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THE NUMBER OF FACES OF BALANCED COHEN—MACAULAY COMPLEXES AND A GENERALIZED MACAULAY THEOREM

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A Cohen—Macaulay complex is said to be balanced of type $a=(a_1, a_2, ..., a_s)$ if its vertices can be colored using s colors so that every maximal face gets exactly a_i vertices of the i:th color. For $b=(b_1, b_2, ..., b_s)$, $0\leqq b\leqq a$, let f_b denote the number of faces having b_i vertices of the i:th color. Our main result gives a characterization of the f-vectors $f=(f_b)_{0\leqq b\leqq a}$ or equivalently the h-vectors, which can arise in this way from balanced Cohen—Macaulay complexes. As part of the proof we establish a generalization of Macaulay's compression theorem to colored multicomplexes. Finally, a combinatorial shifting technique for multicomplexes is used to give a new simple proof of the original Macaulay theorem and another closely related result.

1. Introduction

The purpose of this paper is to obtain information about the number of faces of finite simplicial complexes satisfying certain algebraic or combinatorial conditions. These so called "Cohen—Macaulay" and "shellable" complexes have previously been studied from this point of view by Stanley [8, 10]. The main contribution of this paper is the achievement of a complete characterization of the number arrays which can occur in a refined face number count of so called "balanced" such complexes. The necessity of this characterization was already proved in [10] using methods from commutative algebra. To prove the sufficiency we develop some combinatorial tools, mainly a generalization of the compression method of Macaulay to colored multicomplexes. We now proceed to give a statement of the main result followed by a discussion of the relevant definitions and background in Section 2.

Theorem 1. *Let* $a=(a_1, a_2, ..., a_s)\in \mathbf{Z}^s_+$, *and suppose that* $g=(g_b)_{0\leq b\leq a}$ is *an array of integers. The following are equivalent:*
(1) *g is the h-vector of a balanced Cohen—Macaulay complex,*
(2) *g is the h-vector of a balanced shellable complex,*
(3) *g is the f-vector of a colored multicomplex,*
(4) *g is the f-vector of a compressed colored multi-complex.*

In Sections 4 and 5 below we will prove the implications (3)→(4) and (4)→(2), respectively. The implication (2)→(1) is trivial (since all shellable complexes are Cohen—Macaulay) and (1)→(3) is the main result of [10].

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For the case $s=1$ this result was previously known: the equivalence of (1), (2) and (3) is then a theorem of Stanley [8] (although a proof of the implication (3)→(2) appears here for the first time), and the equivalence of (3) and (4) is Macaulay's theorem [7]. In Section 3 we will give a new proof of this original Macaulay theorem and with the same method establish another closely related result. A different generalization of the Macaulay theorem was given by Clements and Lindström [12].

Some corollaries and remarks are gathered in the final Section.

2. Preliminaries

Definition 2.1. A family $\mathscr{C}$ of subsets of a finite set V is called a *simplicial complex* if

(i) $F \subseteq F' \in \mathscr{C}$ implies $F \in \mathscr{C}$, for all $F, F' \subseteq V$, and

(ii) $\{v\} \in \mathscr{C}$, for all $v \in V$.

The members F of $\mathscr{C}$ are called *faces*, and $\mathscr{C}$ is said to be *pure* if all its maximal faces have the same cardinality. The *dimension* of a face is one less than its cardinality.

Let $a=(a_1, a_2, \ldots, a_s) \in \mathbf{Z}_+^s$ be a sequence of positive integers. A *balanced complex* of type a is a simplicial complex $\mathscr{C}$ together with an ordered partition $V=V_1 \dot{\cup} V_2 \dot{\cup} \ldots \dot{\cup} V_s$ of its vertex set such that $|F \cap V_i| = a_i$ for all maximal faces F and all $1 \leqq i \leqq s$. It follows that a balanced complex is always pure. If $b=(b_1, b_2, \ldots$ $\ldots; b_s) \in \mathbf{Z}^s$ and $0 \leqq b \leqq a$, define f_b to be the number of faces F' of $\mathscr{C}$ such that $|F' \cap V_i| = b_i$ for all $1 \leqq i \leqq s$. The integer array $(f_b)_{0 \leqq b \leqq a}$, called the *f-vector* of the balanced complex $\mathscr{C}$, is a principal object of study in this paper. Let us note two important, and in a sense opposite, special cases. If $s=1$, which we informally call the "*unbalanced*" case, the situation specializes to the study of ordinary f-vectors of pure $(a-1)$-dimensional complexes. The case when $a_i=1, 1 \leqq i \leqq s$, the "*completely balanced*" case, covers several important examples, some of which will be mentioned later in this section.

Suppose that $\mathscr{C}$ is a balanced complex of type a with f-vector $(f_b)_{0 \leqq b \leqq a}$. Define

$$h_b = \sum_{c \leqq b} f_c \prod_{i=1}^{s} (-1)^{b_i - c_i} \binom{a_i - c_i}{b_i - c_i}. \tag{2.1}$$

The integer array $(h_b)_{0 \leqq b \leqq a}$ is called the *h-vector* of $\mathscr{C}$. It is easy to see that the transformation (2.1) is invertible, so that knowledge of the h-vector is equivalent to knowledge of the f-vector. Experience has shown that characterizations of f-vectors for some classes of simplicial complexes are most conveniently expressed in terms of the h-vectors. The integers h_b may in general be negative as well as positive, see e.g. Example 3.4 (b) of [10].

Definition 2.2. Suppose that $\mathscr{C}$ is a simplicial complex on vertex set $V=\{x_1, x_2, \ldots$ $\ldots, x_n\}$. Let k be a field. Define the *face ring* $k[\mathscr{C}]$ of $\mathscr{C}$ (over k) to be the quotient ring $k[x_1, \ldots, x_n]/I_{\mathscr{C}}$, where $I_{\mathscr{C}}$ is the ideal generated by all square-free monomials $x_{i_1} x_{i_2} \ldots$ $\ldots x_{i_k}$ such that $\{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\} \notin \mathscr{C}$, $1 \leqq i_1 < \ldots < i_k \leqq n$. The complex $\mathscr{C}$ is said to be *Cohen—Macaulay* (over k) if $k[\mathscr{C}]$ is a Cohen—Macaulay ring. For a detailed discussion of this concept and its significance, see [11].

Now, suppose that $\mathscr{C}$ is a balanced complex of type $a \in \mathbf{Z}^s_+$, with ordered partition $V = V_1 \dot{\cup} \ldots \dot{\cup} V_s$. The ring $k[\mathscr{C}]$ can be made into an $\mathbf{N}^s$-graded k-algebra by defining the degree of each vertex $x \in V_i$ to be the i-th unit coordinate vector, $1 \leq i \leq s$. The *Hilbert series* of $k[\mathscr{C}]$ is an $\mathbf{N}^s$-graded algebra, $F(k[\mathscr{C}], \lambda) = \sum \{H(k[\mathscr{C}], b)\lambda^b \colon b \in \mathbf{N}^s\}$, is rational as a formal power series in the indeterminates $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s)$. Here $H(k[\mathscr{C}], b)$ is the dimension of the b-homogeneous part of $k[\mathscr{C}]$ as a vector space over k (the "Hilbert function"). A computation [10, Proposition 3.3] shows that

$$F(k[\mathscr{C}], \lambda) = \frac{\sum\limits_{0 \leq b \leq a} h_b \lambda^b}{\prod\limits_{1 \leq i \leq s} (1-\lambda_i)^{a_i}}. \tag{2.2}$$

This formula gives an algebraic interpretation to the h-vector of $\mathscr{C}$, and can alternatively (as in [10]) be taken as its definition. When $\mathscr{C}$ is a Cohen—Macaulay complex the numerator in (2.2) is itself the Hilbert series of an $\mathbf{N}^s$-graded algebra. This fact is the basis for the proof in [10] of the implication (1)→(3) of Theorem 1. Incidentally, this proof also uses one form of the Macaulay theorem.

Definition 2.3. Suppose that $\mathscr{C}$ is a pure simplicial complex. A *shelling* of $\mathscr{C}$ is an ordering $F_1, F_2, \ldots, F_t$ of its maximal faces so that F_i intersects the subcomplex generated by $F_1, \ldots, F_{i-1}$ in a nonempty union of maximal proper faces, for $i = 2, \ldots, t$. Equivalently, for $j < i$ there exists $k < i$ and $x \in F_i$ such that $F_j \cap F_i \subseteq F_k \cap F_i = F_i - \{x\}$. $\mathscr{C}$ is said to be *shellable* if it admits a shelling. Shellable complexes are known to be Cohen—Macaulay [1, 2, 8]. Given a shelling of $\mathscr{C}$, define the *restriction* of maximal face F_i to be its subface $\mathscr{R}(F_i) = \{x \in F_i | F_i - \{x\} \subseteq F_j$ for some $j < i\}$. The restriction map $\mathscr{R}\colon \operatorname{Max}\mathscr{C} \to \mathscr{C}$ characterizes shellings (cf. [2, Proposition 1.2]): Given an ordering $F_1, F_2, \ldots, F_t$ of $\operatorname{Max}\mathscr{C}$ and a map $\mathscr{R}\colon \operatorname{Max}\mathscr{C} \to \mathscr{C}$ the following are equivalent:

(i) the ordering is a shelling and $\mathscr{R}$ its restriction map,
(ii) if $F \subseteq F_i$, then $F \not\supseteq \mathscr{R}(F_i) \Leftrightarrow F \subseteq F_j$, for some $j < i$.

The h-vector of a shellable complex has a combinatorial interpretation which is of importance in this paper. Suppose that $\mathscr{C}$ is a shellable complex which is balanced of type $a \in \mathbf{Z}^s_+$, with ordered partition $V = V_1 \dot{\cup} \ldots \dot{\cup} V_s$ and h-vector $(h_b)_{0 \leq b \leq a}$. It is shown in [10, Proposition 3.6] that h_b equals the number of maximal faces F_i such that $|\mathscr{R}(F_i) \cap V_j| = b_j$, for $1 \leq j \leq s$.

Several classes of pure simplicial complexes arising in combinatorics and algebra are known to be shellable and completely balanced. Let it suffice here to mention on the one hand various central classes of lattices and partially ordered sets (surveyed in [3]) and on the other Tits buildings [2] and some related geometric incidence systems.

Definition 2.4. Let X be a finite set of indeterminates. If $\mathscr{M}$ is a family of monomials in these indeterminates, define its *shadow* $\partial(\mathscr{M})$ to be the set of all non-zero monomials occurring in

$$\left\{\frac{\partial m}{\partial x} \,\middle|\, m\in\mathcal{M}, x\in X\right\}.$$

The family $\mathcal{M}$ is said to be a *multicomplex* (or sometimes just "complex") if $\partial(\mathcal{M})\subseteq\mathcal{M}$. Equivalently, $\mathcal{M}$ is a multicomplex if $m|m'\in\mathcal{M}$ implies $m\in\mathcal{M}$, for all monomials m and m'. Let $\mathcal{M}^d$ denote the set of all monomials in $\mathcal{M}$ of degree d. The *f-vector* of a multicomplex $\mathcal{M}$ is the sequence $(f_0, f_1, f_2, \ldots)$ where $f_j=|\mathcal{M}^j|$, $j=0, 1, 2, \ldots$.

Given an underlying linear ordering of $X=\{x_1, x_2, \ldots, x_n\}$, which we take to be $x_1<x_2<\ldots<x_n$, there is an induced *lexicographic ordering* of the monomials of any fixed degree $d: x_1^{\alpha_1}x_2^{\alpha_2}\ldots x_n^{\alpha_n}<x_1^{\beta_1}x_2^{\beta_2}\ldots x_n^{\beta_n}$ if for some j one has $\alpha_j<\beta_j$ and $\alpha_i=\beta_i$ for all $i>j$. (This is sometimes called "reverse lexicographic order".) A multicomplex $\mathcal{M}$ is said to be *compressed* if $\mathcal{M}^d$ forms an initial segment in the lexicographic ordering of degree d monomials for all $d\geqq 0$ (i.e., if $m<m'\in M^d$, then $m\in M^d$). The significance of compressed multicomplexes was first discovered by Macaulay [7], and later by Lindström et al. [6], [12] and others. See the survey [5] or [14] for more facts and references concerning compression.

Given a subset $Y\subseteq X$ and a monomial $m=\prod\{x^{\alpha(x)}\colon x\in X\}$ define $m_Y=\prod\{x^{\alpha(x)}\colon x\in Y\}$. Let $a=(a_1, a_2, \ldots, a_s)\in\mathbf{Z}_+^s$. A *colored multicomplex* of type a is a multicomplex $\mathcal{M}$ together with an ordered partition $X=X_1\dot\cup X_2\dot\cup\ldots\dot\cup X_s$ of its indeterminates such that $\deg m_{X_i}\leqq a_i$ for all $m\in\mathcal{M}$ and all $1\leqq i\leqq s$. Define the *multidegree* of a monomial $m\in\mathcal{M}$ as the vector $\mathrm{DEG}\, m=(\deg m_{X_1}, \deg m_{X_2}, \ldots, \deg m_{X_s})\in\mathbf{N}^s$. Then the *f-vector* of the colored multicomplex $\mathcal{M}$ is the integer array $(f_b)_{0\leqq b\leqq a}$, where f_b is the number of $m\in\mathcal{M}$ such that $\mathrm{DEG}\, m=b$. This definition is clearly consistent with our earlier definitions of f-vectors for balanced complexes and general multicomplexes in the areas of overlap. Notice that a colored multicomplex of type $(1, 1, \ldots, 1)$ is in fact a simplicial complex, however not necessarily pure.

Suppose that $\mathcal{M}$ is a colored multicomplex of type a as above. For each block X_j of the ordered partition of X, $1\leqq j\leqq s$, fix a linear ordering of its elements $x_{j_1}<x_{j_2}<\ldots$. This induces a *partial ordering* on the monomials of any fixed multidegree: $m\leqq m'$ if and only if $m_{X_i}\leqq m'_{X_i}$ in the lexicographic ordering for each $1\leqq i\leqq s$. A colored multicomplex $\mathcal{M}$ is said to be *compressed* if its monomials of any fixed multidegree form an order ideal in this partial ordering (i.e., $m\leqq m'\in\mathcal{M}$ implies $m\in\mathcal{M}$). This definition specializes to the usual notion of compression for the $s=1$ case. Notice that, in distinction to the $s=1$ case, there may in general exist several compressed colored multi-complexes having the same f-vector.

3. The Macaulay Theorem

Given a sequence of non-negative integers $f=(f_0, f_1, \ldots)$ let $\mathcal{L}_f$ be the family consisting of the first f_i monomials of degree i in the lexicographic ordering for $i=0, 1, \ldots$. Also, define the d-representation of a number m as the unique way of writing

$$m = \binom{a_1}{d} + \binom{a_2}{d-1} + \ldots + \binom{a_r}{d-r+1}$$

with $a_1 > a_2 > \ldots > a_r \geqq d-r+1 \geqq 1$.

Theorem 2. (Macaulay [7]). *Given a sequence $f=(f_0, f_1, \ldots)$ of non-negative integers, the following are equivalent:*

(a) *f is the f-vector of a multicomplex,*

(b) *$\mathcal{L}_f$ is a multicomplex,*

(c) *if $f_d = \binom{a_1}{d} + \ldots + \binom{a_r}{d-r+1}$ is the d-representation of f_d, then*

$$f_{d-1} \geqq \binom{a_1-1}{d-1} + \binom{a_2-1}{d-2} + \ldots + \binom{a_r-1}{d-r},\ d \geqq 1,\ \text{and}\ f_0 = 1.$$

The following analogue of Macaulay's theorem has the advantage of avoiding the sometimes cumbersome d-representation of numbers. It is similar to Lovász's analogue of the Kruskal—Katona theorem (cf. [4]).

Theorem 3. *Suppose $(f_0, f_1, \ldots)$ is the f-vector of some multicomplex. If $f_d = \binom{x}{d}$, where x is a real number, $x \geqq d \geqq 1$, then $f_{d-1} \geqq \binom{x-1}{d-1}$.*

Before giving the proofs we define a useful shifting technique for multicomplexes. Let $\mathcal{M}$ be a multicomplex on indeterminates $x_1, x_2, \ldots, x_n$. For $S \subseteq [1, n]$ denote by $\mathcal{M}(S)$ the subcomplex of all $\mathcal{M}$-monomials in variables $x_{i_1}, x_{i_2}, \ldots, x_{i_r}$ such that $i_1, i_2, \ldots, i_r \in S$. Given $1 \leqq i < j \leqq n$ set $\mathcal{M}(\overline{ij}) = \mathcal{M}([1, n] - \{i, j\})$. For $p \in \mathcal{M}(\overline{ij})$ define $\mathcal{M}(p) = \{m \in \mathcal{M}(\{i, j\}) | p \cdot m \in \mathcal{M}\}$. Note that $\mathcal{M}(p)$ is a subcomplex and that $p | p'$ implies $\mathcal{M}(p) \supseteq \mathcal{M}(p')$, since $\mathcal{M}$ is a complex.

Let us study the complex $\mathcal{M}(p)$ more carefully. Suppose that $(g_0, g_1, \ldots)$ is its f-vector. For given d, the degree d monomials in $\mathcal{M}(p)$ are of the form $x_i^a x_j^{d-a}$. Each of these have two maximal divisors except x_i^d and x_j^d, and a degree $d-1$ monomial has only 2 multiples of degree d. This implies easily that $g_{d-1} \geqq g_d - 1$, and even $g_{d-1} \geqq g_d$ unless $g_d = d+1$ (and $g_{d-1} = d$). Let L^d be the collection of the first (in lexicographic order) g_d monomials of degree d, i.e., $L^d = \{x_i^d, x_i^{d-1}x_j, \ldots, x_i^{d-g_d+1}x_j^{g_d-1}\}$. It follows from the preceding that $\bigcup\{L^d\colon\ d \geqq 0\}$ is a complex. Set $S_{ij}(\mathcal{M}(p)) = \bigcup_d L^d$.

We can now define shifting:

$$S_{ij}(\mathcal{M}) = \bigcup_{p \in M(\overline{ij})} \{pq | q \in S_{ij}(\mathcal{M}(p))\}.$$

By the above discussion it should be clear that $S_{ij}(\mathcal{M})$ is a multicomplex having the same f-vector as $\mathcal{M}$.

Iterated shifting for all pairs $1 \leqq i < j \leqq n$ leads to a multicomplex $\tilde{\mathcal{M}}$ satisfying $S_{ij}(\tilde{\mathcal{M}}) = \tilde{\mathcal{M}}$ for all $1 \leqq i < j \leqq n$. (Actually, one can show that $\binom{n}{2}$ shiftings are sufficient to produce such a stable complex if we proceed in the order that S_{ij} precedes $S_{i'j'}$ if $i < i'$ or $i = i'$ and $j > j'$). Notice that the shifted complex $\tilde{\mathcal{M}}$ by definition satisfies the property:

$$\text{(3.1)} \qquad \begin{aligned} &\textit{if} \quad p = \prod_{\gamma=1}^{n} x_\gamma^{\alpha_\gamma} \in \tilde{\mathcal{M}} \quad \text{and} \quad 1 \leqq i < j \leqq n, \quad \alpha_j \geqq \beta_j > 0, \\ &\qquad\qquad \textit{then} \quad \frac{x_i^{\beta_j}}{x_j^{\beta_j}} p \in \tilde{\mathcal{M}}. \end{aligned}$$

Proof of Theorem 3. Suppose that $\mathcal{M}$ is a multicomplex having f-vector $(f_0, f_1, \ldots)$. Without loss of generality we may assume that $\mathcal{M}$ is shifted, i.e., $\tilde{\mathcal{M}} = \mathcal{M}$. We will use induction on the number $f_1 = n$ of indeterminates. The case $n = 1$ is trivially true.

Let us consider for given d the classes of monomials $\mathcal{M}_i^{j-i} = \{p \in \mathcal{M}([2, n]) \mid x_1^i p \in \mathcal{M}^j\}$, $0 \leqq i \leqq j \leqq d$. From (3.1) it follows that

$$\text{(3.2)} \qquad \partial(\mathcal{M}_i^{(j+1)-i}) \subseteq \mathcal{M}_{i+1}^{(j+1)-(i+1)}, \quad 0 \leqq i \leqq j < d.$$

Let $m_i = |\mathcal{M}_i^{d-i}|$. We shall use (3.2) to show that $m_0 \leqq \binom{x-1}{d}$ holds.

Suppose to the contrary that $m_0 > \binom{x-1}{d}$. The induction hypothesis and (3.2) then yield successively $m_1 > \binom{x-2}{d-1}$, $m_2 > \binom{x-3}{d-2}$, $\ldots$, $m_{d-1} > \binom{x-d}{1}$, and of course $m_d = 1$. Adding up these inequalities gives

$$|\mathcal{M}^d| = \sum_{i=0}^{d} m_i > \sum_{i=0}^{d} \binom{x-1-i}{d-i} = \binom{x}{d} = |\mathcal{M}^d|,$$

a contradiction.

We have now that $m_1 + m_2 + \ldots + m_d = |\mathcal{M}^d| - m_0 \geqq \binom{x}{d} - \binom{x-1}{d} = \binom{x-1}{d-1}$. Therefore the proof will be complete if we show that $f_{d-1} \geqq m_1 + m_2 + \ldots + m_d$. However, this is already clear since for the degree d monomials $p = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$ with $\alpha_1 \geqq 1$, which are counted by the right hand side, the map $p \mapsto p/x_1$ is injective into $\mathcal{M}^{d-1}$. ∎

Proof of Theorem 2. To show the implication (a)$\rightarrow$(c) we can argue just as in the preceding proof. It is sufficient to prove that $m_0 \leqq \binom{a_1-1}{d} + \ldots + \binom{a_r-1}{d-r+1}$. Suppose the contrary. Itt hen follows by induction, using (3.2), that $m_1 \geqq \binom{a_1-2}{d-1} + \ldots + \binom{a_r-2}{d-r}$, and in general:

$$m_j \geqq \binom{a_1-1-j}{d-j} + \ldots + \binom{a_r-1-j}{d-r+1-j}, \quad j = 1, 2, \ldots, d.$$

In interpreting and verifying these inequalities some care has to be taken with regard to degenerate binomial coefficients: recall that by definition $\binom{a}{b}=0$ unless $a\geqq b\geqq 0$. These inequalities lead to the contradiction

$$|\mathcal{M}^d| = \sum_{j=0}^{d} m_j > \sum_{j=0}^{d}\sum_{i=1}^{r}\binom{a_i-1-j}{d-i+1-j} = \sum_{i=1}^{r}\binom{a_i}{d-i+1} = |\mathcal{M}^d|.$$

The implication (b)→(a) is trivial, since f is the f-vector of $\mathcal{L}_f$. To show that (c)→(b) one observes that $\partial\mathcal{L}_f^d$ has cardinality $\binom{a_1-1}{d-1}+\ldots+\binom{a_r-1}{d-r}$ and forms an initial segment in the lexicographic ordering of degree $d-1$ monomials. ∎

For some additional aspects on Macaulay's theorem and further references, see [5] and [9].

4. Proof of the implication (3)→(4)

We will show in this Section that if $\mathcal{M}$ is a colored multicomplex of type $a=(a_1, \ldots, a_s)$, then there exists a compressed multicomplex $\mathcal{N}$ of type a with the same f-vector.

Suppose that $X=X_1\dot{\cup}\ldots\dot{\cup}X_s$ is an ordered partition of a finite set X of indeterminates. For $1\leqq j\leqq s$ fix a linear ordering $x_{j_1}<x_{j_2}<\ldots$ of the elements of X_j. For $l, i\geqq 0$ denote by $\mathcal{L}^{(j)}(l, i)$ the first l monomials of degree i in the lexicographic order with variables from X_j.

Suppose now that $\mathcal{M}$ is a multicomplex over X. For $Y\subseteqq X$, define $\mathcal{M}_Y=\{m_Y|m\in\mathcal{M}\}$. Fix $1\leqq j\leqq s$ and consider the partition

$$\mathcal{M} = \bigcup_{\tilde{m}\in\mathcal{M}_{X-X_j}} \{m\in\mathcal{M}|m_{X-X_j}=\tilde{m}\}.$$

We are going to compress each set in this partition individually. Let $f_i(\tilde{m})$ denote the number of $m\in\mathcal{M}$ with $m_{X-X_j}=\tilde{m}$ and $\deg m/\tilde{m}=i$. Define

$$\mathcal{M}^{(j)}_{\tilde{m},i} = \{\tilde{m}m^*|m^*\in\mathcal{L}^{(j)}(f_i(\tilde{m}), i)\} \quad \text{and}$$

$$\mathcal{M}^{(j)} = \bigcup_{\tilde{m}\in\mathcal{M}_{X-X_j}}\bigcup_i \mathcal{M}^{(j)}_{\tilde{m},i}.$$

The f-vector of $\mathcal{M}^{(j)}$ is by construction the same as that of $\mathcal{M}$. We claim that $\mathcal{M}^{(j)}$ is a complex.

To see this, suppose that p is a monomial and for some $x\in X, p'=px\in\mathcal{M}^{(j)}$ holds. We must show that $p\in\mathcal{M}^{(j)}$. There are two cases to consider.

Case 1: $x\in X-X_j$. Set $q=p_{X-X_j}, q'=p'_{X-X_j}$, and $i=\deg p/q=\deg p'/q'$. Note that $q'=qx$. If $m\in\mathcal{M}$ and $m_{X-X_j}=q'$, then $m/x\in\mathcal{M}$ and $(m/x)_{X-X_j}=q$. Hence, $f_i(q')\leqq f_i(q)$. Therefore $\mathcal{L}^{(j)}(f_i(q'), i)\subseteqq\mathcal{L}^{(j)}(f_i(q), i)$, which yields $p\in\mathcal{M}^{(j)}$.

Case 2: $x\in X_j$. Set again $q=p_{X-X_j}$. Then also $q=p'_{X-X_j}$ holds. Since $\mathcal{M}$ is a complex, so is $\mathcal{M}/q=\{m/q|m\in M, m_{X-X_j}=q\}$. Now the Macaulay theorem says that $\mathcal{M}^{(j)}/q$ is a complex, in particular $p\in\mathcal{M}^{(j)}$.

We have shown that $\mathcal{M}^{(j)}$ is a multicomplex whose f-vector (as a colored multicomplex of type a) is the same as the f-vector of $\mathcal{M}$. The construction can now be repeated for other values of j.

Given a monomial m define $o_i(m)$ as the position of m_{X_i} in the linear order. It is clear that

$$\sum_{m\in\mathcal{M}}\sum_{i=1}^{s} o_i(m) \geqq \sum_{m\in\mathcal{M}^{(j)}}\sum_{i=1}^{s} o_i(m),$$

with equality if and only if $\mathcal{M}=\mathcal{M}^{(j)}$. Thus, iterating the above operation for all $1\leqq j\leqq s$, finally we obtain a colored multicomplex $\mathcal{N}$ with the same f-vector which is invariant for all j under the operation. This means that for all $1\leqq j\leqq s$, all $\tilde{n}\in\mathcal{N}_{X-X_j}$ and all $i\geqq 0$, the set $\{n/\tilde{n} | n\in\mathcal{N}, n_{X-X_j}=\tilde{n}, \deg n/\tilde{n}=i\}$ is initial in the lexicographic ordering. Equivalently, if $m\leqq n\in\mathcal{N}$, then $m\in\mathcal{N}$, i.e., $\mathcal{N}$ is compressed.

5. Proof of the implication (4) → (2)

The argument will first be carried out for the case $s=1$ and then, using this special case, in general.

Let $X=\{x_1, x_2, \dots, x_n\}$ and $V=\{v_1, v_2, \dots, v_{n+a}\}$. We start by defining a canonical bijection between the monomials in X of degree $\leqq a$ and the a-subsets of V. Let $m=x_{i_1}x_{i_2}\dots x_{i_k}$, $1\leqq i_1\leqq i_2\leqq\dots\leqq i_k\leqq n$. Define

$$\delta(m) = \{v_1, v_2, \dots, v_{a-k}\},$$

$$\sigma(m) = \{v_{a-k+i_1+1},\ v_{a-k+i_2+2},\ \dots,\ v_{a-k+i_k+k}\},$$

and finally set

$$\varphi(m) = \delta(m)\cup\sigma(m).$$

It is easily checked that for $0\leqq k\leqq a$, the mapping φ sets up a bijection between the monomials in X of degree k and those a-subsets of V which contain $v_1, v_2, \dots,$ and v_{a-k}, but not v_{a-k+1}.

Now let $\mathcal{M}$ be a compressed multicomplex on X with f-vector $(g_0, g_1, \dots, g_a)$, $g_1=n$. Define a simplicial complex $\mathcal{C}$ on V by letting the maximal faces of $\mathcal{C}$ be the sets $\varphi(m), m\in\mathcal{M}$. Thus $\mathcal{C}$ is pure of dimension $a-1$. Let $m_1<m_2<\dots<m_t$ $(t=\sum g_i)$ be any total ordering of $\mathcal{M}$ satisfying $i\leqq j\Rightarrow\deg m_i\leqq\deg m_j$.

Claim: $\varphi(m_1), \varphi(m_2), \dots, \varphi(m_t)$ *is a shelling of* $\mathcal{C}$ *with restriction map* $\mathcal{R}(\varphi(m_i))=\sigma(m_i)$, $1\leqq i\leqq t$.

It follows from the claim that $|\mathcal{R}(\varphi(m_i))|=\deg m_i$, and hence that $(g_0, g_1, \dots, g_a)$ is the h-vector of the shellable complex $\mathcal{C}$, which completes the proof for the case $s=1$.

To prove the claim we must show:

(a) *If* $i<j$, *then* $\sigma(m_j)\not\subseteq\varphi(m_i)$, *and*

(b) *If* $\sigma(m_j)\subseteq F\subsetneq\varphi(m_j)$, *then* $F\subseteq\varphi(m_i)$ *for some* $i<j$.

Proof of (a): Suppose $i\leqq j$ and $\sigma(m_j)\subseteq\varphi(m_i)$. Since $\deg m_i\leqq\deg m_j$ we must have $\delta(m_j)\subseteq\delta(m_i)\subseteq\varphi(m_i)$. Thus, $\varphi(m_j)=\delta(m_j)\cup\sigma(m_j)\subseteq\varphi(m_i)$, so $i=j$.

Proof of (b): It suffices to consider the case that F is maximal, i.e., $F=\varphi(m_j)-\{v_z\}$, for some $v_z\in\sigma(m_j)$. Suppose that $m_j=x_1^r x_{i_{r+1}}x_{i_{r+2}}\dots x_{i_k}$, $1<i_{r+1}\leqq i_{r+2}\leqq\dots\leqq i_k\leqq n$. In the sequel, for simplicity subsets of V will be denoted by the string of subscripts of their elements. Thus, for instance,

$$\varphi(m_j)=\{1,2,\dots,a-k\}\cup\{a-k+2,a-k+3,\dots,a-k+r+1\}\cup$$
$$\cup\{a-k+i_{r+1}+r+1,a-k+i_{r+2}+r+2,\dots,a-k+i_k+k\}.$$

Case 1: $a-k+2\leqq z\leqq a-k+r+1$. (Void if $r=0$). Set $u=z-a+k-1$, so $1\leqq u\leqq r$. Let $m_i=x_1^{r-u}x_{i_{r+1}}x_{i_{r+2}}\dots x_{i_k}$, so $m_i\in\mathcal{M}$ and $i<j$. Then

$$\varphi(m_i)=\{1,2,\dots,a-k+u\}\cup\{a-k+u+2,\dots,a-k+r+1\}\cup$$
$$\cup\{a-k+i_{r+1}+r+1,\dots,a-k+i_k+k\},$$

so $F=\varphi(m_j)-\{v_z\}\subseteq\varphi(m_i)$, as desired.

Case 2: $z=a-k+i_y+y$ for some $r+1\leqq y\leqq k$. Let $F'=(\varphi(m_j)-\{v_z\})\cup\{v_{a-k+1}\}$, i.e., in terms of subscripts:

$$F'=\{1,2,\dots,a-k+r+1\}\cup\{a-k+i_{r+1}+r+1,\dots,a-k+i_{y-1}+$$
$$+y-1,a-k+i_{y+1}+y+1,\dots,a-k+i_k+k\}.$$

Then

$$\varphi^{-1}(F')=x_{i_{r+1}-1}x_{i_{r+2}-1}\dots x_{i_{y-1}-1}x_{i_{y+1}}x_{i_{y+2}}\dots x_{i_k}.$$

Now, $\varphi^{-1}(F')\leqq x_{i_{r+1}}x_{i_{r+2}}\dots x_{i_{y-1}}x_{i_{y+1}}x_{i_{y+2}}\dots x_{i_k}$ in lexicographic order. Since the right hand side divides m_j and $\mathcal{M}$ is compressed, it follows that $\varphi^{-1}(F')\in\mathcal{M}$, so $F'=\varphi(m_i)$ for some i. Moreover, $i<j$ since $\deg m_i<\deg m_j$. The proof of the $s=1$ case is now complete.

For the general case, suppose that X is a finite set of indeterminates and $X=X_1\dot\cup\dots\dot\cup X_s$ is an ordered partition. Fix some linear ordering $x_{j_1}<x_{j_2}<\dots<x_{j_{n_j}}$ of the elements within each class X_j, $1\leqq j\leqq s$. Given $a=(a_1,\dots,a_s)\in Z_+^s$, we introduce a set of vertices V with an ordered partition $V=V_1\dot\cup\dots\dot\cup V_s$, such that $|V_j|=n_j+a_j$ and give a fixed linear ordering $v_{j_1}<v_{j_2}<\dots<v_{j_{n_j+a_j}}$ to each class V_j, $1\leqq j\leqq s$. For each j, let $\varphi_j(\cdot)=\delta_j(\cdot)\cup\sigma_j(\cdot)$ denote the bijection previously defined between monomials in X_j of degree $\leqq a_j$ and a_j-element subsets of V_j. Now, if m is a monomial in X such that $\deg m_{X_j}\leqq a_j$, $1\leqq j\leqq s$, define

$$\delta(m)=\bigcup_j\delta_j(m_{X_j}),$$
$$\sigma(m)=\bigcup_j\sigma_j(m_{X_j}),$$
$$\varphi(m)=\delta(m)\cup\sigma(m)=\bigcup_j\varphi_j(m_{X_j}).$$

It is clear from the construction that φ gives a bijection between monomials m in X such that $\deg m_{X_j}\leqq a_j$, $1\leqq j\leqq s$, and $(\sum a_j)$-element subsets S of V such that $|S\cap V_j|=a_j$, $1\leqq j\leqq s$.

Suppose that $\mathcal{M}$ is a compressed colored multicomplex of type $a=(a_1, \ldots, a_s)$ over the indeterminates $X=X_1\dot\cup\ldots\dot\cup X_s$. Also, suppose that $g=(g_b)_{0\leqq b\leqq a}$ is the f-vector of $\mathcal{M}$. Construct a simplicial complex $\mathcal{C}$ on vertices $V=V_1\dot\cup\ldots\dot\cup V_s$ by letting $\varphi(m)$, $m\in\mathcal{M}$, be its maximal faces. Then $\mathcal{C}$ is a balanced complex of type a. Let $m_1<m_2<\ldots<m_t$ be a total ordering of $\mathcal{M}$ such that $i\leqq j\Rightarrow \deg m_i\leqq \deg m_j$.

Claim: $\varphi(m_1), \varphi(m_2), \ldots, \varphi(m_t)$ is a shelling of $\mathcal{C}$ with restriction map $\mathcal{R}(\varphi(m_i))=$ $=\sigma(m_i)$, $1\leqq i\leqq t$.

Since then $|\mathcal{R}(\varphi(m_i))\cap V_j|=|\sigma_j((m_i)_{X_j})|=\deg (m_i)_{X_j}$, it follows that g is the h-vector of $\mathcal{C}$, and the proof is complete.

To prove the claim we must show the same assertions (a) and (b) that were stated above for the $s=1$ case.

Proof of (a): Suppose $i\leqq j$ and $\sigma(m_j)\subseteq\varphi(m_i)$. Then, in fact, $\sigma(m_j)\subseteq\sigma(m_i)$ and $\delta(m_i)\subseteq\delta(m_j)$. If the second inclusion were strict, then $\deg m_i>\deg m_j$, which would violate $i\leqq j$. Hence, $\delta(m_i)=\delta(m_j)$, so $\sigma(m_i)=\sigma(m_j)$, thus $\varphi(m_i)=\varphi(m_j)$, and therefore $i=j$.

Proof of (b): Suppose that $F=\varphi(m_j)-\{v\}$, $v\in\sigma(m_j)\cap V_k$. When proving part (b) for the $s=1$ case we showed that there exist $m', m''\in\mathcal{M}_{X_k}$ such that $\varphi_k((m_j)_{X_k})-$ $-\{v\}\subseteq\varphi_k(m')$, where $m'\leqq m''$ in lexicographic order and m'' is a proper divisor of $(m_j)_{X_k}$. Now, let $m^*=m'\cdot(m_j)_{X-X_k}$. Then $m^*\in\mathcal{M}$, since $\mathcal{M}$ is compressed, $\deg m^*<\deg m_j$, since $\deg m'<\deg (m_j)_{X_k}$, and $\varphi(m_j)-\{v\}\subseteq\varphi(m^*)$. ∎

6. Comments

6.1. Let V be a finite set and let $\pi=\{V_1, V_2, \ldots, V_k\}$ and $\sigma=\{V_1', V_2', \ldots$ $\ldots, V_m'\}$ be two unordered partitions of V into disjoint subsets (called blocks). Then π is said to be a *refinement* of σ, written $\pi\leqq\sigma$, if for each $1\leqq i\leqq k$ there exist $1\leqq j\leqq m$ such that $V_i\subseteq V_j'$. As is well known, the refinement partial ordering of the partitions of V is a *lattice,* meaning that *meets* (greatest lower bounds) and *joins* (least upper bounds) exist for all pairs of partitions.

Suppose that $\mathcal{C}$ is a Cohen—Macaulay complex over the vertex set V. Let an unordered partition of V be called *$\mathcal{C}$-balancing* if $\mathcal{C}$ is a balanced complex with respect to some (and hence every) permutation of its blocks. If π and σ are $\mathcal{C}$-balancing partitions and $\pi<\sigma$, then Theorem 1 shows that π imposes stricter requirements on the f-vector of $\mathcal{C}$ than does σ. Hence, it is of interest to seek a minimal $\mathcal{C}$-balancing partition. It turns out that there is in fact a unique minimal $\mathcal{C}$-balancing partition of V. More generally the following can be proven.

Let $\mathcal{C}$ be a pure simplicial complex which is *strongly connected,* in the sense that any two maximal faces F and F' can be connected by a sequence of maximal faces $F=F_0, F_1, \ldots, F_r=F'$ such that successively F_{i-1} and F_i intersect in a maximal proper face, $i=1, 2, \ldots, r$. If π and σ are two $\mathcal{C}$-balancing partitions of V, then so is their meet $\pi\wedge\sigma$. Hence, the $\mathcal{C}$-balancing partitions form a lattice under the refinement ordering.

It is known that Cohen—Macaulay complexes are strongly connected. For shellable complexes this is a direct consequence of the definition, whereas for general

Cohen—Macaulay complexes a more involved proof (e.g., using rankselection for the two top levels in the face-lattice) seems unavoidable.

6.2. In the $s=1$ case, conditions (3) and (4) of Theorem 1 correspond exactly to conditions (a) and (b) of Theorem 2. In this case, the original Macaulay theorem also has a third equivalent condition (c), namely the purely numerical condition in terms of d-representations of numbers. When $s\geqq 2$ we do not know of any similar numerical formulation. Part of the difficulty seems to lie in the non-uniqueness of compressed multicomplexes with a given f-vector, when $s\geqq 2$. In this connection it would be of interest to seek a numerical characterization of f-vectors $(f_b)_{0\leqq b\leqq a}$ of colored multicomplexes of type $a\in \mathbf{Z}_+^s$, when $s\geqq 2$.

6.3. In the completely balanced case, i.e. when $a=(1,1, \dots, 1)$, it is possible to prove the implication (3)→(2) directly without passing through (4), i.e. without first compressing the multicomplex. In all other cases compression seems indispensable.

6.4. Let us call a simplicial complex *k-chromatic* if its 1-skeleton is k-chromatic in the sense of graph theory, i.e., if its vertices can be colored using at most k colors so that every face receives distinct colors at all its vertices. Viewing a completely balanced complex temporarily as unbalanced we derive the following characterization of ordinary h-vectors of completely balanced Cohen—Macaulay complexes.

Corollary. *Let $g=(g_0, g_1, \dots, g_k)$ be a string of integers. The following are equivalent:*
(1) *g is the (ordinary) h-vector of a completely balanced $(k-1)$-dimensional Cohen—Macaulay (or shellable) complex,*
(2) *g is the f-vector of a k-chromatic simplicial complex.*

A numerical characterization of the f-vectors of general simplicial complexes is known from the Kruskal—Katona theorem (see [4, 5]). Being k-chromatic is clearly a considerable constraint, and it would be of interest to have a numerical characterization of the f-vectors of k-chromatic simplicial complexes. A necessary condition is given by the following result of the second author, Füredi and Kalai [13]: If $f=(f_0, f_1, \dots, f_d)$ is the f-vector of a k-chromatic $(d-1)$-dimensional simplicial complex and $f_d=\binom{k}{d}x^d$ for some real x, then $f_i\geqq\binom{k}{i}x^i$ for $0\leqq i\leqq d-1$.

6.5. A special class of completely balanced Cohen—Macaulay complexes is given by the order complexes of Cohen—Macaulay posets. See [1, 3, 10] for definitions and further details. It is easily seen by direct construction or from the preceding Corollary, that (1, 3, 3, 0) is the h-vector of a completely balanced 2-dimensional Cohen—Macaulay complex. However, (1, 3, 3, 0) is not the h-vector of any Cohen—Macaulay poset. Hence, the question remains open to characterize the h-vectors of Cohen—Macaulay posets, either viewed as unbalanced or as completely balanced complexes. We remark that in the latter case the h-vector of a Cohen—Macaulay poset P coincides with the "rank-selected Möbius invariant" $\beta(P, S)$ as discussed in [3] and [10].

6.6. Suppose that $\mathscr{C}$ is a completely balanced $(k-1)$-dimensional shellable complex. We know that its h-vector is realized on the one hand as the f-vector of some k-chromatic simplicial complex, and on the other hand as the f-vector of the family of restrictions $\mathcal{R}(F)$ of maximal faces $F\in\operatorname{Max}\mathcal{C}$ for any shelling. This raises the question: Can always some shelling be found for which $\{\mathcal{R}(F)|F\in\operatorname{Max}\mathcal{C}\}$ forms a simplicial complex? For the class of shellable posets which admit an "SL-labeling" in the sense of [1] we can prove an affirmative answer. This includes all semimodular and supersolvable finite lattices.

6.7. It is mistakenly claimed in [10, p. 152] that the implication (3)→(2) of Theorem 1 is not generally true. The claim is based on an incorrect counterexample: the bipartite graph would have 9 (and not 6) vertices and 10 edges.

Note added in proof. In the very recent work [13] the numerical characterization problem of h-vectors of completely balanced Cohen—Macaulay complexes is solved.

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