THE NUMBER OF FACES OF SIMPLICIAL POLYTOPES AND SPHERES[a]

Richard P. Stanley

Department of Mathematics
Massachusetts Institute of Technology
Cambridge, Massachusetts 02139

Let $\mathscr{P}$ be a d-polytope, i.e., a d-dimensional convex polytope [17, 30]. Let $f_i = f_i(\mathscr{P})$ denote the number of i-dimensional faces of $\mathscr{P}$. Thus, f_0 is the number of vertices of $\mathscr{P}$. The vector $f(\mathscr{P}) = (f_0, f_1, \ldots, f_{d-1})$ is called the *f-vector* of $\mathscr{P}$. What can be said about such a vector? In 1893 Poincaré proved (with an error corrected in 1899) the formula

$$f_0 - f_1 + f_2 - \cdots + (-1)^{d-1} f_{d-1} = 1 + (-1)^{d-1}, \tag{1}$$

generalizing Euler's famous formula $V - E + F = 2$. Proofs of (1) were published by several nineteenth century geometers, but they all contained unjustified assumptions about the structure of $\mathscr{P}$. It was not until 1955 that Hadwiger gave a nontopological proof of (1), and finally in 1971 Bruggesser and Mani [8] vindicated some of the nineteenth century efforts by showing that their assumptions were valid. For further information and references, see [17, Section 8.6; 19].

We can go on to ask, What more can be said about the f-vector of a d-polytope? The acquisition of any sort of definitive result for arbitrary d-polytopes seems hopeless. However, if we assume that $\mathscr{P}$ is *simplicial*, i.e., every proper face of $\mathscr{P}$ is a simplex, then much more can be said. For instance, if $\mathscr{P}$ is simplicial, then every $(d-2)$-face is contained in exactly two $(d-1)$-faces and every $(d-1)$-face contains d $(d-2)$-faces. Hence, $2f_{d-2} = df_{d-1}$, which is a linear relation among the f_i's independent from (1). Before we consider the most general linear relations that can hold among the f_i's, we first discuss a generalization of simplicial polytopes to which much of what we say applies.

Let $\Delta(\mathscr{P})$ denote the *boundary complex* of a simplicial d-polytope $\mathscr{P}$. By definition, it is the abstract simplicial complex whose vertices are the vertices of $\mathscr{P}$, and whose faces are those sets of vertices that span a proper face of $\mathscr{P}$. The geometric realization $|\Delta(\mathscr{P})|$ of $\mathscr{P}$ (as defined, e.g., in [34, pp. 110–111]) is therefore a $(d-1)$-sphere S^{d-1}. Thus we may consider more generally any (finite) simplicial complex Δ on a vertex set $V = \{x_1, \ldots, x_n\}$ whose geometric realization is S^{d-1}. We call Δ a *simplicial* $(d-1)$*-sphere*. It is by no means apparent that there exists a simplicial $(d-1)$-sphere Δ which is not the boundary complex of a simplicial d-polytope. Indeed, many earlier geometers simply assumed the two concepts were equivalent. It was not until 1965 that Grünbaum [16] gave the first example of such a Δ. The smallest possible example has $d = 4$ and $n = 8$. (It follows from Steinitz's theorem [17, Section 13.1] that an example cannot exist for $d = 3$.)

We can define the f-vector $f(\Delta)$ of a simplicial sphere (or of any simplicial complex) just as we did for polytopes. Thus $f_0 = n$, the number of vertices. The

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Euler characteristic formula (1) continues to hold for simplicial $(d-1)$-spheres, as does the formula $2f_{d-2} = df_{d-1}$. To state the most general linear relation holding among the f_i's, it is convenient to introduce the numbers

$$h_i = h_i(\Delta) = \sum_{j=0}^{i} \binom{d-j}{d-i} (-1)^{i-j} f_{j-1}, \qquad 0 \le i \le d, \tag{2}$$

where we set $f_{-1} = 1$. Thus $h_0 = 1$, $h_1 = n - d$, and $h_d = (-1)^d(1 - f_0 + f_1 - \cdots + (-1)^d f_{d-1}) = 1$. The vector $h(\Delta) = (h_0, h_1, \ldots, h_d)$ is called the *h-vector* of Δ. It is easy to invert the equations (2) and solve for the f_i's, yielding

$$f_i = \sum_{j=0}^{d} \binom{d-j}{d-i-1} h_j.$$

Thus, knowing $h(\Delta)$ is equivalent to knowing $f(\Delta)$.

To compute the h-vector for particular examples, the following procedure can be used. We illustrate it for the four-dimensional cross-polytope, whose f-vector is (8, 24, 32, 16). Write down the f-vector on a diagonal, and put a 1 to the left of f_0.

$$\begin{array}{ccccc} 1 & 8 & & & \\ & & 24 & & \\ & & & 32 & \\ & & & & 16 \end{array}$$

Complete this array to construct a "difference table," by placing underneath a pair of consecutive entries their difference (and bringing down 1 on the left edge).

$$\begin{array}{ccccc} & & & 1 & 8 \\ & & 1 & 7 & 24 \\ & 1 & 6 & 17 & 32 \\ 1 & 5 & 11 & 15 & 16 \end{array}$$

The next row of differences will be $h(\Delta)$.

$$\begin{array}{cccccc} & & & & 1 & 8 \\ & & & 1 & 7 & 24 \\ & & 1 & 6 & 17 & 32 \\ & 1 & 5 & 11 & 15 & 16 \\ \hline h(\Delta) = (1, & 4, & 6, & 4, & 1). & \end{array}$$

Note that in this example we have $h_i = h_{d-i}$ $(d = 4)$. In general:

The Dehn–Sommerville equations: For any simplicial $(d-1)$-sphere Δ with h-vector $(h_0, h_1, \ldots, h_d)$, we have $h_i = h_{d-i}$.

This result was proved by Dehn in 1905 for $d \le 5$, and for general d by Sommerville in 1927. See [17, Sections 9.2 and 9.8; 30, Sections 2.4 and 5.1] for further details.

Little further study of the f-vectors of polytopes and spheres was undertaken until 1957, when Motzkin [32], motivated by the development of linear programming, asked how large $f_i(\mathscr{P})$ could be, given the dimension d and number of vertices n of $\mathscr{P}$. More precisely, for $n \geq d + 1$ let $C(n, d)$ denote the convex hull of any n points on the *moment curve* $\{(\tau, \tau^2, \ldots, \tau^d) \in \mathbb{R}^d : \tau \in \mathbb{R}\}$. The convex polytope $C(n, d)$ is known as a *cyclic polytope*, and its combinatorial type is independent of the choice of n extreme points. Cyclic polytopes were first investigated by Carathéodory in 1907 and 1911 and rediscovered by Gale in 1955 and Motzkin in 1957 (see [17, Section 7.4]). The following two properties concerning $C(n, d)$ can be proved without great difficulty:

(i) $C(n, d)$ is *simplicial*,

(ii) $f_i(C(n, d)) = \binom{n}{i+1}$, $0 \leq i \leq [\frac{1}{2}d] - 1$.

Clearly, no polytope or sphere with n vertices can have more than $\binom{n}{i+1}$ i-faces, so cyclic polytopes maximize $f_0, f_1, \ldots, f_{[d/2]-1}$ over all simplicial d-polytopes with n vertices [or simplicial $(d - 1)$-spheres with n vertices]. On the other hand, the Dehn–Sommerville equations uniquely determine $f_{[d/2]}, \ldots, f_{d-1}$ once $f_0, \ldots, f_{[d/2]-1}$ are known. It is certainly reasonable to suppose that maximizing $f_0, \ldots, f_{[d/2]-1}$ will also maximize the other f_i's. We therefore say that a simplicial $(d - 1)$-sphere Δ with n vertices satisfies the *Upper Bound Conjecture* (UBC) if

$$f_i(\Delta) \leq f_i(C(n, d)), \qquad 0 \leq i \leq d - 1.$$

In 1957 Motzkin [32] conjectured that all d-polytopes satisfy the UBC (and it is easy to reduce this conjecture to the simplicial case), and in 1964 Klee [25] pointed out that the UBC might as well be made for arbitrary simplicial spheres.

In 1970 McMullen [28] proved the UBC for simplicial polytopes. He first showed that the UBC for a $(d - 1)$-sphere Δ with n vertices followed from the inequality

$$h_i \leq \binom{n - d + i - 1}{i}, \qquad 0 \leq i \leq d. \tag{3}$$

He then used the recently proved result of Bruggesser and Mani [8] that the boundary complex of a convex polytope is shellable. Since simplicial spheres need not be shellable (the first example being given in [12], based on the fact that shellable spheres are combinatorial[b]), McMullen's techniques do not extend to simplicial spheres.

In order to extend the UBC for polytopes to arbitrary simplicial spheres, it is necessary to introduce the machinery of commutative algebra. Given an abstract simplicial complex Δ on a vertex set $V = \{x_1, \ldots, x_n\}$, let $A = k[x_1, \ldots, x_n]$ be the polynomial ring over the field k in the variables $x_1, \ldots, x_n$. Define I_Δ to be the ideal of A generated by all squarefree monomials $x_{i_1} x_{i_2} \cdots x_{i_r}$ such that $\{x_{i_1}, x_{i_2}, \ldots, x_{i_r}\} \notin \Delta$. Set $k[\Delta] = A/I_\Delta$. If we let $k[\Delta]_i$ denote the vector space of all homogeneous

[b] Danaraj and Klee [9] asked whether all combinatorial spheres are shellable. This was answered in the negative by Mandel [27, p. 197].

polynomials of degree i in $k[\Delta]$, then $k[\Delta]$ becomes what we will call a *standard graded k-algebra*, viz., a commutative, associative k-algebra R with identity, together with a collection of subspaces R_i, $i \geq 0$, satisfying:

(i) $R = \coprod_{i \geq 0} R_i$ (vector space direct sum),
(ii) $R_0 = k$,
(iii) $R_i R_j \subset R_{i+j}$,
(iv) R is generated as a k-algebra by R_1,
(v) R is a finitely generated k-algebra [which in view of (iv) is equivalent to $\dim_k R_1 < \infty$].

Given a standard graded k-algebra R, an element of R_i is said to be *homogeneous of degree i*. We also define the *Hilbert function* $H(R, \cdot): \mathbb{N} \to \mathbb{N}$ of R (where $\mathbb{N} = \{0, 1, 2, \ldots\}$) by

$$H(R, i) = \dim_k R_i.$$

[Condition (v) guarantees that $H(R, i) < \infty$.] An easy computation [36, Proposition 3.2] shows that

$$\begin{aligned} H(k[\Delta], i) &= 1, & i &= 0, \\ &= \sum_{j=0}^{d-1} f_j \binom{i-1}{j}, & i &> 0, \end{aligned} \tag{4}$$

where $(f_0, f_1, \ldots, f_{d-1})$ is the f-vector of Δ. The generating function

$$F(R, \lambda) = \sum_{i \geq 0} H(R, i)\lambda^i$$

is called the *Hilbert series* of R. A basic result of commutative algebra due to Hilbert (see [1, Chapter 11] for a simple proof) asserts that

$$F(R, \lambda) = (h_0 + h_1\lambda + \cdots + h_s\lambda^s)(1 - \lambda)^{-d}, \tag{5}$$

for certain integers $h_i = h_i(R)$ and d. We assume d has been chosen as small as possible (i.e., $h_0 + h_1 + \cdots + h_s \neq 0$), and we then call d the *Krull dimension* of R, denoted $d = \dim R$. It is not hard to show that $\dim R$ is equal to the largest number of elements of R (or of R_1) that are algebraically independent over k. It follows easily from (2) and (4) or otherwise [40, p. 63] that

$$\dim k[\Delta] = 1 + \dim \Delta$$

and

$$h_i(k[\Delta]) = h_i(\Delta).$$

A collection of d homogeneous elements $\theta_1, \ldots, \theta_d$ of R (where $d = \dim R$) of positive degree is called a *homogeneous system of parameters* (h.s.o.p.) if $\dim_k R/(\theta_1, \ldots, \theta_d) < \infty$. The Noether normalization lemma asserts that an h.s.o.p. $\theta_1, \ldots, \theta_d$ for R always exists, and each θ_i can be chosen to have degree one if k is infinite. (This graded version of the Noether normalization lemma is due to Hilbert.)

We are now in a position to make the crucial definition from commutative algebra. We will choose the most convenient definition for our purpose, though on the surface it does not seem to be related to the usual definition found in papers and textbooks. However, it is an easy matter to reconcile our definition with the "standard" one involving R-sequences [38, Corollary 3.2].

DEFINITION. A standard graded k-algebra R (where k is infinite) is called *Cohen–Macaulay* if for some (equivalently, every) h.s.o.p. $\theta_1, \ldots, \theta_d$ of degree one, we have

$$H(R/(\theta_1, \ldots, \theta_d), i) = h_i(R), \qquad i \geq 0.$$

In particular, $k[\Delta]$ is Cohen–Macaulay if for some h.s.o.p. $\theta_1, \ldots, \theta_d$ of degree one, we have

$$H(k[\Delta]/(\theta_1, \ldots, \theta_d), i) = h_i(\Delta), \qquad i \geq 0.$$

EXAMPLE. The algebra $k[\Delta] = k[x, y, z]/(xy, xz)$ is not Cohen–Macaulay, since $h(\Delta) = (1, 1, -1)$, and we can hardly have $H(k[\Delta]/(\theta_1, \theta_2), 2) = -1$.

THEOREM 1 [36, Corollary 4.4]. *If* $|\Delta| \approx S^{d-1}$ *and* $k[\Delta]$ *is Cohen–Macaulay, then the UBC holds for* Δ.

Proof. Let $\theta_1, \ldots, \theta_d$ be an h.s.o.p. for $k[\Delta]$ of degree one, and set $S = k[\Delta]/(\theta_1, \ldots, \theta_d)$. Since $H(k[\Delta], 1) = n$, we have $H(S, 1) = n - d$, so S is generated by $n - d$ linear forms. Hence, $H(S, i)$ cannot exceed the total number of monomials of degree i in $n - d$ variables. This number is $\binom{n-d+i-1}{i}$. Since $k[\Delta]$ is Cohen–Macaulay, we have $H(S, i) = h_i(\Delta)$. Thus $h_i(\Delta) \leq \binom{n-d+i-1}{i}$, which is (3). ▮

In order to prove the UBC for spheres, it remains to show that $k[\Delta]$ is Cohen–Macaulay whenever Δ is a sphere. It turns out that at about the same time that THEOREM 1 was proved, Reisner, working without knowledge of the UBC, found a complete characterization of those Δ for which $k[\Delta]$ is Cohen–Macaulay.[c] In order to state Reisner's result, recall that for any $F \in \Delta$ (including $F = \varnothing$) the *link* of F is defined by

$$\text{lk } F = \{G \in \Delta : G \cap F = \varnothing, G \cup F \in \Delta\}.$$

In particular, lk $\varnothing = \Delta$. Let $\tilde{H}_i(\Delta)$ denote reduced simplicial homology with coefficients in k.

THEOREM 2 [33]. *Let* Δ *be any finite simplicial complex. The algebra* $k[\Delta]$ *is Cohen–Macaulay if and only if for all* $F \in \Delta$, $\tilde{H}_i(\text{lk } F) = 0$ *unless* $i = \dim(\text{lk } F)$.

It is well known from topology that if $|\Delta| \approx S^{d-1}$, then for all $F \in \Delta$,

$$\begin{aligned} \tilde{H}_i(\text{lk } F) &\cong 0, \qquad i \neq \dim(\text{lk } F), \\ &\cong k, \qquad i = \dim(\text{lk } F). \end{aligned}$$

Hence, $k[\Delta]$ is Cohen–Macaulay for all $|\Delta| \approx S^{d-1}$, and we conclude:

THEOREM 3 [36, Corollary 5.3]. *The UBC is valid for all spheres* Δ.

[c] The simultaneity of THEOREMS 1 and 2 is not quite as coincidental as it may at first appear, since both Reisner and I were heavily influenced by Hochster's paper [20].

Let us mention for readers interested in understanding the proof of THEOREM 2 that in addition to Reisner's original argument, proofs also appear in [2, 21, 40]. All the proofs, except that in [2], use homological algebra. For readers familiar with homological algebra, the most straightforward proof is the one in [40]. (This proof is due to Hochster but is unpublished by him.)

While we now know that the UBC holds for spheres, we still can ask, What more can be said about the f-vector of simplicial polytopes or spheres? To this end, given integers $h, i > 0$, write

$$h = \binom{n_i}{i} + \binom{n_{i-1}}{i-1} + \cdots + \binom{n_j}{j},$$

where $n_i > n_{i-1} > \cdots > n_j \geq j \geq 1$. Such a representation always exists and is unique. For a nice discussion of the significance of this representation, see [13, Section 8]. Now define

$$h^{\langle i\rangle} = \binom{n_i + 1}{i+1} + \binom{n_{i-1}+1}{i} + \cdots + \binom{n_j+1}{j+1},$$

and set $0^{\langle i\rangle} = 0$. Call a vector $(h_0, \ldots, h_d) \in \mathbb{Z}^{d+1}$ an *M-vector* (after F. S. Macaulay) if $h_0 = 1$ and $0 \leq h_{i+1} \leq h_i^{\langle i\rangle}$, $1 \leq i \leq d-1$.

The next result is due essentially to Macaulay [26] but was first stated as such in [38, Theorem 2.2].

THEOREM 4. *Let k be a field. A vector $(h_0, \ldots, h_d) \in \mathbb{Z}^{d+1}$ is an M-vector if and only if there exists a standard graded k-algebra R such that $H(R, i) = h_i$, $0 \leq i \leq d$.*

COROLLARY. *If $|\Delta| \approx S^{d-1}$, then $h(\Delta)$ is an* M-*vector.*

Proof. Follows from THEOREM 2 and our definition of Cohen–Macaulay. ∎

The conditions imposed on the h-vector of a sphere by the previous corollary and the Dehn–Sommerville equations are by no means sufficient to characterize such h-vectors. In 1971 McMullen [29], putting together all results concerning f-vectors known at that time, produced the following remarkable conjecture, called the "g-conjecture" after McMullen's notation g_i for our $h_i - h_{i-1}$.

Conjecture. A vector $(h_0, \ldots, h_d) \in \mathbb{Z}^{d+1}$ is the h-vector of some simplicial d-polytope $\mathscr{P}$ if and only if $h_i = h_{d-i}$ and $(h_0, h_1 - h_0, h_2 - h_1, \ldots, h_{[d/2]} - h_{[d/2]-1})$ is an M-vector.

As McMullen noted in [29], the above conjecture could conceivably be valid for simplicial $(d-1)$-spheres, but he felt he did not understand spheres well enough to have any confidence in such a generalization.

Note that the g-conjecture implies in particular that $h_0 \leq h_1 \leq \cdots \leq h_{[d/2]}$ for the h-vector of a simplicial polytope. This weaker result had earlier been conjectured by McMullen and Walkup [31] and had been called by them the "Generalized Lower-Bound Conjecture" (GLBC). This terminology was used because the GLBC implies the earlier "Lower-Bound Conjecture," which gives the minimum value of f_i for a simplicial d-polytope with n vertices. The Lower-Bound Conjecture was subsequently proved by Barnette [3, 4], but the GLBC remained open. The GLBC also included a condition as to when $h_i = h_{i+1}$. This part of the GLBC remains open at present.

It turns out that McMullen's g-conjecture (and therefore the inequality $h_0 \leq h_1 \leq \cdots \leq h_{[d/2]}$ of the GLBC) is indeed true (for polytopes). The "if" part of the conjecture was proved by Billera and Lee [5, 6] in 1979. They explicitly constructed the desired $\mathscr{P}$ by cleverly taking a sequence of stellar and inverse stellar subdivisions of a cyclic polytope. We will now sketch the proof of the "only if" part of the conjecture [39]. The proof consists basically of putting together some results from algebraic geometry. First note that we can assume that our simplicial polytope satisfies: (a) if $\dim \mathscr{P} = d$, then $\mathscr{P} \subset \mathbb{R}^d$ (simply restrict one's attention to the affine span of $\mathscr{P}$), (b) the vertices of $\mathscr{P}$ lie in $\mathbb{Q}^d$, i.e., $\mathscr{P}$ is a *rational* polytope (because small perturbations of the vertices of $\mathscr{P}$ do not change the combinatorial type of $\mathscr{P}$ since $\mathscr{P}$ is simplicial), and (c) the origin of $\mathbb{R}^d$ lies in the interior of $\mathscr{P}$. We now note the following:

1. In 1970 Demazure [11] and in 1973 Mumford and co-workers [24] defined a certain class of complex varieties now known as *toric varieties*. Briefly, they may be described as follows [10]. Let Σ be a collection of convex polyhedral cones in a $\mathbb{Q}$-vector space V, satisfying:

(1) every cone in Σ has a vertex at the origin,

(2) if τ is a face of a cone $\sigma \in \Sigma$, then $\tau \in \Sigma$,

(3) if $\sigma, \sigma' \in \Sigma$, then $\sigma \cap \sigma'$ is a face of both σ and σ'.

We call Σ a *fan*. Σ is *complete* if $\bigcup_{\sigma \in \Sigma} \sigma$ is the whole space V, and *simplicial* if each $\sigma \in \Sigma$ is a simplicial cone. Let V^* be the dual space to V. For a cone $\sigma \in \Sigma$, define

$$\sigma^* = \{f \in V^*: f(\sigma) \geq 0\}.$$

Then σ^* is a convex polyhedral cone in V^*, called the *dual cone* to σ. Choose a lattice L in V, and let L^* be the dual lattice in V^*. Then $\sigma^* \cap L^*$ is an additive monoid in V^*; let $\mathbb{C}[\sigma^* \cap L^*]$ denote the monoid algebra of $\sigma^* \cap L^*$ over $\mathbb{C}$. Define an affine variety

$$X_{\sigma^*} = X_{(\sigma^*, L^*)} = \operatorname{Spec} \mathbb{C}[\sigma^* \cap L^*].$$

[Naively, view $\mathbb{C}[\sigma^* \cap L^*]$ as the quotient of a polynomial ring (whose variables correspond to the generators of $\sigma^* \cap L^*$, which are always finite in number) by some ideal I. Then $X_{(\sigma^*, L^*)}$ may be thought of as the variety of zeros of polynomials in I.] If τ is a face of σ, then X_{τ^*} can be identified with an open subvariety of X_{σ^*}. These identifications allow us to glue together from the X_{σ^*} (as σ ranges over Σ) a variety over $\mathbb{C}$, which is denoted by X_Σ and is called the *toric variety associated with* Σ. (Note that we have defined X_Σ as an *abstract variety*, in the sense of Weil. See, e.g., [18, p. 105].) The variety X_Σ does not a priori have an embedding into affine or projective space, and one of the main results stated below gives a criterion for X_Σ to be projective.

Now suppose $\mathscr{P}$ is a convex polytope (not necessarily simplicial) satisfying conditions (a)–(c) above. For any proper face F of $\mathscr{P}$, define a cone σ_F to consist of all rational points that lie on some ray with vertex 0 and passing through F. The set Σ of all such σ_F (together with 0) forms a complete fan. The corresponding toric variety X_Σ (say for the lattice $L = \mathbb{Z}^d$) is denoted $X(\mathscr{P})$.

2. A basic result of the theory of toric varieties [10, Proposition 6.9.1; 11; 24] asserts essentially that for any fan Σ, X is projective if and only if $X = X(\mathscr{P})$ for some convex polytope.

A second (easier) result states that when $\mathscr{P}$ is simplicial, $X(\mathscr{P})$, while not necessarily smooth, has very nice singularities. Namely, $X(\mathscr{P})$ looks locally like affine space $\mathbb{C}^d$ modulo a finite (Abelian) group of linear transformations. Varieties with these types of singularities are known as *V-varieties.*

3. In 1978 Danilov [10, Theorem 10.8 and Remark 10.9], extending work of Jurkiewicz, computed the cohomology ring $H^*(X(\mathscr{P}), \mathbb{C})$ when $\mathscr{P}$ is simplicial. All the cohomology occurs in even degree, i.e.,

$$H^*(X(\mathscr{P}), \mathbb{C}) = \coprod_{i=0}^{d} H^{2i}(X(\mathscr{P}), \mathbb{C}),$$

and there is an isomorphism

$$H^*(X(\mathscr{P}), \mathbb{C}) \xrightarrow{\cong} \mathbb{C}[\Delta(\mathscr{P})]/(\theta_1, \ldots, \theta_d) \tag{6}$$

for a certain h.s.o.p. of $\mathbb{C}[\Delta(\mathscr{P})]$ of degree one (which depends on how $\mathscr{P}$ is embedded in $\mathbb{R}^d$). In order for the isomorphism to be degree preserving one must consider elements of $H^{2i}(X, \mathbb{C})$ as having degree i, so $\theta_1, \ldots, \theta_d$ may be regarded as elements of $H^2(X, \mathbb{C})$. Note in particular that by THEOREM 2, the fact that $|\Delta(\mathscr{P})| \approx S^{d-1}$, and our definition of Cohen–Macaulay, we have

$$\dim_{\mathbb{C}} H^{2i}(X(\mathscr{P}), \mathbb{C}) = h_i(\mathscr{P}). \tag{7}$$

4. In 1976 Steenbrink [41] proved that projective V-varieties Y [in particular $Y = X(\mathscr{P})$ for $\mathscr{P}$ simplicial] satisfy the hard Lefschetz theorem. This means the following. Since Y is projective, we can embed it (as a closed subvariety) in some projective space $\mathbb{P}^N$. Let H be a generic hyperplane in $\mathbb{P}^N$. Then $Y \cap H$ is a closed subvariety of Y of complex codimension 1. With any closed subvariety Z of complex codimension i of an irreducible complex projective variety Y, a standard construction of algebraic geometry (e.g., [15] or [14, Chapter 5, Section 4]) associates a cohomology class $[Z] \in H^{2i}(Y, \mathbb{C})$. In particular, let $\omega \in H^2(Y, \mathbb{C})$ denote the class of the hyperplane section $Y \cap H$. The classical hard Lefschetz theorem (first proved completely by Hodge) states that if Y is smooth and irreducible of complex dimension d, and if $0 \le i \le d$, then the linear transformation

$$H^i(Y, \mathbb{C}) \xrightarrow{\omega^{d-i}} H^{2d-i}(Y, \mathbb{C}) \tag{8}$$

defined as multiplication by ω^{d-i} is a bijection (e.g., [15, Chapter 0, Section 7]). Steenbrink showed that this result continues to hold when Y is a projective V-variety, though Y need not be smooth. In general, the singularities of V-varieties are so "tame" that V-varieties behave almost like smooth varieties.

Note that if for all $0 \le i \le d$ the map (8) is a bijection, then it follows that for all $0 \le i < d$ the map $H^i(Y, \mathbb{C}) \xrightarrow{\omega} H^{i+2}(Y, \mathbb{C})$ is injective (and surjective for $d - 1 \le i \le 2d - 2$).

5. We now have at our disposal all the tools to prove the necessity of McMullen's g-conjecture. Let $\mathscr{P}$ be a simplicial d-polytope satisfying conditions (a)–(c)

above, and let $X = X(\mathscr{P})$ be the corresponding toric variety. Define the graded algebra

$$R = H^*(X, \mathbb{C})/(\omega),$$

where (ω) denotes the ideal of $H^*(X, \mathbb{C})$ generated by the class of a hyperplane section. The ith degree part of R is given by the quotient vector space

$$R_i = H^{2i}(X, \mathbb{C})/\omega H^{2i-2}(X, \mathbb{C}). \tag{9}$$

By (6), R is indeed a standard graded algebra and, in particular, is generated by R_1. From (9) we have

$$H(R, i) = \dim_{\mathbb{C}} R_i = \dim_{\mathbb{C}} H^{2i}(X, \mathbb{C}) - \dim_{\mathbb{C}} \omega H^{2i-2}(X, \mathbb{C}).$$

By (7), $\dim_{\mathbb{C}} H^{2i}(X, \mathbb{C}) = h_i(\mathscr{P})$. By the hard Lefschetz theorem, if $0 \le i \le [d/2]$, then

$$\dim_{\mathbb{C}} \omega H^{2i-2}(X, \mathbb{C}) = \dim_{\mathbb{C}} H^{2i-2}(X, \mathbb{C}) = h_{i-1}(\mathscr{P}).$$

Thus $H(R, i) = h_i(\mathscr{P}) - h_{i-1}(\mathscr{P})$, $0 \le i \le [d/2]$. Hence by THEOREM 4, $(h_0, h_1 - h_0, h_2 - h_1, \ldots, h_{[d/2]} - h_{[d/2]-1})$ is an M-vector, and the proof is complete. ▌

Two questions immediately suggest themselves, the second of which was alluded to earlier:

1. Is there a simpler proof? By the very nature of M-vectors it seems essential to consider graded algebras, in particular, the algebra $S = \mathbb{C}[\Delta(\mathscr{P})]/(\theta_1, \ldots, \theta_d)$. However, it might be possible to bypass the theory of toric varieties and prove directly the existence of a suitable $\omega \in S_1$. If one examines the "standard" proof of the hard Lefschetz theorem, the following can be seen. Define three linear transformations α, β, γ: $S \to S$ as follows: $\alpha(x) = \omega x$, $\beta(x) = (2i - d)x$ if $x \in S_i$, and γ is the adjoint of α with respect to a certain scalar product on S. Then α, β, γ generate (under the bracket operation) the Lie algebra $\mathrm{sl}_2(\mathbb{C})$. This is equivalent in the present situation to saying $\alpha\gamma - \gamma\alpha = \beta$. Thus to prove the necessity of the g-conjecture, it would suffice to find α (i.e., $\omega \in S_1$), β (which is automatic), and γ (which involves a suitable choice of a scalar product on S). Some recent work of Kalai [23] might also lead to a simpler proof.

2. Does McMullen's g-conjecture hold for more general simplicial complexes? In particular, we have the following hierarchy:

$$\text{simplicial polytope} \Rightarrow P_1\text{-sphere} \Rightarrow \text{combinatorial sphere}$$

$$\Rightarrow P\text{-sphere} \Rightarrow \text{simplicial sphere} \Rightarrow \text{Gorenstein complex}. \tag{10}$$

The relevant definitions are as follows. A *P_1-sphere* is an abstract simplicial complex Δ which can be realized in $\mathbb{R}^d$ by a collection of rectilinear simplices whose union is the boundary of a convex polytope. In other words, triangulate the boundary of a convex polytope into (rectilinear) simplices. (Similarly define *P_1-ball*.) A *combinatorial sphere* is a triangulation of the sphere S^{d-1} which possesses a subdivision isomorphic to a subdivision of the boundary complex of a d-simplex. The notion of a P-sphere is due to J. Edmonds and seems to be the most general class of simplicial spheres avoiding certain pathologies and which lend themselves to proofs by induction. A simplicial complex Δ satisfying $|\Delta| \approx S^{d-1}$ is a *P-sphere* if it possesses a subdivision Δ' which is a P_1-sphere, such that the subdivision induced on each cell of Δ is itself a P_1-ball. (See [27, Chapter 5.VII].) Finally, a *Gorenstein complex* is an abstract simplicial complex Δ for which the ring $k[\Delta]$ is a Gorenstein ring (see [21, 40]).

It is known that all the implications in **(10)** are strict, and we can ask, For which of them does the g-conjecture hold? In the case of P_1-spheres Δ [possessing a realization $\mathscr{P}$ satisfying (a)–(c)] we still have a toric variety $X(\mathscr{P})$, but it need not be projective. Thus Steenbrink's theorem does not apply. Can one somehow "induce" the hard Lefschetz theorem on $X(\mathscr{P})$ from a subdivision $\mathscr{P}'$ of $\mathscr{P}$ for which $X(\mathscr{P}')$ is projective? (Such subdivisions always exist [10, Section 6.9.2; 24, p. 161].) Let us also note that Gorenstein complexes are precisely those simplicial complexes for which the ring $S = \mathbb{C}[\Delta]/(\theta_1, \ldots, \theta_d)$ satisfies (abstract) Poincaré duality. Since Poincaré duality is essentially a consequence of the hard Lefschetz theorem, it follows that Gorenstein complexes are the most general ones for which we could expect the g-conjecture to hold.

Historical Note. The following comments on how the proof of the necessity of the g-conjecture was found may be of interest. I had realized from my first work on the UBC (see the sentence after Conjecture 2 in [35]) that the necessity of the g-conjecture would follow from the existence of a suitable element $\omega \in S_1$, where $S = \mathbb{C}[\Delta(\mathscr{P})]/(\theta_1, \ldots, \theta_d)$. The proof of a rather uninteresting special case was announced in [37, Theorem 8]. In 1976, Tony Iarrobino brought the hard Lefschetz theorem to my attention. It was now apparent that one needed a smooth projective variety X whose cohomology ring was isomorphic to S [with the grading given by $S_i \cong H^{2i}(X, \mathbb{C})$]. I had been aware for some time of the theory of toroidal embeddings [24] and checked this reference to see whether the varieties $X(\mathscr{P})$ had the right properties. Three problems arose: (i) I could not understand [24] well enough to tell whether $X(\mathscr{P})$ was projective, (ii) in any case, no mention was made of $H^*(X(\mathscr{P}), \mathbb{C})$, and (iii) worst of all, $X(\mathscr{P})$ was not necessarily smooth, so the hard Lefschetz theorem (seemingly) did not apply. There matters rested until the spring or summer of 1979, when I stumbled upon the paper of Danilov [10] on the new journal shelf of the MIT library. Remark 3.8 immediately caught my attention. It asserted that for any triangulation Δ of the sphere S^{d-1}, the ring $S = \mathbb{C}[\Delta]/(\theta_1, \ldots, \theta_d)$ was the cohomology ring of a smooth variety $X(\mathscr{P})$! Thus, the necessity of the g-conjecture is valid for all spheres for which $X(\mathscr{P})$ is projective. But in reading [10] more carefully it became apparent that Remark 3.8 was stated rather carelessly. One needs to assume Δ is a P_1-sphere, not an arbitrary sphere, in order to define $X(\mathscr{P})$. Moreover, the corresponding variety $X(\mathscr{P})$ is not always smooth. I therefore asked some algebraic geometers whether $X(\mathscr{P})$, when projective, could satisfy the hard Lefschetz theorem, but none knew. Shortly thereafter I took the book [22] out of the library in order to look at a paper related to a completely different topic in which I was interested. In browsing through this book I discovered the paper of Steenbrink [41], with its proof of the hard Lefschetz theorem for projective V-varieties.[d] It remained only to ascertain that for convex polytopes the varieties $X(\mathscr{P})$ were projective. This was accomplished via a conversation with David Mumford on September 13, 1979, and the proof was complete.

[d] Danilov [10] refers to this paper of Steenbrink, but since he does not mention the hard Lefschetz theorem it did not occur to me to pursue this reference.

REFERENCES

1. ATIYAH, M. F. & I. G. MACDONALD. 1969. Introduction to Commutative Algebra. Addison–Wesley. Reading, Mass.
2. BACLAWSKI, K. & A. M. GARSIA. 1981. Combinatorial decompositions of a class of rings. Adv. in Math. **39:** 155–184.
3. BARNETTE, D. W. 1971. The minimum number of vertices of a simple polytope. Israel J. Math. **9:** 121–125.
4. BARNETTE, D. W. 1973. A proof of the lower bound conjecture for convex polytopes. Pacific J. Math. **46:** 349–354.
5. BILLERA, L. J. & C. W. LEE. 1980. Sufficiency of McMullen's condition for *f*-vectors of simplicial polytopes. Bull. Amer. Math. Soc. **2:** 181–185.
6. BILLERA, L. J. & C. W. LEE. 1981. A proof of the sufficiency of McMullen's conditions for *f*-vectors of simplicial convex polytopes. J. Combin. Theory Ser. A **31:** 237–255.
7. BOREL, A. & A. HAEFLIGER. 1961. La class d'homologie fondamentale d'un espace analytique. Bull. Soc. Math. France **89:** 461–513.
8. BRUGGESSER, M. & P. MANI. 1971. Shellable decompositions of cells and spheres. Math. Scand. **29:** 197–205.
9. DANARAJ, G. & V. KLEE. 1978. Which spheres are shellable? Ann. Discrete Math. **2:** 33–52.
10. DANILOV, V. I. 1978. The geometry of toric varieties. Russian Math. Surveys **33:** 97–154. [Translated from: Uspekhi Mat. Nauk **33** (1978): 85–134.]
11. DEMAZURE, M. 1970. Sous-groupes algébriques de rang maximum du groupe de Cremona. Ann. Sci. École Norm. Sup. **3:** 507–588.
12. EDWARDS, R. D. 1975. The double suspension of a certain homology 3-sphere is S^5. Notices Amer. Math. Soc. **22:** A-334.
13. GREENE, C. & D. J. KLEITMAN. 1978. Proof techniques in the theory of finite sets. *In* Studies in Combinatorics. G.-C. Rota, Ed.: 22–79. Mathematical Association of America. Washington, D.C.
14. GRIFFITHS, P. & J. ADAMS. 1974. Topics in Algebraic and Analytic Geometry. Princeton University Press. Princeton, N.J.
15. GRIFFITHS, P. & J. HARRIS. 1978. Principles of Algebraic Geometry. John Wiley & Sons, Inc. New York, N.Y.
16. GRÜNBAUM, B. 1965. Diagrams and Schlegel diagrams. Notices Amer. Math. Soc. **12:** 578. (Abstract 625-112.)
17. GRÜNBAUM, B. 1967. Convex Polytopes. John Wiley & Sons, Inc. New York, N.Y.
18. HARTSHORNE, R. 1977. Algebraic Geometry. Springer-Verlag. New York, N.Y.
19. HILTON, P. & J. PEDERSEN. 1981. Euler, Poincaré, Pólya- and polyhedra. Enseign. Math. **27:** 326–343.
20. HOCHSTER, M. 1972. Rings of invariants of tori, Cohen–Macaulay rings generated by monomials, and polytopes. Ann. of Math. **96:** 318–337.
21. HOCHSTER, M. 1977. Cohen–Macaulay rings, combinatorics, and simplicial complexes. *In* Ring Theory II. B. R. McDonald & R. Morris, Eds.: 171–223. Marcel Dekker, Inc. New York, N.Y.
22. HOLM, P. (Ed.). 1977. Real and Complex Singularities, Oslo, 1976. Sijthoff & Noordhoff. Alphen aan den Rijn, the Netherlands.
23. KALAI, G. Characterization of *f*-vectors of families of convex sets in R^d. I. Necessity of Eckhoff's conditions. Israel J. Math. In press.
24. KEMPF, G., F. KNUDSEN, D. MUMFORD & B. SAINT-DONAT. 1973. Toroidal Embeddings I. Lecture Notes in Mathematics No. 339. Springer-Verlag. West Berlin, FRG.
25. KLEE, V. 1964. The number of vertices of a convex polytope. Canad. J. Math. **16:** 701–720.
26. MACAULAY, F. S. 1927. Some properties of enumeration in the theory of modular systems. Proc. London Math. Soc. **26:** 531–555.
27. MANDEL, A. 1982. Topology of Oriented Matroids. Ph.D. Thesis. University of Waterloo. Waterloo, Ontario, Canada.
28. MCMULLEN, P. 1970. The maximum numbers of faces of a convex polytope. Mathematika **17:** 179–184.

29. McMullen, P. 1971. The numbers of faces of simplicial polytopes. Israel J. Math. **9:** 559–570.
30. McMullen, P. & G. C. Shephard. 1971. Convex Polytopes and the Upper Bound Conjecture. Cambridge University Press. Cambridge, England.
31. McMullen, P. & D. W. Walkup. 1971. A generalized lower-bound conjecture for simplicial polytopes. Mathematika **18:** 264–273.
32. Motzkin, T. S. 1957. Comonotone curves and polyhedra. Bull. Amer. Math. Soc. **63:** 35. (Abstract 111.)
33. Reisner, G. 1976. Cohen–Macaulay quotients of polynomial rings. Adv. in Math. **21:** 30–49.
34. Spanier, E. H. 1966. Algebraic Topology. McGraw–Hill Book Co. New York, N.Y.
35. Stanley, R. 1975. Cohen–Macaulay rings and constructible polytopes. Bull. Amer. Math. Soc. **81:** 133–135.
36. Stanley, R. 1975. The upper bound conjecture and Cohen–Macaulay rings. Stud. Appl. Math. **54:** 135–142.
37. Stanley, R. 1977. Cohen–Macaulay complexes. *In* Higher Combinatorics. M. Aigner, Ed.: 51–62. D. Reidel Publishing Co. Dordrecht, the Netherlands.
38. Stanley, R. 1978. Hilbert functions of graded algebras. Adv. in Math. **28:** 57–83.
39. Stanley, R. 1980. The number of faces of a simplicial convex polytope. Adv. in Math. **35:** 236–238.
40. Stanley, R. 1983. Commutative Algebra and Combinatorics. Progress in Mathematics Vol. 41. Birkhäuser Boston, Inc. Cambridge, Mass.
41. Steenbrink, J. H. M. 1977. Mixed Hodge structure on the vanishing cohomology. *In* Real and Complex Singularities, Oslo, 1976. P. Holm, Ed.: 525–563. Sijthoff & Noordhoff. Alphen aan den Rijn, the Netherlands.