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Symmetries of Plane Partitions

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We introduce a new symmetry operation, called *complementation*, on plane partitions whose three-dimensional diagram is contained in a given box. This operation was suggested by work of Mills, Robbins, and Rumsey. There then arise a total of ten inequivalent problems concerned with the enumeration of plane partitions with a given symmetry. Four of these ten problems had been previously considered. We survey what is known about the ten problems and give a solution to one of them. The proof is based on the theory of Schur functions, in particular the Littlewood–Richardson rule. Of the ten problems, seven are now solved while the remaining three have conjectured simple solutions.

1. INTRODUCTION

Plane partitions are generalizations of ordinary partitions of integers first considered by P. A. MacMahon. MacMahon defined six symmetry operations on plane partitions and raised the problem of enumerating plane partitions with given symmetries. (Precise statements and references are given below.) The work of Mills–Robbins–Rumsey suggests a further symmetry operation which has been previously overlooked. There then arise in a natural way a total of ten inequivalent problems concerned with the enumeration of plane partitions with given symmetries. In the next section of this paper we discuss the ten symmetry classes and what is known about them. In Section 3 we solve one of the ten enumeration problems. Previously six have been solved, so now there are seven solved problems and three conjectures.

Our proof in Section 3 is based on the theory of symmetric functions and especially Schur functions, whose connection with plane partitions is first explicitly mentioned in [15]. For an introduction to the theory of Schur functions, see [15, Part 1] or [7, Chap. I].

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2. Symmetry Classes of Plane Partitions

Fix positive integers r, s, t. A *plane partition* with $\leqslant r$ rows, $\leqslant s$ columns, and largest part $\leqslant t$ is an $r \times s$ matrix $\pi = (\pi_{ij})$, $1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant s$, such that each π_{ij} is an integer satisfying $0 \leqslant \pi_{ij} \leqslant t$, and such that the rows and columns of π are weakly decreasing. See [15, Part 2]. Define $|\pi| = \sum_{i,j} \pi_{ij}$. If $|\pi| = n$ then we say π is a plane partition *of* n. If m is a positive integer, then write $[m] = \{1, 2, ..., m\}$. The *diagram* $D(\pi)$ of π is the subset of the *box* $B(r, s, t) = [r] \times [s] \times [t]$ defined by

$$D(\pi) = \{(i, j, k) \colon \pi_{ij} \geqslant k\}.$$

Thus $|\pi|$ is equal to the number $|D(\pi)|$ of elements of $D(\pi)$. Frequently we identify π with its diagram $D(\pi)$, and will say that π is *contained* in $B(r, s, t)$, denoted $\pi \subseteq B(r, s, t)$. Similarly we write $x \in \pi$ instead of $x \in D(\pi)$. If we regard $B(r, s, t)$ as a poset (partially ordered set) with the usual product order, then a plane partition contained in $B(r, s, t)$ is just an order ideal (also called semi-ideal, decreasing subset, or down-set) of $B(r, s, t)$.

Let $\mathbb{P}$ denote the set of positive integers. The symmetric group S_3 acts on $\mathbb{P}^3$ be permuting coordinates, and therefore on the set of all (diagrams of) plane partitions. For each subgroup G of S_3 we are interested in the number $N_G(r, s, t)$ of plane partitions contained in $B(r, s, t)$ and invariant under G. Clearly we can assume that $B(r, s, t)$ is G-invariant, so certain choices of G will cause certain of the numbers r, s, t to be equal. The six symmetries of plane partitions just defined were first considered by MacMahon [8, 9, Sects. 425, 509ff]. References to the problem of determining $N_G(r, s, t)$ will be given later.

There is an additional symmetry of plane partitions contained in $B(r, s, t)$ which is suggested by work of Mills–Robbins–Rumsey [11, Conjecture 3S]. If $\pi \subseteq B(r, s, t)$, then define the *complement* π^c of π by

$$\pi^c = \{(r+1-i, s+1-j, t+1-k) \colon (i, j, k) \notin \pi\}.$$

Clearly π^c is a plane partition, and $|\pi| + |\pi^c| = rst$. Thus if $\pi = \pi^c$ then $|\pi| = |\pi^c| = rst/2$, so rst is even. The transformation ^{c} and the group S_3 generate a group T of order 12. For every subgroup G of T we may again ask for the number $N_G(r, s, t) = N_G(B)$ of plane partitions $\pi \subseteq B(r, s, t) = B$ invariant under G (i.e., $w \cdot \pi = \pi$ for all $w \in G$). Again we may assume $B(r, s, t)$ is G-invariant. If G and G' are conjugate subgroups of T then clearly $N_G(r, s, t) = N_{G'}(r, s, t)$. One can check that the group T has ten conjugacy classes of subgroups, giving rise to ten enumeration problems. We now explicitly list these ten classes of plane partitions (where we have chosen a particular group G in each conjugacy class). The following terminology will

TABLE I

Symmetry Classes of Plane Partitions

	B	Class
1.	$B(r, s, t)$	Any
2.	$B(r, r, t)$	Symmetric
3.	$B(r, r, r)$	Cyclically symmetric
4.	$B(r, r, r)$	Totally symmetric
5.	$B(r, s, t)$	Self-complementary
6.	$B(r, r, t)$	Complement = transpose
7.	$B(r, r, t)$	Symmetric and self-complementary
8.	$B(r, r, r)$	Cyclically symmetric and complement = transpose
9.	$B(r, r, r)$	Cyclically symmetric and self-complementary
10.	$B(r, r, r)$	Totally symmetric and self-complementary

be used. The *transpose* π^* of the plane partition $\pi = (\pi_{ij})$ is defined by $\pi^* = (\pi_{ij})$. We say π is *symmetric* if $\pi = \pi^*$. We say π is *cyclically symmetric* if whenever $(i, j, k) \in \pi$ then $(j, k, i) \in \pi$. In other words π is invariant under the (unique) 3-element subgroup G of S_3. This condition is equivalent to saying that for every i, the ith row of the matrix (π_{ij}) is conjugate (in the sense of [7, p. 2]) to the ith column. For example,

$$\begin{array}{ccc} 3 & 2 & 2 \\ 3 & 2 & 0 \\ 1 & 1 & 0 \end{array}$$

is cyclically symmetric. A plane partition π is called *totally symmetric* if it is S_3-invariant, i.e., if it is cyclically symmetric and symmetric. Equivalently, π is symmetric and every row of π is a self-conjugate partition. Of course, by a *self-complementary* plane partition π we mean that $\pi = \pi^c$. We now give our list of the ten symmetry classes of plane partitions contained in $B = B(r, s, t)$. (See Table I.)

We now briefly discuss what is known about enumerating the ten classes. Remarkably, in every case there is a simple formula either known or conjectured. At the present writing seven of the formulas are proved and three are conjectured. In particular, in the next section we establish a formula for Case 5.

Cases 1–4. If $x = (i, j, k) \in B$, then define the height $ht(x) = i + j + k - 2$. If G acts on B and η is an orbit of this action, then define $ht(\eta) = ht(x)$ for any $x \in \eta$. (This definition differs from the original one of Macdonald [7,

p. 52] and seems more natural for our purposes.) Now if G is a subgroup of T corresponding to Cases 1–4, then define a polynomial

$$N_G(B; q) = N_G(r, s, t; q) = \sum_{\substack{\pi \subseteq B \\ \pi^G = \pi}} q^{|\pi|},$$

where the sum is over all G-invariant plane partitions π contained in $B = B(r, s, t)$ (where $B^G = B$). Thus $N_G(B; 1) = N_G(B)$. Macdonald [7, pp. 52–53] observed that previously known results concerning plane partitions can be given the unified statement

$$N_G(B; q) = \prod_{\eta \in B/G} \frac{1 - q^{|\eta|(1 + ht(\eta))}}{1 - q^{|\eta| \cdot ht(\eta)}} \tag{1}$$

where G corresponds to Cases 1 or 2, and where B/G is the set of orbits of G acting on B. For Case 1, Eq. (1) is equivalent to a famous result of MacMahon [9, Sect. 495]. A simple proof based on Schur functions is given in [7, p. 48], and many additional proofs have been given. For Case 2 Eq. (1) is equivalent to a conjecture of MacMahon [8; 9, Sect. 520], shown by G. Andrews [1] to be equivalent to a conjecture of Bender–Knuth [5]. Subsequently the Bender–Knuth conjecture (and therefore (1) when G corresponds to Case 2) was proved independently by Andrews [2], Gordon [6], Macdonald [7, p. 52], and Proctor [13]. Macdonald [7, pp. 52–53] also conjectured that (1) was valid for Case 3, and this conjecture was proved by Mills–Robbins–Rumsey [10]. However, (1) is certainly not true for Case 4; in fact, the right-hand side is not even a polynomial in q. Nevertheless, several persons independently conjectured that (1) is true in Case 4 for $q = 1$; this conjecture remains open. (Thus Case 4 is one of the three open cases.) Andrews and Robbins independently gave a "q-analogue" of Case 4 (alluded to in [4]). We now state an equivalent formula. For G corresponding to Cases 1–4 define another polynomial

$$N'_G(B; q) = \sum_{\substack{\pi \subseteq B \\ \pi^G = \pi}} q^{|\pi/G|}$$

where π/G is the set of orbits of G acting on π. Then in Case 4 it is conjectured that

$$N'_G(B; q) = \prod_{\eta \in B/G} \frac{1 - q^{1 + ht(\eta)}}{1 - q^{ht(\eta)}} \tag{2}$$

Note that (1) and (2) coincide when $q = 1$. Of course also (1) and (2) are identical in Case 1. Remarkably Eq. (2) is also valid in Case 2; it is a restatement of the Bender–Knuth conjecture mentioned above. On the other hand, Eq. (2) is false in Case 3; again the right-hand side is not a polynomial. The known formulas and conjectures concerning Cases 1–4 have such a uniform statement that they demand a uniform proof, but so far only Cases 1 and 2 have achieved any sort of unification, viz., in terms of the representation theory of Lie algebras [13].

Case 5. Let $1 \leqslant i \leqslant 10$. We write $N_i(B)$ for $N_G(B)$ when G corresponds to Case i. It was conjectured independently by this author and D. Robbins (in a different but equivalent form) that

$$N_5(2r, 2s, 2t) = N_1(r, s, t)^2 \tag{3a}$$

$$N_5(2r+1, 2s, 2t) = N_1(r, s, t)\, N_1(r+1, s, t) \tag{3b}$$

$$N_5(2r+1, 2s+1, 2t) = N_1(r+1, s, t)\, N_1(r, s+1, t). \tag{3c}$$

In the next section we prove this conjecture. In fact, we give a generalization involving Schur functions which yields a q-analogue of (3).

Case 6. If $\pi \subseteq B(r, s, t)$ satisfies $\pi^* = \pi^c$, then $r = s$ and $t = 2k$. The anti-diagonal elements $\pi_{i, r+1-i}$ are all equal to k, and we can specify the elements π_{ij} below the anti-diagonal (i.e., $i+j > r+1$) in any way with the values $0, 1, ..., k$ provided they are weakly decreasing in rows and columns. The entire matrix (π_{ij}) is then uniquely determined. Hence (replacing π_{ij} with $k - \pi_{ij}$) $N_6(r, r, 2k)$ is equal to the number of plane partitions contained in the shape $(r-1, r-2, ..., 1)$ with largest part at most k. A simple formula for this number is given by Proctor [14] and may be written

$$N_6(r, r, 2k) = \binom{k+r-1}{r-1} \prod_{i=1}^{r-2} \prod_{j=i}^{r-2} \frac{2k+i+j+1}{i+j+1}.$$

Case 7. If $\pi \subseteq B(r, s, t)$ satisfies $\pi = \pi^* = \pi^c$, then $r = s$ and $t = 2k$. Again the anti-diagonal elements are equal to k. We can specify the elements π_{ij} satisfying $i+j > r+1$ and $i \leqslant j$ in any way with the values $0, 1, ..., k$ provided they are weakly decreasing in rows and columns, and then π is uniquely determined. Thus $N_7(r, r, 2k)$ is equal to the number of plane partitions of the shifted shape $(r-1, r-3, ...)$ (ending in 1 or 2) with largest part at most k. A result proved by Proctor in [12] (see also [17, Sect. 6]), based on the representation theory of the symplectic group, is equivalent to the formulas

$$N_7(2r, 2r, 2k) = N_1(r, r, k)$$

$$N_7(2r+1, 2r+1, 2k) = N_1(r, r+1, k).$$

Case 8. Here $r=2k$, and Mill–Robbins–Rumsey have shown [18] using the ideas of [10], that

$$N_8(2k, 2k, 2k)=\prod_{i=0}^{k-1}\frac{(3i+1)(6i)!\,(2i)!}{(4i+1)!\,(4i)!}.$$

Case 9. Again we must have $r=2k$. Define

$$D(k)=\frac{1!\,4!\,7!\cdots(3k-2)!}{k!\,(k+1)!\cdots(2k-1)!}.$$

Thus $D(1)=1$, $D(2)=2$, $D(3)=7$, $D(4)=42$, $D(5)=429$, $D(6)=7436$, $D(7)=218348$, etc. G. Andrews has shown [3] that $D(k)$ is equal to the number of "descending plane partitions" with largest part at most k. Mills, Robbins, and Rumsey conjecture [11] that $D(k)$ is equal to the number of $k\times k$ "alternating sign matrices" (or "monotone triangles"). Robbins conjectures (private communication) that

$$N_9(2k, 2k, 2k)=D(k)^2.$$

For instance, when $k=2$ the four plane partitions being counted are given by

$$\begin{array}{cccc} 4422 & 4432 & 4333 & 4441 \\ 4422 & 4321 & 4320 & 3321 \\ 2200 & 3210 & 4210 & 3211 \\ 2200 & 2100 & 1110 & 3000. \end{array} \tag{4}$$

Case 10. Again $r=2k$, and Robbins conjectures that $N_{10}(2k, 2k, 2k)=D(k)$. E. g., when $k=2$ the first two plane partitions of (4) are being enumerated. Of the three sets {descending plane partitions with largest part $\leqslant k$}, {$k\times k$ alternating sign matrices}, and {totally symmetric self-complementary $\pi\subseteq B(2k, 2k, 2k)$}, no two are known to have the same cardinality (for all k). Possibly one could establish the equivalence of the conjectures for N_9 and N_{10} by showing

$$N_9(2k, 2k, 2k)=N_{10}(2k, 2k, 2k)^2,$$

without evaluating either case explicitly.

3. Self-complementary Plane Partitions

In order to prove (3), we first review two basic properties of the Schur functions $s_\lambda(x)=s_\lambda(x_1, x_2,\ldots)$. Let $\lambda=(\lambda_1, \lambda_2,\ldots)$ be a partition of $n=$

$\lambda_1+\lambda_2+\cdots$, denoted $|\lambda|=n$ or $\lambda \vdash n$. A *column-strict plane partition* (or *semistandard tableau*) of shape λ is an array $\tau=(\tau_{ij})$ of positive integers τ_{ij}, where $0<j\leqslant\lambda_i$, which is weakly decreasing along rows (i.e., $\tau_{ij}>\tau_{i+1,j}$ when defined) and strictly decreasing along columns ($\tau_{ij}>\tau_{i,j+1}$ when defined). The τ_{ij}'s are called the *parts* of τ. Define $x^\tau=x_1^{m_1}x_2^{m_2}\cdots$, where m_k of the parts of τ are equal to k. We then have the following combinatorial interpretation of the Schur functions [15, Sect. 5] [7, p. 42].

3.1. THEOREM. *The Schur function $s_\lambda(x)$ is given by*

$$s_\lambda(x)=\sum_\tau x^\tau,$$

where τ ranges over all column-strict plane partitions of shape λ.

The product $s_\lambda(x)\,s_\mu(x)$ of Schur functions can be expanded as a linear combination of Schur functions, say

$$s_\lambda(x)\,s_\mu(x)=\sum_\nu c^\nu_{\lambda\mu}s_\nu(x),\qquad c^\nu_{\lambda\mu}\in\mathbb{Z}.$$

The *Littlewood–Richardson* (*L–R*) *rule* gives a combinatorial interpretation of the (nonnegative) integers $c^\nu_{\lambda\mu}$. We give a brief statement of it below; see, e.g., [16, Theorem 2.4] or [7, Chap. I.9] for more details.

3.2. THEOREM (*L–R* rule). *The integer $c^\nu_{\lambda\mu}$ is equal to zero unless $\lambda\subseteq\nu$* (i.e., $\lambda_i\leqslant\nu_i$ *for all* i). *If $\lambda\subseteq\nu$ then $c^\nu_{\lambda\mu}$ is equal to the number of ways of inserting μ_1* 1's, μ_2 2's,... *into a skew diagram Δ of shape ν/λ subject to the following two conditions*:

(a) *The rows are weakly increasing and columns strictly increasing,*

(b) *If a_1, a_2,... is the order of the numbers reading from right to left along the first row, next right to left along the second row, etc., then for any i and j, the number of* i's *among $a_1, a_2,\ldots, a_j$ is not less than the number of* $i+1$'s *among $a_1, a_2,\ldots, a_j$.*

EXAMPLE. Let $\lambda=(3,2,1)$, $\mu=(4,2,1)$, $\nu=(5,4,3,2)$. The arrays A satisfying (a) and (b) are given by

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   11      11      11
  12      22      12
 1       1       2
23      23      13
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Hence $c^\nu_{\lambda\mu}=3$.

The following lemma can easily be generalized, but its present form is adequate for our purposes.

3.3. LEMMA. (a) *Let* $\alpha = \langle s^r \rangle$, *the partition with r parts equal to s. Then* $s_\alpha^2 = \sum_\gamma s_\gamma$, *where* γ *ranges over all partitions of the form*

$$\gamma = (s+\delta_1, s+\delta_2, \ldots, s+\delta_r, s-\delta_r, s-\delta_{r-1}, \ldots, s-\delta_1), \tag{5}$$

where $\delta = (\delta_1, \ldots, \delta_r)$ *is a partition contained in* α, i.e., $s \geqslant \delta_1 \geqslant \cdots \geqslant \delta_r \geqslant 0$.

(b) *Let* $\alpha = \langle s^r \rangle$ *and* $\beta = \langle s^{r+1} \rangle$. *Then* $s_\alpha s_\beta = \sum_\gamma s_\gamma$, *where* γ *ranges over all partitions of the form*

$$\gamma = (s+\delta_1, s+\delta_2, \ldots, s+\delta_r, s, s-\delta_r, s-\delta_{r-1}, \ldots, s-\delta_1),$$

where δ *is a partition contained in* α.

(c) *Let* $\alpha = \langle s^{r+1} \rangle$ *and* $\beta = \langle (s+1)^r \rangle$. *Then* $s_\alpha s_\beta = \sum_\gamma s_\gamma$, *where* γ *ranges over all partitions of the form*

$$\gamma = (s+1+\delta_1, s+1+\delta_2, \ldots, s+1+\delta_r, s, s-\delta_r, s-\delta_{r-1}, \ldots, s-\delta_1),$$

where δ *is a partition contained in* $\langle s^r \rangle$.

Proof. We prove only (a), the other two cases being analogous. Apply Theorem 3.2 to the case $\lambda = \mu = \langle s^r \rangle$. Suppose $c_{\lambda\mu}^\nu \neq 0$. Then ν has the form $\nu = (s+\delta_1, \ldots, s+\delta_r, \varepsilon_1, \varepsilon_2, \ldots)$. In order to satisfy conditions (a) and (b) of Theorem 3.2, the kth row of Δ, for $1 \leqslant k \leqslant r$, must consist of δ_k k's, so δ is contained in $\langle s^r \rangle$. Since the columns of Δ are strictly increasing we must have $\varepsilon_1 \leqslant s$. In order for conditions (a) and (b) of Theorem 3.2 to be satisfied, the first (left-most) column of Δ must consist of the entries j, $j+1, \ldots, r$ where $\delta_{j-1} = s$, $\delta_j < s$. The second column of Δ must consist of j, $j+1, \ldots, r$ where $\delta_{j-1} = s-1$, $\delta_j < s-1$, etc. It follows that $\varepsilon_i = s - \delta_{r+1-i}$. Thus any choice of δ yields a unique Δ of the desired shape, and the proof is complete. ∎

Now let π be a self-complementary plane partition contained in $B(r, s, t')$. At least one of r, s, t' must be even, so suppose without loss of generality that $t' = 2t$. Thus π may be regarded as an $r \times s$ matrix (π_{ij}) with entries contained in $\{0, 1, \ldots, 2t\}$. Define $\tilde{\pi} = (\tilde{\pi}_{ij})$, where $\tilde{\pi}_{ij} = \pi_{ij} + r - i + 1$. Thus $\tilde{\pi}$ is a column-strict plane partition of shape $\langle s^r \rangle$ with entries contained in $\{1, 2, \ldots, 2t+r\}$, and the self-complementarity of π yields $\tilde{\pi}_{ij} + \tilde{\pi}_{r-i+1, s-j+1} = 2t+r+1$. Conversely any such matrix $\tilde{\pi}$ corresponds to a self-complementary π. Now define $w(\pi) = x_1^{m_1} x_2^{m_2} \cdots x_d^{m_d}$, where $d = t + [r/2]$ and m_i is the number of parts of $\tilde{\pi}$ equal to i. (Note that the values

$m_1, \ldots, m_d$ determine m_j for $j = d+1, \ldots, 2t+r$.) For instance, suppose $r = 4$, $s = 5$, $2t = 6$. Let π be given by

$$\pi = \begin{bmatrix} 6 & 6 & 5 & 4 & 3 \\ 6 & 5 & 5 & 4 & 2 \\ 4 & 2 & 1 & 1 & 0 \\ 3 & 2 & 1 & 0 & 0 \end{bmatrix}$$

Then

$$\tilde{\pi} = \begin{bmatrix} 10 & 10 & 9 & 8 & 7 \\ 9 & 8 & 8 & 7 & 5 \\ 6 & 4 & 3 & 3 & 2 \\ 4 & 3 & 2 & 1 & 1 \end{bmatrix},$$

and $w(\pi) = x_1^2 x_2^2 x_3^3 x_4^2 x_5$.

3.4. THEOREM. *Define*

$$F(r, s, 2t; x) = F(r, s, 2t; x_1, \ldots, x_d) = \sum_\pi w(\pi),$$

where π ranges over all self-complementary plane partitions contained in the box $B(r, s, 2t)$ and $d = t + [r/2]$ as above. Then:

(a) $F(2r, 2s, 2t; x) = s_\alpha(x_1, \ldots, x_d)^2$, *where* $\alpha = \langle s^r \rangle$.

(b) $F(2r+1, 2s, 2t; x) = s_\alpha(x_1, \ldots, x_d)\, s_\beta(x_1, \ldots, x_d)$, *where* $\alpha = \langle s^r \rangle$ *and* $\beta = \langle s^{r+1} \rangle$.

(c) $F(2r+1, 2s+1, 2t; x) = s_\beta(x_1, \ldots, x_d)\, s_\gamma(x_1, \ldots, x_d)$, *where* $\beta = \langle s^{r+1} \rangle$ *and* $\gamma = \langle (s+1)^r \rangle$.

Proof. We prove only (a) (using Lemma 3.3(a)), the proof of (b) and (c) being analogous (using Lemma 3.3(bc)). Consider the entries of $\tilde{\pi}$ equal to $d+1, d+2, \ldots, 2t+2r$ (where $d = t+r$). They occupy a diagram of some shape $\gamma \vdash 2rs$ and we set $\gamma = S(\pi)$. By the self-complementarity of $\tilde{\pi}$, the shape γ has the form (5) for some partition δ contained in $\langle s^r \rangle$. Moreover, given any γ of the form (5) choose any column-strict plane partition σ of shape γ with parts contained in $\{t+r, \ldots, 2t+2r\}$, and $\tilde{\pi}$ is then uniquely determined. Since $m_i = m_{2t+2r-i+1}$, it follows from Theorem 3.1 that

$$s_\gamma(x_1, \ldots, x_d) = \sum_\pi w(\pi),$$

where π ranges over all self-complementary plane partitions contained in $B(2r, 2s, 2t)$ satisfying $\gamma = S(\pi)$. Hence

$$F(2r, 2s, 2t; x) = \sum_{\gamma} s_{\gamma}(x_1, ..., x_d),$$

summed over all γ satisfying (5). The proof follows from Lemma 3.3(a). ∎

To obtain equation (3) from Theorem 3.4 note that the number $N_1(r, s, t)$ of plane partitions $\pi = (\pi_{ij})$ contained in $B(r, s, t)$ is equal to the number of column-strict plane partitions $\tilde{\pi}$ of shape $\langle s^r \rangle$ and largest part $\leqslant r + t$ (viz., $\tilde{\pi}_{ij} = \pi_{ij} + r - i + 1$). Hence if $\alpha = \langle s^r \rangle$ then

$$N_1(r, s, t) = s_{\alpha}(1, ..., 1) \qquad (r + t \text{ 1's}),$$

so (3) follows from Theorem 3.4. More generally, we get a "q-analogue" of (3) by substituting $x_i = q^i$ in Theorem 3.4. By (1) applied to Case 1 (or because $s_{\lambda}(q, q^2, ..., q^d)$ can in general be written as a simple product; see [7, p. 27, Example 1]) we have

$$q^{-r(r+1)s} \sum_{\pi} q^{v(\pi)} = \left[\prod_{x \in B'} \frac{1 - q^{1 + ht(x)}}{1 - q^{ht(x)}} \right]^2,$$

where π ranges over all self-complementary plane partitions contained in $B = B(2r, 2s, 2t)$, where $B' = B(r, s, t)$, and where $v(\pi)$ is the sum of those entries i of $\tilde{\pi}$ satisfying $1 \leqslant i \leqslant r + t$. Similar formulas hold for $B(2r + 1, 2s, 2t)$ and $B(2r + 1, 2s + 1, 2t)$.

Let us mention that Eq. (3) might also be proved by exhibiting an explicit bijection between self-complementary plane partitions and suitable pairs of ordinary plane partitions. The various proofs of the Littlewood–Richardson rule can be used to give a simple bijection, but in order to prove the validity of the bijection one must invoke the validity of the proof of the Littlewood–Richardson rule used to define the bijection. Is there a simple bijection which avoids the Littlewood–Richardson rule entirely?

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