A Bound on the Spectral Radius of Graphs with e Edges

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ABSTRACT

The spectral radius $\rho(A)$ of the adjacency matrix A of a graph G with e edges satisfies $\rho(A) \leqslant \frac{1}{2}(-1+\sqrt{1+8e})$. Equality occurs if and only if $e = \binom{k}{2}$ and G is a disjoint union of the complete graph K_k and isolated vertices.

Let A be a symmetric $(0,1)$ matrix with zero trace (i.e., the adjacency matrix of a graph G). Let the number of 1's of A be $2\binom{k}{2}$ (so G has $\binom{k}{2}$ edges). R. A. Brualdi and A. J. Hoffmann [1, Theorem 2.2] showed that the spectral radius $\rho(A)$ satisfies $\rho(A) \leq k-1$, with equality if and only if there exists a permutation matrix P such that PAP^T has the form

$$\begin{bmatrix} J_k^0 & 0 \\ 0 & 0 \end{bmatrix}, \tag{1}$$

where J_k^0 is the $k \times k$ matrix with 0's on the main diagonal and 1's elsewhere. (In other words, G is isomorphic to the disjoint union of the complete graph K_k and isolated vertices.) Here we obtain a bound on the spectral radius of any graph with e edges, which implies the Brualdi-Hoffman bound when $e = \binom{k}{2}$. We also obtain the conditions for equality. Our proofs are simpler than those of Brualdi and Hoffman.

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THEOREM. *Let $A=(a_{ij})$ be a symmetric* (0,1) *matrix with zero trace. Let the number of* 1's *of A be* $2e$. *Then*

$$\rho(A) \leq \tfrac{1}{2}(-1+\sqrt{1+8e}). \tag{2}$$

Equality holds if and only if

$$e=\binom{k}{2}$$

and PAP^T has the form (1) *for some permutation matrix P.*

Proof. Let A_i denote the ith row of A, and r_i the ith row sum. Let $x=(x_1,\dots,x_n)^T$ be an eigenvector of A of length one corresponding to the eigenvalue $\rho(A)$. Let $x(i)$ denote the vector obtained from x by replacing x_i with 0. Since $Ax=\rho(A)x$, we have $A_i x=\rho(A)x_i$. Since the diagonal elements of A are 0, we have $A_i x = A_i x(i)$. Hence, by the Cauchy-Schwartz inequality,

$$\begin{aligned}\rho(A)^2 x_i^2 = |A_i x(i)|^2 &\leq |A_i|^2 \cdot |x(i)|^2 \\ &= r_i(1-x_i^2).\end{aligned}$$

Sum on i to obtain

$$\rho(A)^2 \leq 2e-\sum r_i x_i^2. \tag{3}$$

Now

$$\begin{aligned}\sum r_i x_i^2 &= \sum_{i,j} x_i^2 a_{ij} \\ &= \sum_{i<j}\left(x_i^2+x_j^2\right)a_{ij} \\ &\geqslant \sum_{i<j} 2x_i x_j a_{ij} \\ &= \sum_{i,j} x_i a_{ij} x_j \\ &= x^T A x \\ &= \rho(A).\end{aligned} \tag{4}$$

Hence, from (3),

$$\rho(A)^2 \leq 2e - \rho(A),$$

which implies (2).

In order for equality to hold in (2), all inequalities in the above argument must be equalities. In particular, from (4) we have

$$\left(x_i^2 + x_j^2\right)a_{ij} = 2x_i x_j a_{ij}$$

for all $i < j$. Hence either $a_{ij} = 0$ or $x_i = x_j$. Thus, choosing P so that Px has the form

$$Px = \left(y_1, y_1, \dots, y_1, y_2, y_2, \dots, y_2, \dots, y_j, y_j, \dots, y_j\right)$$

where $y_1, y_2, \dots, y_j$ are distinct, it follows that PAP^T has block diagonal form,

$$PAP^T = \begin{pmatrix} B_1 & & & 0 \\ & B_2 & & \\ & & \ddots & \\ 0 & & & B_j \end{pmatrix},$$

where each B_i has an eigenvector $(1, 1, \dots, 1)^T$. Hence each B_i has equal row sums, so $\rho(A)$ is the maximum row sum of A. Therefore $\sqrt{1+8e}$ is an integer, so

$$e = \binom{k}{2}.$$

Then $\rho(A) = k - 1$, and it follows easily that there is one nonzero block $B_1 = J_k^0$. This completes the proof. ∎

REFERENCES

1. R. A. Brualdi and A. J. Hoffman, On the spectral radius of (0, 1)-matrices, *Linear Algebra Appl.* 65:133–146 (1985).

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