECHT, Die Characktere der symmetrichen Gruppe, *Math. Zeit.* **73** (1960), 312–329.

[T] A. TAKEMURA, "Zonal Polynomials," Institute of Mathematical Statistics, Lecture Notes-Monograph Series (S. S. Gupta, Series Ed.), Vol. 4, Hayward, CA, 1984.