A CHROMATIC-LIKE POLYNOMIAL FOR ORDERED SETS

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This paper surveys some results appearing in a section of the author's doctoral dissertation [4, Ch. IV, Section 5]. For further details, generalizations, and applications, see [4].

Let P be a finite (partially) ordered set with $p > 0$ elements and longest chain of length ℓ (or cardinality $\ell + 1$). A chain (totally ordered set) with p elements is denoted C_p.

Definition (i) A map $\sigma : P \longrightarrow C_n$ is said to be order-preserving if $X \leq Y \Rightarrow \sigma(X) \leq \sigma(Y)$. Define $\Omega(n)$ to be the number of order-preserving maps $\sigma : P \longrightarrow C_n$.

(ii) A map $\tau : P \longrightarrow C_n$ is said to be strictly order-preserving if $X < Y \Rightarrow \tau(X) < \tau(Y)$. Define $\overline{\Omega}(n)$ to be the number of strict order-preserving maps $\tau : P \longrightarrow C_n$.

(iii) e_n denotes the number of surjective order-preserving maps $\sigma : P \longrightarrow C_n$.

(iv) $\overline{e}_n$ denotes the number of surjective strict order-preserving maps $\tau : P \longrightarrow C_n$.

For instance, if $P = C_p$, then $\Omega(n) = \binom{n+p-1}{p}$ and $\overline{\Omega}(n) = \binom{n}{p}$; while if P is a disjoint union of p points, then $\Omega(n) = \overline{\Omega}(n) = n^p$. For any P, the number $e_p = \overline{e}_p$ is equal to the number of ways of extending P to a total order and is an important numerical invariant of P. It is not hard to see that $\Omega(n)$ is equal to the number of semi-ideals in the direct product $P \times C_{n-1}$ (see [1] for definitions). In particular, $\Omega(1) = 1$ and $\Omega(2)$ is the number of semi-ideals of P.

Theorem 1. $\Omega(n)$ and $\overline{\Omega}(n)$ are polynomials in n of degree p and leading coefficient $e_p/p!$ given by

$$\Omega(n) = \sum_{s=1}^{p} e_s \binom{n}{s}$$

$$\overline{\Omega}(n) = \sum_{s=1}^{p} \overline{e_s} \binom{n}{s}$$

Proof. For each of the $\binom{n}{s}$ subsets S of C_n of size s, there are e_s (resp. $\overline{e}_s$) order-preserving (resp. strict order-preserving) maps of P onto S, and the theorem follows. □

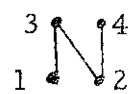
In the language of the calculus of finite differences,

$$e_s = \Delta^s \, \Omega(0) \quad ,$$

$$\overline{e_s} = \Delta^s \, \overline{\Omega}(0) \quad .$$

The polynomial $\overline{\Omega}(n)$ is an ordered set analog of the chromatic polynomial of a graph. $\overline{\Omega}(n)$ counts the number of ways of "coloring" P with the colors $1, 2, \ldots, n$ such that no two comparable elements of P have the same color, and such that this coloring is "compatible" with the ordering of P. One point at which the analogy breaks down is that the coefficients of $\overline{\Omega}(n)$ need not alternate in sign, the smallest such P having five elements.

We now come to the crucial lemma (whose proof will not be given here) in analyzing the polynomials $\Omega(n)$ and $\overline{\Omega}(n)$. Let ω be any surjective order-preserving map $P \longrightarrow \{1, 2, \ldots, p\}$, i.e., ω is an extension of P to a total order. We denote the elements of P by $X_1, \ldots, X_p$, where $\omega(X_i) = i$. List all permutations $i_1, i_2, \ldots, i_p$ of $1, 2, \ldots, p$ with the property that if $X<Y$ in P, then $\omega(X)$ appears before $\omega(Y)$ in $i_1, i_2, \ldots, i_p$. There are e_p such permutations. Put a "$\leq$" between two consecutive terms i_j and i_{j+1} if $i_j \leq i_{j+1}$; otherwise put a "$<$" sign. Denote the array thus obtained by $\mathcal{L}$. Denote by $\bar{\mathcal{L}}$ the array obtained from $\mathcal{L}$ by changing all "$<$" signs to "$\leq$" signs and "$\leq$" signs to "$<$" signs. We say a map $\sigma : P \longrightarrow C_n$ is <u>compatible</u> with a permutation $i_1, \ldots, i_p$ appearing in $\mathcal{L}$ (or $\bar{\mathcal{L}}$) if $\sigma(X_{i_1}) \leq \sigma(X_{i_2}) \leq \ldots \leq \sigma(X_{i_p})$ and $\sigma(X_{i_j}) < (X_{i_{j+1}})$ whenever a "$<$" sign appears in ~~$\mathcal{L}$ (or $\bar{\mathcal{L}}$)~~ this permutation between i_j and i_{j+1}.

<u>Example</u>: Let P and ω be given by 

. Then $\mathcal{L}$ and $\bar{\mathcal{L}}$ are given by

$\mathcal{L}$	$\bar{\mathcal{L}}$
$1 \leq 2 \leq 3 \leq 4$	$1 < 2 < 3 < 4$
$2 < 1 \leq 3 \leq 4$	$2 \leq 1 < 3 < 4$
$1 \leq 2 \leq 4 < 3$	$1 < 2 < 4 \leq 3$
$2 < 1 \leq 4 < 3$	$2 \leq 1 < 4 \leq 3$
$2 \leq 4 < 1 \leq 3$	$2 < 4 \leq 1 < 3$

<u>Lemma</u> (i) Every order-preserving map $\sigma : P \longrightarrow C_n$ is compatible with exactly one permutation in $\mathcal{L}$.

(ii) Every strict order-preserving map $\tau : P \longrightarrow C_n$ is compatible with exactly one permutation in $\bar{\mathcal{L}}$. □

Thus we obtain alternative expressions for $\Omega(n)$ and $\bar{\Omega}(n)$ by summing the contributions coming from each permutation in $\mathcal{L}$ and $\bar{\mathcal{L}}$. If exactly s "$<$" signs appear in a given permutation, then this permutation is easily seen to contribute a term $\binom{n+p-1-s}{p}$ to $\Omega(n)$ or $\bar{\Omega}(n)$. Thus by the lemma, we obtain

Theorem 2 : Let w_s (resp. $\bar{w}_s$) be the number of permutations in $\mathcal{L}$ (resp. $\bar{\mathcal{L}}$) with exactly s "<" signs. Then

$$\Omega(n) = \sum_{s=0}^{p-1} w_s \binom{p+n-1-s}{p}$$

$$\bar{\Omega}(n) = \sum_{s=0}^{p-1} \bar{w}_s \binom{p+n-1-s}{p} \quad . \ \square$$

But clearly $\bar{w}_s = w_{p-1-s}$. Substituting into Theorem 2 and comparing the resulting expression for $\bar{\Omega}(n)$ with the expression for $\Omega(n)$, we obtain the following fundamental result.

Theorem 3 : $\bar{\Omega}(n) = (-1)^p \, \Omega(-n)$.

The numbers w_s are natural generalizations of the Eulerian numbers [3, pp214-215] . When P is a disjoint union of p points, then w_s is equal to the number of permutations of $1, 2, \ldots, p$ with exactly s decreases between consecutive terms. This is the combinatorial definition of the Eulerian numbers $A_{p,s+1}$. We also have the generating functions

$$\sum_{n=0}^{\infty} \Omega(n)\, x^n = \left(\sum_{s=0}^{p-1} w_s\, x^{s+1}\right) \Big/ (1-x)^{p+1}$$

$$\sum_{n=0}^{\infty} \bar{\Omega}(n)\, x^n = \left(\sum_{s=0}^{p-1} \bar{w}_s\, x^{s+1}\right) \Big/ (1-x)^{p+1}$$

Theorem 3 allows the determination of all integer zeros of $\Omega(n)$. We state a slightly stronger result.

Corollary 1 We have $\Omega(0) = \Omega(-1) = \ldots = \Omega(-\ell) = 0$, while for $n > 0$,

$$(-1)^p \Omega(-\ell - n) \geq \Omega(n) > 0 \quad .$$

One can ask when equality holds in the inequality at the end of Corollary 1. A complete answer is provided by the following two theorems. They are proved by constructing in an obvious way a strict order-preserving map $\tau : P \longrightarrow C_{n+\ell}$ corresponding to a given order-preserving map $\sigma : P \longrightarrow C_n$, and analyzing when this correspondence is bijective.

Theorem 4. $\Omega(-\ell - 1) = (-1)^p$ if and only if every element of P is contained in a chain of length ℓ . □

Theorem 5 . The following three conditions are equivalent.

(i) $\Omega(-\ell - n) = (-1)^p \Omega(n)$ for some integer $n > 1$.

(ii) $\Omega(-\ell - n) = (-1)^p \Omega(n)$ for all n .

(iii) Every maximal chain of P has length ℓ .

It is not difficult to find ordered sets satisfying the conditions of Theorem 4 but not of Theorem 5. There are five such non-isomorphic ordered sets with six elements and none smaller. Theorem 5 leads to some interesting identities which appear to be difficult to prove by purely combinatorial reasoning.

Corollary 2. If every maximal chain of P has length ℓ , then

(i) $2e_{p-1} = (p+\ell-1)\, e_p$

(ii) $2\overline{e}_{p-1} = (p-\ell-1)\, e_p$

(iii) The coefficient of n^{p-1} in $\Omega(n)$ is $\ell e_p / 2(p-1)!$.

(iv) $\sum_{s=1}^{p} e_s = 2^{\ell} \sum_{s=1}^{p} \overline{e}_s$.

Proof By Theorem 5, we have

$$\Omega(n) = \sum_{s=1}^{p} e_s \binom{n}{s} = (-1)^p \sum_{s=1}^{p} e_s \binom{-\ell-n}{s}$$

Equating coefficients of n^{p-1} gives (i) while (ii) is proved similarly using $\bar{\Omega}(n)$. (iii) is then an immediate consequence of (i). We omit the proof of (iv) which involves a somewhat more complicated manipulation. □

As a consequence of formula (i) or (ii) of the previous corollary, we get a curious though not very significant result. I have been unable to find a direct combinatorial proof of this fact.

Corollary 3 If every maximal chain of P has length ℓ, then either $p-\ell$ is odd or e_p is even.

The preceding corollary motivates the following conjecture: Let P be any finite ordered set. If the length of every maximal chain of P has the same parity as p, then e_p is even.

In conclusion we mention that various methods are available for explicitly determining $\Omega(n)$ for special classes of ordered sets P. For instance, one of the more interesting such classes consists of those P which are the direct product of two chains, say $P = C_r \times C_s$. It can then be shown that

$$\Omega(n) = \frac{\binom{r+n-1}{r}\binom{r+n}{r} \cdots \binom{r+s+n-2}{r}}{\binom{r}{r}\binom{r+1}{r} \cdots \binom{r+s-1}{r}}$$

This formula is closely related to MacMahon's solution of the "generalized ballot problem". [2, Section 103].

References

1. Garrett Birkhoff, Lattice Theory, 3rd ed., American Mathematical Society (Providence, 1967)

2. P. A. MacMahon, Combinatory Analysis, Cambridge (1916), reprinted by Chelsea (1960)

3. John Riordan, An Introduction to Combinatorial Analysis, John Wiley and Sons, Inc. (New York, 1958)

4. Richard Stanley, Ordered Structures and Partitions, Ph. D. dissertation, Harvard University, 1970 or 1971