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THE Q-DYSON CONJECTURE, GENERALIZED EXPONENTS, AND THE INTERNAL PRODUCT OF SCHUR FUNCTIONS

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ABSTRACT. The q-Dyson conjecture is a combinatorial problem posed by G. Andrews in 1975. The conjecture can be formulated in terms of symmetric functions, and it is shown how the theory of symmetric functions can be used to prove a limiting form of the conjecture. The proof uses a new identity involving the internal product of Schur functions. The same techniques yield information about a limiting form of the "generalized exponents" of $SL(n,\mathbb{C})$, as defined by Kostant. Complete details will appear in a forthcoming paper in Linear and Multilinear Algebra.

1. THE Q-DYSON CONJECTURE. Let $a_1,\cdots, a_n$ be nonnegative integers. In 1962 Dyson [2] conjectured that when the product

$$\prod_{\substack{i,j=1\\ i\neq j}}^{n} \left(1-x_ix_j^{-1}\right)^{a_i}$$

is expanded as a Laurent polynomial in the variables $x_1,\cdots, x_n$, then the constant term is equal to the multinomial coefficient $(a_1+\cdots+a_n)!/a_1!\cdots a_n!$. This conjecture was proved in 1962 by Gunson [4] and Wilson [15], and in 1970 an exceptionally elegant proof was given by Good [3].

In 1975 G. Andrews [1, p. 216] formulated a "q-analogue" of the Dyson conjecture, which reduces to the original conjecture when $q = 1$. Write $(a)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$, so $(q)_n = (1-q)(1-q^2)\cdots(1-q^n)$.

Q-DYSON CONJECTURE. When the product

$$\prod_{1\le i<j\le n} (qx_ix_j^{-1})_{a_i}\,(x_jx_i^{-1})_{a_j}$$

is expanded as a Laurent polynomial in the variables $x_1,\cdots, x_n$, then the constant term is equal to the q-multinomial coefficient

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$$(q)_{a_1+\cdots+a_n}/(q)_{a_1}\cdots(q)_{a_n}.$$

This conjecture was proved for $n \le 3$ by Andrews [1] and for $n = 4$ by Kadell [6]. It was also proved for $a_1 = \cdots = a_n = 1, 2$, or ∞ by Macdonald [12], who formulated a far-reaching generalization. Further work appears in [5]. Here we will establish, as a corollary to a more general result, the case $a_1 = a_2 = \cdots = \ell$ in the limit $n \longrightarrow \infty$. In the form stated above, the q-Dyson conjecture becomes meaningless when $n \longrightarrow \infty$. However, it can be restated to make sense in this limit. We will give a restatement in terms of representation theory due to Macdonald [12, Conj. 3.1']. Let $SL(n,\mathbb{C})$ denote the group of $n \times n$ complex matrices of determinant one, and $s\ell(n,\mathbb{C})$ its Lie algebra of all $n \times n$ complex matrices of trace zero. Let

$$\mathrm{ad}\colon SL(n,\mathbb{C}) \longrightarrow GL(s\ell(n,\mathbb{C}))$$

denote the adjoint representation of $SL(n,\mathbb{C})$, defined by

$$(\mathrm{ad}\ X)(A) = XAX^{-1},$$

for $X \in SL(n,\mathbb{C})$ and $A \in s\ell(n,\mathbb{C})$.

Q-DYSON CONJECTURE FOR $a_1 = \cdots = a_n = \ell$ (reformulated). The multiplicity of the trivial character of $SL(n,\mathbb{C})$ in the virtual character

$$\det(1-q\cdot \mathrm{ad}\ X)(1-q^2\cdot \mathrm{ad}\ X)\cdots(1-q^{\ell-1}\cdot \mathrm{ad}\ X)$$

is equal to

$$\prod_{i=1}^{n-1}\prod_{j=1}^{\ell-1}\left(1-q^{\ell i+j}\right). \tag{1}$$

2. SYMMETRIC FUNCTIONS. The above conjecture can be formulated in terms of symmetric functions, which we will now briefly review. Let $\lambda = (\lambda_1,\lambda_2,\cdots)$ be a <u>partition</u>, i.e., a decreasing sequence $\lambda_1 \ge \lambda_2 \ge \cdots \ge 0$ of nonnegative integers with only finitely many λ_i unequal to zero. If $\lambda_{n+1} = \lambda_{n+2} = \cdots = 0$ then we also write $\lambda = (\lambda_1,\cdots,\lambda_n)$. The number of nonzero λ_i is the <u>length</u> of λ, denoted $\ell(\lambda)$. If $m = \lambda_1 + \lambda_2 + \cdots$ then we write $\lambda \vdash m$ or $|\lambda| = m$. The <u>conjugate</u> partition $\lambda' = (\lambda_1', \lambda_2',\cdots)$ to λ has $\lambda_i - \lambda_{i+1}$ parts equal to i.

Let $\Lambda_n = \Lambda_n(x)$ denote the ring of all symmetric polynomials with rational coefficients in the variables $x = (x_1,\cdots, x_n)$, and let Ω_n denote Λ_n modulo the ideal generated by $x_1x_2\cdots x_n - 1$. A vector space basis for Ω_n consists of all <u>Schur functions</u> $s_\lambda(x) = s_\lambda(x_1,\cdots, x_n)$, where λ ranges over all partitions of length $\le n - 1$. For the definition and basic properties of Schur functions, see [11].

If φ: $SL(n,\mathbb{C}) \longrightarrow GL(N,\mathbb{C})$ is a (polynomial) representation of $SL(n,\mathbb{C})$, then the character of φ is the unique polynomial char $\varphi \in \Omega_n$ satisfying char $\varphi = \operatorname{tr} \varphi(X)$ for any $X \in SL(n,\mathbb{C})$ with eigenvalues $x_1,\cdots, x_n$. A basic theorem (e.g. [14]) on the representations of $SL(n,\mathbb{C})$ states that the irreducible (polynomial) characters of $SL(n,\mathbb{C})$ are precisely the Schur functions $s_\lambda(x) \in \Omega_n$, $\ell(\lambda) \le n-1$. Thus the problem of decomposing char φ into irreducible characters is equivalent to expanding char φ as a linear combination of Schur functions in the ring Ω_n.

Sometimes it is convenient to work with symmetric functions (= formal power series) in infinitely many variables $x = (x_1,x_2,\cdots)$. We let $\Lambda = \Lambda(x)$ denote the ring of all symmetric formal power series of bounded degree with rational coefficients in the variables x. Λ is the inverse limit of the rings Λ_n in the category of graded rings. The Schur functions $s_\lambda(x)$, for all partitions λ, form a vector space basis for $\Lambda(x)$. The completion $\hat{\Lambda}$ of Λ (with respect to the ideal of symmetric functions with zero constant term) consists of all symmetric formal power series with no restriction on the degree. $\hat{\Lambda}$ is the inverse limit of the rings Λ_n in the category of rings. For further information, see [11, Ch. I.2]. Let us remark that in [11] the elements of Λ_n, Λ, and $\hat{\Lambda}$ have integer coefficients, but we will find it convenient to allow rational coefficients from the start.

Now suppose $X \in SL(n,\mathbb{C})$ has eigenvalues $x_1,\cdots, x_n$, i.e.,

$$\det(1-qX) = \prod_{i=1}^{n} (1-qx_i) .$$

Since $\operatorname{ad} X$ has eigenvalues $x_i x_j^{-1}$ (once each for $i \neq j$) and 1 ($n-1$ times) (e.g., [14, eqn. (8)]), we have

$$\begin{aligned} &\det(1-q\cdot\operatorname{ad} X)\cdots(1-q^{\ell-1}\cdot\operatorname{ad} X) \\ &= (q)_{\ell-1}^{-1} \prod_{i,j=1}^{n} \prod_{k=1}^{\ell-1} (1-q^k x_i x_j^{-1}) . \end{aligned} \tag{2}$$

Q-DYSON CONJECTURE FOR $a_1 =\cdots= a_n = \ell - 1$ (again reformulated). When equation (2) is expanded in the ring $\Omega_n \otimes \mathbb{Q}[[q]]$ as a linear combination of Schur functions $s_\lambda(x)$ with $\ell(\lambda) \le n - 1$, then the coefficient of the trivial Schur function $s_\phi(x)$ is given by (1).

The above conjecture makes sense as $n \longrightarrow \infty$. We might as well consider a much more general situation, and later specialize to the case at hand . Thus introduce two new sets $u = (u_1,u_2,\cdots)$ and $v = (v_1,v_2,\cdots)$ of variables, and write (in the ring $\Omega_n \otimes \mathbb{Q}[[u,v]]$)

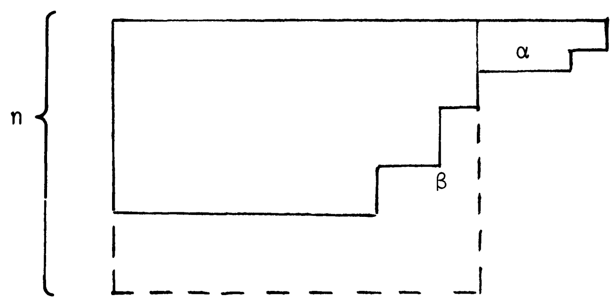
$$\left[\prod_k \frac{1-u_k}{1-v_k}\right] \det \prod_k \frac{1-u_k \mathrm{ad}\ X}{1-v_k \mathrm{ad}\ X} = \sum_\lambda c_\lambda^n(u;v) s_\lambda(x), \tag{3}$$

where λ ranges over all partitions with $\ell(\lambda) \le n-1$. Clearly $c_\lambda^n(u;v) \in \mathbb{Z}[[u,v]]$, the ring of formal power series with integer coefficients in the variables $u = (u_1, u_2, \cdots)$ and $v = (v_1, v_2, \cdots)$.

We wish to consider $c_\lambda^n(u;v)$ as $n \to \infty$. To do so, one must vary λ with n or else the limit becomes zero or undefined. The correct way of passing to the limit was suggested by R. Gupta (in the somewhat less general context of Section 5). Given any two partitions α and β of lengths k and ℓ of the same integer m, and given $n \ge k + \ell$, define the partition

$$[\alpha, \beta]_n = (\beta_1+\alpha_1, \beta_1+\alpha_2, \cdots, \beta_1+\alpha_k, \underbrace{\beta_1, \cdots, \beta_1}_{n-k-\ell}, \beta_1-\beta_\ell, \beta_1-\beta_{\ell-1}, \cdots, \beta_1-\beta_2)$$

of length $\le n-1$.

$[\alpha,\beta]_n$ = 

It follows from Gupta's work that

$$\lim_{n\to\infty} c^n_{[\alpha,\beta]_n}(u;v)$$

exists as a formal power series, which we denote by $c_{\alpha\beta}(u;v)$. Our main goal here is a formula for $c_{\alpha\beta}(u;v)$. The q-Dyson conjecture in the case $n \to \infty$ corresponds to taking $\alpha = \beta = \phi$ (the void or trivial partition) and $v_i = 0$, $u_1 = q$, $u_2 = q^2, \cdots, u_{\ell-1} = q^{\ell-1}$, $u_\ell = u_{\ell+1} = \cdots = 0$.

In order to state our result, we first review some more background from the theory of symmetric functions. The irreducible characters χ^λ of the symmetric group S_m are indexed by partitions λ of m. If $w \in S_m$, then define $\rho(w)$ to be the partition of m whose parts are the cycle lengths of w. For any

$\lambda \vdash m$ with $\ell = \ell(\lambda)$, define the power-sum symmetric function $p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_\ell}$, where $p_n(x) = \sum_i x_i^n$. The Schur functions and power-sums are related by (e.g., [11, Ch. I.7])

$$s_\lambda = \frac{1}{m!} \sum_{w \in S_m} \chi^\lambda(w) p_{\rho(w)} . \tag{4}$$

Now let

$$\chi^\alpha \chi^\beta = \sum_\gamma g_{\alpha\beta\gamma} \chi^\gamma ,$$

where each $g_{\alpha\beta\gamma}$ is a nonnegative integer. It is an important open problem to obtain a nice combinatorial interpretation of $g_{\alpha\beta\gamma}$. D.E. Littlewood [10], in order to incorporate the Kronecker product $\chi^\alpha \chi^\beta$ into the theory of symmetric functions, defined an associative (and commutative) product $f*g$ on symmetric functions by

$$s_\alpha * s_\beta = \sum_\gamma g_{\alpha\beta\gamma} s_\gamma ,$$

called the internal product. (Littlewood uses the term "inner product". Since the product $f*g$ has nothing to do with the usual definition of inner product in linear algebra, we have followed a suggestion of I.G. Macdonald in calling it the internal product. Littlewood uses the notation $f \circ g$ for internal product. Since we are adhering to the notation of [11], where $f \circ g$ denotes plethysm, we have introduced the new notation $f*g$.) Note that $s_\alpha * s_m = s_\alpha$ and $s_\alpha * s_{1^m} = s_{\alpha'}$, where α' denotes the conjugate partition to α.

In terms of the power-sums we have the expansion

$$s_\alpha * s_\beta = \frac{1}{m!} \sum_{w \in S_m} \chi^\alpha(w) \chi^\beta(w) p_{\rho(w)} . \tag{5}$$

The following basic property of the internal product is due to Schur [13] (p.69 of Dissertation; p.65 of GA):

2.1. PROPOSITION. We have

$$\prod_{i,j,k} (1 - x_i y_j v_k)^{-1} = \sum_{\alpha,\beta} s_\alpha * s_\beta(v) s_\alpha(x) s_\beta(y). \tag{6}$$

Now define a scalar product $\langle f,g \rangle$ in the ring Λ by letting the Schur functions form an orthonormal basis, i.e.,

$$\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu} .$$

Given partitions α, β, define a symmetric function $s_{\alpha/\beta} \in \Lambda$, called a skew Schur function, by the rule

$$\langle s_{\alpha/\beta}, s_\gamma\rangle = \langle s_\alpha, s_\beta s_\gamma\rangle .$$

In other words, multiplication by s_β is adjoint to the linear transformation sending s_α to $s_{\alpha/\beta}$. It is not difficult to show that $s_{\alpha/\beta} = 0$ unless $\beta \le \alpha$, i.e., $\beta_i \le \alpha_i$ for all i. For further information, see [11, Ch.I.5].

Let us remark that the Schur function $s_{[\alpha,\beta]_n}(x)$ was considered by D.E. Littlewood [8], who essentially showed that in the ring Ω_n we have

$$s_{[\alpha,\beta]_n}(x) = \sum_\lambda (-1)^{|\lambda|} s_{\alpha/\lambda}(x) s_{\beta/\lambda'}(1/x) ,$$

where $x = (x_1,\dots, x_n)$ and $1/x = (1/x_1,\dots, 1/x_n)$. For instance, the adjoint representation of $SL(n,\mathbb{C})$ corresponds to the partition $[1,1]_n$, with character

$$\begin{aligned} s_{[1,1]_n}(x) &= s_1(x)s_1(1/x) - 1 \\ &= (x_1 +\dots+ x_n)(x_1^{-1} +\dots+ x_n^{-1}) - 1 \\ &= n - 1 + \sum_{i\neq j} x_i x_j^{-1} . \end{aligned}$$

3. A FORMULA FOR $c_{\alpha\beta}(u;v)$. In order to evaluate $c_{\alpha\beta}(u;v)$ we first obtain a formula for the generating function

$$C(x,y) = \sum_{\alpha,\beta} c_{\alpha\beta}(u;v)s_\alpha(x)s_\beta(y) \in \mathbb{Q}[[u,v]] \otimes \hat{\Lambda}(x) \otimes \hat{\Lambda}(y) .$$

It will be more convenient to work with

$$C_0(x,y) = \sum_{\alpha,\beta} c_{\alpha\beta}(0;v)s_\alpha(x)s_\beta(y) ,$$

and later to apply a standard trick to obtain $C(x,y)$ from $C_0(x,y)$. (Here $c_{\alpha\beta}(0;v)$ denotes the substitution $u_k = 0$ in $c_{\alpha\beta}(u;v)$.)

3.1. LEMMA. We have

$$C_0(x,y) = \sum_{\lambda,\mu,\alpha} s_\lambda * s_\mu(v) s_{\lambda/\alpha}(x) s_{\mu/\alpha}(y)$$

Sketch of proof. One begins by setting $y_j = x_j^{-1}$ in (6) so that the left-hand side of (6) coincides with the left-hand side of (3) when $u_k = 0$. The proof then proceeds by standard manipulations of symmetric functions, which we omit. □

Next we find a more tractable expression for $C_0(x,y)$ by establishing the following symmetric function identity, which apparently is new.

3.2. LEMMA. We have

$$\prod_{i,j}\prod_{r>0}\prod_{a_1,\cdots,a_r}(1 - x_i y_j v_{a_1}\cdots v_{a_r})^{-1}$$

$$= \left[\prod_{k\geq 1}(1 - p_k(v))\right]\sum_{\lambda,\mu,\alpha} s_\lambda * s_\mu(v) s_{\lambda/\alpha}(x) s_{\mu/\alpha}(y). \qquad (7)$$

(Here $a_1,\cdots, a_r$ range independently over all indices of the v's.)

Sketch of proof. We work in the ring $R = \mathbb{Q}((v)) \otimes \Lambda(x) \otimes \Lambda(y)$, which should be regarded as consisting of formal power series of bounded degree, symmetric in the x's and y's separately, with coefficients in the field $\mathbb{Q}((v))$ (the quotient field of $\mathbb{Q}[[v]]$). Define a scalar product on R by letting the elements $s_\alpha(x)s_\beta(y)$ form an orthonormal basis.

If $f \in R$ then let $D(f)$ denote the linear transformation which is adjoint to multiplication by f, i.e.,

$$\langle D(f)g,\ h\rangle = \langle g,\ fh\rangle \ .$$

Note that $D(f+g) = D(f) + D(g)$. Let $P(v) = \prod_{k\geq 1}(1 - p_k(v))$. The right-hand of (7) is given by

$$P(v)\sum_\alpha D(s_\alpha(x)s_\alpha(y))\sum_{\lambda,\mu} s_\lambda * s_\mu(v) s_\lambda(x) s_\mu(y)$$

$$= P(v)\ D(\prod_{i,j}(1 - x_i y_j)^{-1})\prod_{i,j,k}(1 - x_i y_j v_k)^{-1} \ .$$

Thus writing LHS for the left-hand side of (7), we need to show that for all $f \in R$,

$$\langle \text{LHS},\ f\rangle = \langle P(v)\ \Pi(1-x_i y_j v_k)^{-1},\ f\ \Pi(1-x_i y_j)^{-1}\rangle.$$

It suffices to check this for all f forming a $\mathbb{Q}((v))$-basis for R. Choose $f = p_\alpha(x)p_\beta(y)$, and the verification becomes a routine computation using standard symmetric function techniques. □

If we now compare Lemmas 3.1 and 3.2 and expand the right-hand side of (7) in terms of the $p_\lambda(v)$'s, we obtain:

3.3. LEMMA. We have

$$c_{\alpha\beta}(0;v) = P(v)^{-1}\ s_\alpha * s_\beta\left[p_k \to \frac{p_k(v)}{1-p_k(v)}\right] \ . \qquad (8)$$

The notation indicates that we are to expand $s_\alpha * s_\beta$ in terms of the p_k's, as given explicitly by (5), and substitute $p_k(v)/(1 - p_k(v))$ for p_k. □

In order to find a similar formula for $c_{\alpha\beta}(u;v)$, we first replace the variables v in (8) by the two sets of variables u and v. Now let ω_u denote the algebra automorphism described in [11, pp. 14-17, 26] acting on symmetric functions in u (regard all other variables as scalars commuting with ω_u). By standard properties of ω_u,

$$\omega_u\left[\prod_k (1+u_k)^{-1}(1-v_k)^{-1}\right]\det\prod_k (1+u_k \text{ ad } X)^{-1}(1-v_k \text{ ad } X)^{-1}$$

$$= \left[\prod_k \frac{1-u_k}{1-v_k}\right] \det\prod_k \frac{1-u_k \text{ ad } X}{1-v_k \text{ ad } X} .$$

Hence from (3) we get

$$\omega_u\, c_{\alpha\beta}(0;-u,v) = c_{\alpha\beta}(u;v) .$$

On the other hand, from [11, (2.13)] there follows

$$\omega_u\, p_k(-u,v) = p_k(v) - p_k(u).$$

We deduce from Lemma 3.3 our main result:

3.4. THEOREM. We have

$$c_{\alpha\beta}(u;v) = \left[\prod_k (1+p_k(u)-p_k(v))^{-1}\right] s_\alpha * s_\beta\left(p_k \to \frac{-p_k(u)+p_k(v)}{1+p_k(u)-p_k(v)}\right) . \quad \square$$

The above theorem is essentially implicit in the work of P. Hanlon. He computed maximal weight vectors for certain virtual representations of $SL(n,\mathbb{C})$, and it was apparent that his result implied an identity involving symmetric functions. The actual identity turned out to be a special case of Theorem 3.4, but there is no difficulty in obtaining all of Theorem 3.4 from Hanlon's technique. Earlier I had proved some special cases of Theorem 3.4, and the proof sketched here uses similar techniques.

4. APPLICATION TO THE Q-DYSON CONJECTURE. Recall that the q-Dyson conjecture corresponds to the substitution

$$u_i = \begin{cases} q^i, & 1 \le i \le \ell - 1 \\ 0, & i \ge \ell \end{cases}$$

$$v_i = 0,$$

$$\alpha = \beta = \phi .$$

Let us first state the result for arbitrary α,β .

4.1. COROLLARY. Let $\alpha,\beta \vdash m$. The coefficient of the character $s_{[\alpha,\beta]_n}$ in the expansion of the virtual character

$$\det(1-q \cdot \mathrm{ad}\ X)(1-q^2 \cdot \mathrm{ad}\ X) \cdots (1-q^{\ell-1} \cdot \mathrm{ad}\ X) \tag{9}$$

of $SL(n,\mathbb{C})$ approaches, as $n \to \infty$, the value

$$\left[\prod_{k=1}^{\ell-1} (1-q^k)^{-1}\right] c_{\alpha\beta}(q,q^2,\cdots, q^{\ell-1}; 0)$$

$$= \left[\prod_{i\ge 1} \prod_{j=1}^{\ell-1} (1-q^{\ell i+j})\right] s_\alpha * s_\beta\left(p_k \to \frac{-q^k(1-q^{(\ell-1)k})}{1-q^{\ell k}}\right) . \quad \square$$

If we let $\alpha = \beta = \phi$ (the void partition) above, then $s_\phi * s_\phi = s_\phi = 1$, so we obtain:

4.2. COROLLARY (q-Dyson conjecture for $a_i = \ell - 1$ and $n = \infty$). The coefficient of the trivial character s_ϕ in (9) approaches, as $n \to \infty$, the value

$$\prod_{i\ge 1} \prod_{j=1}^{\ell-1} (1-q^{\ell i+j}) . \quad \square$$

What can be said about the form of the generating function $c_{\alpha\beta}(q,q^2,\cdots, q^{\ell-1};0)$ appearing in Corollary 4.1? We will state a few results along these lines. Empirical evidence suggests that much stronger statements are possible, and that a fairly simple explicit formula may exist in many cases, if not in general.

For any partition λ, define the <u>hook-length</u> of λ at $x = (i,j) \in \lambda$ to be

$$h(x) = h(i,j) = \lambda_i + \lambda_j' - i - j + 1 .$$

Here we identify $\lambda = (\lambda_1,\lambda_2,\cdots)$ with its Young diagram $\{(i,j)\colon 1 \le i \le \lambda_1' = \ell(\lambda),\ 1 \le j \le \lambda_i\}$. Following [11, p. 28], let

$$H_\lambda(q) = \prod_{x\in\lambda} (1-q^{h(x)}) ,$$

the "hook polynomial" of λ .

The following lemma appears to be new. Its proof is based upon the combinatorial description due to Littlewood and Richardson [9, p.70] [11, Ex.5, p. 64] (equivalent to the "Murnagham-Nakayama formula") for computing the irreducible characters of the symmetric group S_m . If $w \in S_m$ has cycle type $\rho = \rho(w)$, then we write $\chi^\lambda(\rho)$ for $\chi^\lambda(w)$.

4.3. LEMMA. Let $\lambda,\rho \vdash m$. Let χ^λ denote the irreducible character of S_m corresponding to λ . If $\chi^\lambda(\rho) \neq 0$, then $H_\lambda(q)$ is divisible by $\prod_{i=1}^{\ell(\rho)} (1 - q^{\rho_i})$. □

4.4. PROPOSITION. Define a formal power series $D_{\alpha\beta}(q)$ (which depends on ℓ) by

$$c_{\alpha\beta}(q,q^2,\cdots, q^{\ell-1}; 0) = \left[\prod_{k\geq 1} \frac{1-q^k}{1-q^{k\ell}}\right] D_{\alpha\beta}(q) .$$

Then for some polynomial $L_{\alpha\beta}(q) \in \mathbb{Z}[q]$ (which depends on ℓ), we have

$$D_{\alpha\beta}(q) = L_{\alpha\beta}(q)H_\alpha(q^\ell)^{-1} .$$

Proof. By Corollary 4.1 and (5), we have

$$D_{\alpha\beta}(q) = \frac{1}{m!} \sum_{w\in S_m} \chi^\alpha(w)\chi^\beta(w) \prod_{k\geq 1} \left(\frac{-q^k(1-q^{(\ell-1)k})}{1-q^{\ell k}}\right)^{m_k(w)} ,$$

where $m_k(w)$ parts of $\rho(w)$ are equal to k. By Lemma 4.3, every term of the above sum for which $\chi^\alpha(w) \neq 0$ is a rational function whose denominator divides $H_\alpha(q^\ell)$. Hence the entire sum is a rational function whose denominator can be taken to be $H_\alpha(q^\ell)$, and it is easily seen that the numerator has integer coefficients. □

4.5. PROPOSITION. Let β consist of the single part m. Then

$$L_{\alpha m}(q) = \prod_{(i,j)\in\alpha} (q^{i\ell} - q^{(j-1)\ell+1}) .$$

Equivalently, $L_{\alpha m}(q)$ is obtained by multiplying together for all i the product of the first α_i terms from the i-th row of the array

$$\begin{array}{lll} q^{\ell} - q & q^{\ell} - q^{\ell+1} & q^{\ell} - q^{2\ell+1} \ \cdots \\ q^{2\ell} - q & q^{2\ell} - q^{\ell+1} & q^{2\ell} - q^{2\ell+1} \ \cdots \\ q^{3\ell} - q & q^{3\ell} - q^{\ell+1} & q^{3\ell} - q^{2\ell+1} \ \cdots \\ \vdots & \vdots & \vdots \end{array}$$

The proof is essentially a consequence of Littlewood's work on "S-functions of special series", in particular, Theorem II on page 125 of [9].

5. GENERALIZED EXPONENTS. There is an additional specialization of $c_{\alpha\beta}(u;v)$ of independent interest. Let $\mathfrak{g} = s\ell(n,\mathbb{C})$. The adjoint action of $SL(n,\mathbb{C})$ extends to an action on the symmetric algebra $S(\mathfrak{g}) = \coprod_{k\geq 0} S^k(\mathfrak{g})$, where S^k denotes the k-th symmetric power. It is well-known that the ring $J = S(\mathfrak{g})^{SL(n,\mathbb{C})}$ of invariants of this action is a polynomial ring in $n-1$ variables $\theta_2,\cdots,\theta_n$, where θ_i is homogeneous of degree i. Namely, for $A \in \mathfrak{g}$, $\theta_i(A)$ is the coefficient of t^{n-i} in the characteristic polynomial $\det(A-tI)$ of A.

By a theorem of Kostant [7, Thm. 0.2], we can write

$$S(\mathfrak{g}) = J \otimes H ,$$

where $H = \coprod H^k$ is a graded subspace of $S(\mathfrak{g})$ invariant under $SL(n,\mathbb{C})$. Let H_λ denote the isotypic component of H corresponding to λ, i.e., the sum of all subspaces of H which afford the character $s_\lambda(x)$. We may then decompose H_λ into irreducible subspaces H_λ^i,

$$H_\lambda = \coprod_i H_\lambda^i ,$$

where each H_λ^i can be chosen to be <u>homogeneous</u>, i.e., to lie in $S^{d_i}(\mathfrak{g})$ for some d_i. The numbers d_i are called the <u>generalized exponents</u> of λ. Define

$$G_\lambda(q) = \sum_i q^{d_i} ,$$

the generating function for the generalized exponents of λ. Kostant also shows in [7, Thm. 0.11] (when applied to $SL(n,\mathbb{C})$) that $G_\lambda(1)$ is equal to the dimension of the zero-weight space of the representation λ and is therefore finite. Thus $G_\lambda(q)$ is a polynomial in q.

In terms of generating functions it is easy to see from the above discussion that

$$\det(1-q \cdot \mathrm{ad}\, X)^{-1} = \frac{1}{(1-q^2)\cdots(1-q^n)} \sum_\lambda G_\lambda(q) s_\lambda(x_1,\cdots,x_n)$$

$$(\text{modulo } x_1\cdots x_n - 1). \qquad (10)$$

Ranee Gupta conceived the idea of studying $G_{[\alpha,\beta]_n}(q)$ as $n \to \infty$, and showed that

$$G_{\alpha\beta}(q) := \lim_{n\to\infty} G_{[\alpha,\beta]_n}(q)$$

exists as a formal power series. She conjectured that $G_{\alpha\beta}(q)$ is a rational function $P_{\alpha\beta}(q)H_\alpha(q)^{-1}$, where $P_{\alpha\beta}(q)$ is a polynomial with nonnegative integer coefficients satisfying $P_{\alpha\beta}(1) = \chi^\beta(1)$. Later she and I conjectured on the basis of numerical evidence that $G_{\alpha\beta}(q) = s_\alpha * s_\beta(q,q^2,\cdots)$. We will indicate how all these conjectures follow immediately from our previous discussion, except for the nonnegativity of the coefficients of $P_{\alpha\beta}(q)$, which remains open.

Comparing (3) with (10), we see that

$$G_{\alpha\beta}(q) = \left[\prod_{k\geq 1} (1-q^k)\right] c_{\alpha\beta}(0;q) .$$

From Theorem 3.4 we deduce:

5.1. PROPOSITION. We have

$$G_{\alpha\beta}(q) = s_\alpha * s_\beta(q,q^2,\cdots) .$$

Additional properties of $G_{\alpha\beta}(q)$ follow from Proposition 5.1 in the same way Proposition 4.4 follows from Corollary 4.1. We merely state the results here.

5.2. PROPOSITION. (i) There is a polynomial $P_{\alpha\beta}(q) \in \mathbb{Z}[q]$ for which

$$G_{\alpha\beta}(q) = P_{\alpha\beta}(q)H_\alpha(q)^{-1} .$$

(ii) $P_{\alpha\beta}(1) = \chi^\beta(1)$, the number of standard Young tableaux of shape β.

(iii) $P_{\alpha'\beta'}(q) = P_{\alpha\beta}(q)$.

(iv) $q^{m+h(\alpha)}P_{\alpha\beta}(1/q) = P_{\alpha\beta'}(q)$, where $|\alpha| = |\beta| = m$ and $h(\alpha) = \sum_{x\in\alpha} h(x)$.

(v) $\deg P_{\alpha\beta}(q) \leq h(\alpha)$, and the coefficient of $q^{h(\alpha)}$ is the Kronecker delta $\delta_{\alpha\beta'}$.

(vi) $P_{\alpha\beta}(q)$ is divisible by q^m, and the coefficient of q^m is $\delta_{\alpha\beta}$.

(vii) $P_{\beta\alpha}(q) = P_{\alpha\beta}(q)H_\beta(q)H_\alpha(q)^{-1}$.

(viii) Let β consist of the single part m, and write $P_{\alpha m}(q)$ for $P_{\alpha\beta}(q)$. Then $P_{\alpha m}(q) = q^{m+n(\alpha)}$, where $n(\alpha) = \Sigma(i-1)\alpha_i = \sum\binom{\alpha_i'}{2}$.

Finally we state explicitly the conjecture mentioned above.

5.3. CONJECTURE (Gupta-Stanley). The coefficients of $P_{\alpha\beta}(q)$ are nonnegative.

Alain Lascoux has proved the above conjecture when β is a "hook", i.e., a partition of the form $(m-k, 1^k)$ for some $0 \le k \le m-1$. He has shown that in this case $P_{\alpha\beta}(q)$ is the coefficient of t^k in the product

$$q \prod_{\substack{(i,j)\in\alpha \\ (i,j)\neq(1,1)}} (q^i + tq^j) .$$

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Late note. The entire q-Dyson conjecture has been proved by D. Zeilberger, A proof of Andrew's q-Dyson conjecture, Discrete Math., submitted.

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