AN INTRODUCTION TO COHEN-MACAULAY
PARTIALLY ORDERED SETS

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0. INTRODUCTION

The theory of Cohen-Macaulay posets derives much of its appeal from its interactions with three distinct branches of mathematics - combinatorics, commutative algebra, and algebraic topology. Our aim here is to convey the flavor of this subject by discussing the basic results and some of their main applications. A comprehensive account of Cohen-Macaulay posets would be too lengthy to include here and will appear elsewhere [BGS], but it is hoped that this introduction will stimulate the reader to look further into the subject.

The combinatorial aspects of Cohen-Macaulay posets have their origins in the following natural problem. Given some set S of objects which one wants to enumerate or in some way "determine," is it possible to partition S into (finitely many) "nice" subsets $S_1,\ldots,S_r$ which can easily be handled individually? The first significant example of such an approach toward counting a set of objects may be found in MacMahon [MM] (especially vivid in Section 439), and was later systematized in [St_1] (and less comprehensively in [Kn]). The theory of P-partitions developed in [St_1] may be formulated in terms of distributive lattices, and is outlined here in Section 1. It is then natural to generalize this theory to a wider class of lattices. Successive generalizations appear in [St_2] and [St_3], finally culminating in the theory of lexicographic shellability [Bj_1]. Section 2 is devoted to this topic.

Section 3 is devoted to the basic topological concepts associated with posets. The prototype for this subject is the theorem of P. Hall [Ha,(2.21)][Ro, p. 346], which may be interpreted as expressing the values of the Möbius function of a poset P as a reduced Euler characteristic. Since Euler characteristics may be computed from certain homology groups, it is natural to go further and look at the actual homology groups. This idea was suggested by Rota [Ro, pp. 355-6] and further considered by Mather [Mat], Folkman [Fo], and Lakser [La]. The work of Folkman on geometric lattices was made part of a general theory by Baclawski [$Ba_{2,4}$], who was the first to consider Cohen-Macaulay posets from a purely topological point of view. Subsequent topological investigations related to the Cohen-Macaulay property appear in [Ba_6],[$Bj_{2,3}$], [BW_1], [BWr], [Fa], [Mu], [Qu] and [Wr].

In Section 4 we introduce the fundamental commutative ring R_P associated with a poset P, and show some connections with the preceding combinatorial concepts. In particular we give a ring-theoretic definition of a Cohen-Macaulay poset. Although Cohen-Macaulay rings date back to the pioneering work of Macaulay [Mac], it was not until the paper [Hoc_1] of Hochster that it became apparent that Cohen-Macaulay rings (or, at least, special classes of them) were closely related to combinatorics. This connection was made more explicit in [St_5] and [Re], where a ring R_Δ was attached to an arbitrary simplicial complex Δ. A poset P may be regarded as a special kind of simplicial complex (see Section 3), and the rings R_P were first singled out for special study in [St_6, Section 8]. Subsequently the poset ring, *per se*, was studied in [Ba_7], [BG], [Ga_1], [Ga_2], [Hoc_2], and [St_8].

In Section 5 we discuss what is perhaps the main theorem on Cohen-Macaulay posets (due to G. Reisner [Re]), which demonstrates the equivalence of the algebraic and topological approaches. Finally in Section 6 we present a brief glimpse of further results in order to illustrate the many fascinating ramifications and applications of the theory, and to bring the reader near some exciting areas of current research.

The following notation will be adhered to throughout. $\mathbb{N}$ denotes the set of nonnegative integers and $\mathbb{P}$ the positive integers. If $d \in \mathbb{N}$, then $[d] = \{1,2,\ldots,d\}$. A disjoint union is denoted by the symbol $\uplus$. All posets P will be finite throughout this paper. If y covers x in P (i.e., if $x < y$ and no $z \in P$ satisfies $x < z < y$) then we write $x \to y$. A poset P is said to be *pure* if every maximal chain has the same length $d+1$ (or cardinality $d+2$). If P is pure and has a $\hat{0}$ and $\hat{1}$ (i.e., $\hat{0} \le x \le \hat{1}$ for all $x \in P$), then P is said to be *graded* of rank $d+1$. If P is any poset, then $\hat{P}$ denotes P with a $\hat{0}$ and $\hat{1}$ adjoined. If P has a $\hat{0}$ and $\hat{1}$, then $\bar{P}$ denotes P with the $\hat{0}$ and $\hat{1}$ removed and is called the *proper part* of P. The set of maximal chains of P is denoted $M(P)$, and the set of all chains $C(P)$.

Assume throughout the remainder of this section that P is graded of rank $d+1$. If $x \in P$, then $r(x)$ denotes the *rank* of x, i.e., the length ℓ of the longest chain $\hat{0} = x_0 < x_1 < \cdots < x_\ell = x$ with top element x. In particular, $r(\hat{0}) = 0$ and $r(\hat{1}) = d+1$. If $x \le y$ in P we write $r(x,y) = r(y) - r(x)$. The set of elements of P of rank i is denoted P_i, and the cardinality $|P_i|$ of this set is denoted W_i and is known as a *Whitney number of the second kind*. If $c: \hat{0} = x_0 < x_1 < \cdots < x_{i+1} = \hat{1}$ is a chain of P, then the set $\{r(x_1),\ldots,r(x_i)\}$ is called the *rank set* of c.

Let μ denote the Möbius function of P [Ro]. For our purposes the most convenient definition of μ is the following. If $x \le y$ in

P, then define $\mu(x,y) = c_0 - c_1 + c_2 - \cdots$, where c_i is the number of chains $x = x_0 < x_1 < \cdots < x_i = y$ of length i between x and y. We write $\mu(P)$ for $\mu(\hat{0},\hat{1})$. The *Whitney numbers* w_i *of the first kind* are defined by

$$w_i = \sum_{x\in P_i} \mu(\hat{0},x) \quad .$$

In particular, $w_0 = 1$ and $w_{d+1} = \mu(P)$. Recall that the *characteristic polynomial* $p_P(t)$ is defined by

$$p_P(t) = \sum_{i=0}^{d+1} w_i t^{d+1-i}$$

Now suppose that S is any subset of $[d] = \{1,2,\ldots,d\}$. Define the S-*rank-selected subposet* $P_S = \{x\varepsilon P\colon x = \hat{0},\ x = \hat{1},$ or $r(x)\varepsilon S\}$. Thus for instance $P_{[d]} = P$ and $P_i = \bar{P}_{\{i\}}$. Let $\alpha(S) = \alpha_P(S)$ denote the number of maximal chains of P_S. Equivalently, $\alpha(S)$ is the number of chains in P whose rank set is S. Note in particular that $\alpha(i)$ (short for $\alpha(\{i\})$) is equal to W_i. Define

$$\beta(S) = \beta_P(S) = \sum_{T\subseteq S} (-1)^{|S-T|}\alpha(T). \tag{1}$$

Thus by the Principle of Inclusion-Exclusion,

$$\alpha(S) = \sum_{T\subseteq S} \beta(T) \quad . \tag{2}$$

Note that it follows from our definition of μ that $\beta(S) = (-1)^{|S|-1}\mu(P_S)$. In particular, for $i = 1,2,\ldots,d$,

$$(-1)^i w_i = \beta(1,2,\ldots,i-1) + \beta(1,2,\ldots,i).$$

Recall also that the *zeta polynomial* $Z(P,n)$ of P, as defined in [St$_4$] and further studied in [Ed$_1$], may be defined for graded posets by either of the two equivalent conditions

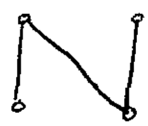
$$Z(P,n) = \sum_{S\subseteq[d]} \alpha(S)\binom{n}{|S|+1}$$

$$\sum_{n\geq 0} Z(P,n)t^n = \frac{\sum_{S\subseteq[d]} \beta(S)t^{|S|+1}}{(1-t)^{d+2}} \quad . \tag{3}$$

Thus we see that the numbers $\beta(S)$ (or equivalently, $\alpha(S)$) are closely related to many properties of P of known interest and importance. The prominence of the numbers $\beta(S)$ will become even more apparent in the following pages, where they will count certain maximal chains in P and will appear in the Hilbert series of the poset ring, as Betti numbers of rank-selected subposets, and as group characters.

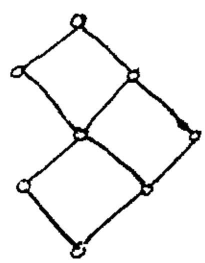
1. DISTRIBUTIVE LATTICES

Let P be a finite poset with $d+1$ elements. An *order ideal* (also called a *semi-ideal* or *decreasing subset*) of P is a subset I of P such that if $x \in I$ and $y < x$, then $y \in I$. Let J(P) denote the set of all order ideals of P, ordered by inclusion. Then J(P) is graded of rank $d+1$, and is in fact a distributive lattice. Conversely, any finite distributive lattice is of the form J(P) for a unique poset P [Bi, p. 59, Thm. 3]. Consider, for instance, the poset P given by



(4)

Then J(P) looks like



(5)

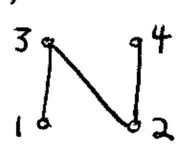
The values of $\alpha_{J(P)}(S)$ and $\beta_{J(P)}(S)$ are given by

S	$\alpha(S)$	$\beta(S)$
ϕ	1	1
1	2	1
2	2	1
3	2	1
12	3	0
13	4	1
23	3	0
123	5	0

Several empirical facts are evident. For instance, $\beta(S) \ge 0$ and several values of $\beta(S)$ are equal to 0. The fact that $\beta(1,2,3) = 0$ follows from the well-known fact that if L is a finite distributive lattice then $\mu(L) = 0$ unless L is a boolean algebra [Ro, pp. 349-50]. But the reason, say, for $\beta(1,3) = 1$ above is not so clear. What is needed is a suitable combinatorial interpretation of the numbers $\beta(S)$.

Consider the general case $L = J(P)$, where $|P| = d+1$. We first associate with every maximal chain of L a permutation of the set $[d+1]$. Fix an order-preserving bijection $\omega: P \to [d+1]$. Let $\phi = I_0 \subset I_1 \subset \cdots \subset I_{d+1} = P$ be a maximal chain m of L. Each I_i is an i-element order ideal of P. Hence $I_i - I_{i-1}$ contains a unique element x_i of P. Associate with m the permutation $\pi(m) = (\omega(x_1), \omega(x_2), \ldots, \omega(x_{d+1}))$ of $[d+1]$. The set $\mathcal{L}(P) = \mathcal{L}(P,\omega) := \{\omega(m) : m \in \mathcal{M}(L)\}$ is called the *Jordan-Hölder set* of P. If $\pi = (a_1, \ldots, a_{d+1})$ is any permutation of $[d+1]$, then the *descent set* of π is defined by $\mathcal{D}(\pi) = \{i : a_i > a_{i+1}\}$. It is then easy to verify that $\alpha_L(S)$ is equal to the number of permutations $\pi \in \mathcal{L}(P)$ whose descent set is contained in S.

For instance, if we define ω for the poset P of (4) by



then we have the following table:

$\pi \in \mathcal{L}(P)$	$\mathcal{D}(\pi)$
1234	ϕ
2134	1
1243	3
2143	1,3
2413	2

Since, for example, four of the above descent sets are contained in $\{1,3\}$, we conclude $\alpha(1,3) = 4$.

Now let $\gamma(S) = \gamma_P(S) = |\{\pi \in \mathcal{L}(P) : \mathcal{D}(\pi) = S\}|$. Clearly

$$\sum_{T \subseteq S} \gamma(T) = |\{\pi \in \mathcal{L}(P) : \mathcal{D}(\pi) \subseteq S\}| = \alpha(S).$$

But the numbers $\beta_L(T)$ are uniquely defined by (2), so we conclude $\beta_L(S) = \gamma_P(S)$. Hence:

1.1 THEOREM: *Let* $L = J(P)$. *Then* $\beta_L(S)$ *is equal to the number of permutations* $\pi \in L(P)$ *whose descent set is* S.

This shows in particular that $\beta_L(S) \geq 0$ for a distributive lattice L, and much additional information can be gleaned from Theorem 1.1 [St_1]. Moreover, the numbers $\beta(S)$ play a central role in the theory of P-partitions [St_1], which deals with order-preserving (or order-reversing) maps of a poset P into a chain. For instance, if $\Omega(P,n)$ denotes the number of order-preserving maps $\sigma: P \to [n]$, then

$$\sum_{n\geq 0} \Omega(P,n)t^n = \Big(\sum_{\pi\in L(P)} t^{1+|\mathcal{D}(\pi)|}\Big)(1-t)^{-d-2} \qquad (6)$$

$$= \Big(\sum_{S\subseteq[d]} \beta(S)t^{1+|S|}\Big)(1-t)^{-d-2} \quad .$$

Note that it follows from (3) or otherwise that $\Omega(P,n) = Z(J(P),n)$.

At this point it is natural to ask for a generalization of the preceding theory to wider classes of graded posets Q. What structural properties of Q guarantee that $\beta_Q(S) \geq 0$? Is there a natural way of associating a permutation (or possibly a sequence) with each maximal chain of Q so that Theorem 1.1 remains valid? These and related questions will be considered in the next section.

2. LEXICOGRAPHICALLY SHELLABLE POSETS

Let P be a graded poset of rank $d+1$. Assume that each edge $x \to y$ in P has been labeled by an integer $\lambda(x\to y)$. Then each k-chain $x_0 \to x_1 \to \cdots \to x_k$ is naturally labeled by a k-tuple $\lambda(x_0 \to x_1 \to \cdots \to x_k) = (\lambda(x_0\to x_1), \lambda(x_1\to x_2), \ldots, \lambda(x_{k-1}\to x_k)) \in \mathbb{Z}^k$. We will compare such k-tuples by their *lexicographic order*: $(a_1, a_2, \ldots, a_k) <_L (b_1, b_2, \ldots, b_k)$ if and only if $a_i < b_i$ in the first coordinate where they differ.

2.1 DEFINITION: The edge-labeling λ will be called an *EL-labeling* if for every interval $[x,y]$ in P:

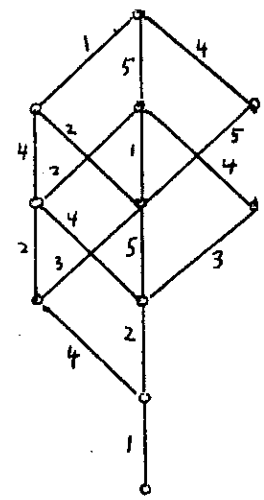
(i) There is a unique chain $a_{x,y}: x = x_0 \to x_1 \to \cdots \to x_k = y$

such that $\lambda(x_0\to x_1) \leq \lambda(x_1 \to x_2) \leq \cdots \leq \lambda(x_{k-1}, x_k)$,

(ii) for every other chain $b: x = y_0 \to y_1 \to \cdots \to y_k = y$ we have $\lambda(b) >_L \lambda(a_{x,y})$.

The poset P is said to be *edge-wise lexicographically shellable*, or *EL-shellable*, if it admits an EL-labeling.

The reader is invited to check that the following edge-labeling is an EL-labeling.



(7)

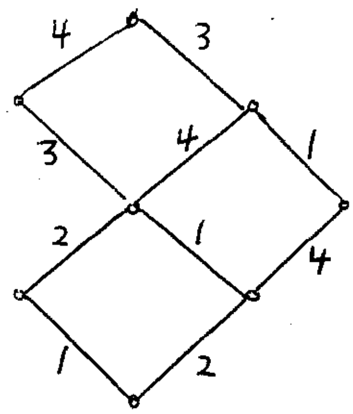
For each maximal chain $m: \hat{0} = x_0 \to x_1 \to \cdots \to x_{d+1} = \hat{1}$ we define the *descent set* $\mathcal{D}(m) = \{i \in [d] : \lambda(x_{i-1} \to x_i) > \lambda(x_i \to x_{i+1})\}$. There is the following combinatorial interpretation of the β-invariant of an EL-shellable poset P.

2.2 THEOREM. *For* $S \subseteq [d]$, $\beta(S)$ *equals the number of maximal chains* m *in* P *such that* $\mathcal{D}(m) = S$. *In particular,* $\beta(S) \geq 0$.

Every interval of P is EL-shellable under the same edge-labeling, so we deduce the following interpretation of the Möbius function of P.

2.3 COROLLARY: *When* $x < y$ *in* P, $(-1)^{r(x,y)}\mu(x,y)$ *equals the number of chains* $x = x_0 \to x_1 \to \cdots \to x_k = y$ *such that* $\lambda(x_0 \to x_1) > \lambda(x_1 \to x_2) > \cdots > \lambda(x_{k-1} \to x_k)$. *In particular,* $(-1)^{r(x,y)}\mu(x,y) \geq 0$.

2.4 EXAMPLE: Distributive lattices. Recall the discussion of Section 1. Let $L = J(P)$ be a finite distributive lattice of rank $d+1$, and let $\omega: P \to [d+1]$ be an order-preserving bijection. If $I \to I'$ is an edge in $J(P)$ then $I' - I = \{x\}$, and we can label $\lambda_\omega(I \to I') = \omega(x)$. This is an EL-labeling. The lattice (5) considered in Section 1 gets the following labeling λ_ω from the choice of ω considered there.

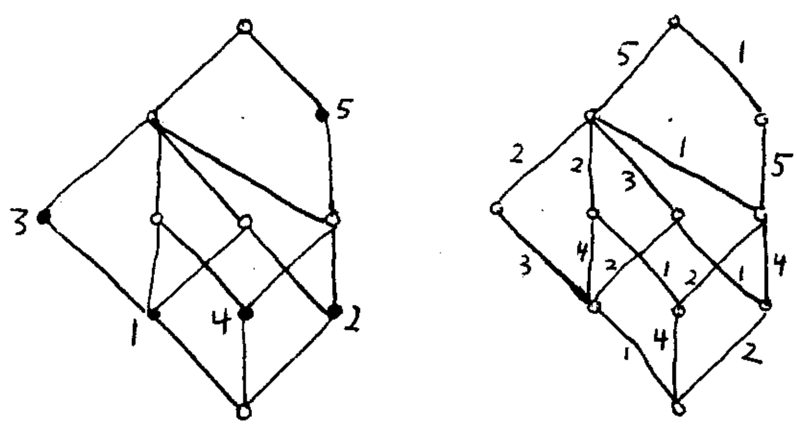


(8)

2.5 EXAMPLE: Semimodular lattices. Let L be a finite (upper) semimodular lattice. Give the join-irreducibles of L a linear order $e_1, e_2, \ldots, e_s$ which extends their partial order (i.e., $e_i < e_j$ implies $i < j$). Then

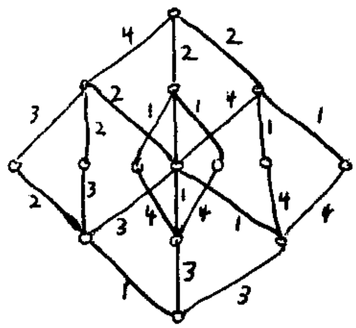
$$\lambda(x \to y) = \min \{i \mid x < x \vee e_i = y\}$$

defines an EL-labeling of L. Clearly, if L is distributive this coincides with the labeling method described in Example 2.4. The first picture below shows an ordering of the join-irreducibles of a semimodular lattice, the second picture shows the induced EL-labeling.



(9)

2.5 EXAMPLE: Supersolvable lattices. A finite lattice L is said to be *supersolvable* if it contains a maximal chain M which together with any other chain in L generates a distributive sublattice. Each such distributive sublattice can be labeled by the method previously described so that M receives the increasing label $(1,2,\ldots,d+1)$. It turns out that this will assign a unique label to each edge of L, and the resulting global labeling of L is an EL-labeling. If G is a supersolvable finite group then the lattice L(G) of subgroups of G is a supersolvable lattice (hence the terminology). The following picture shows an EL-labeling of the subgroup lattice of the Abelian group $\mathbb{Z}_4 \times \mathbb{Z}_4$ produced by the method described.



(10)

The notion of a lexicographically shellable poset as defined above was introduced in [Bj_1], the motivating examples being the finite distributive [St_1], supersolvable [St_2], and semimodular lattices [St_3]. In an EL-shellable poset the maximal chains derive their labels from the edges traversed. In some situations it is not known how to assign labels to the individual edges in a coherent manner, but it is nevertheless possible to assign labels directly to the maximal chains in such a way that the basic properties encountered above are preserved. This leads to the more general definition of lexicographic shellability considered in [BW_1] and [BW_2].

Let P be a graded poset of rank $d+1$. To each maximal chain m: $\hat{0} = x_0 \to x_1 \to \cdots \to x_{d+1} = \hat{1}$ we assign a label $\lambda(m) = (\lambda_1(m), \lambda_2(m), \ldots, \lambda_{d+1}(m)) \in \mathbb{Z}^{d+1}$, satisfying the following condition: If two maximal chains m and m' coincide along their first k edges, then $\lambda_i(m) = \lambda_i(m')$ for $i = 1,2,\ldots,k$. If $[x,y]$ is an interval and c a saturated chain from $\hat{0}$ to x the pair $([x,y],c)$ will be called a *rooted interval* and denoted $[x,y]_c$. Every maximal chain b in a rooted interval $[x,y]_c$ has a well-defined induced label $\lambda^c(b) \in \mathbb{Z}^{r(x,y)}$, namely $\lambda^c_i(b) = \lambda^c_i(b) = \lambda_{r(x)+i}(m)$ for $i = 1,2, \ldots, r(x,y)$ and any maximal chain m containing c and b.

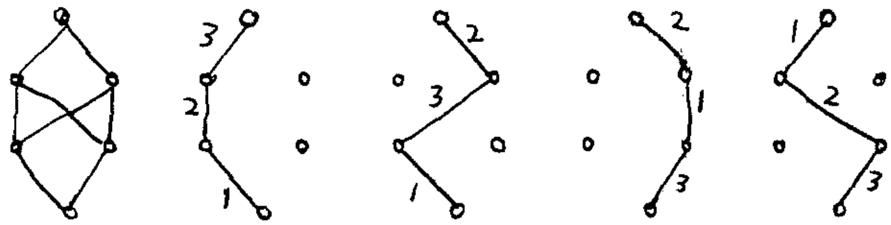
2.7 DEFINITION: A labeling λ of the maximal chains of P as above is a *CL-labeling* if for every rooted interval $[x,y]_c$ in P:

(i) there is a unique maximal chain $a = a_{x,y,c}$ in $[x,y]$ such that $\lambda_1^c(a) \le \lambda_2^c(a) \le \cdots \le \lambda^c_{r(x,y)}(a)$,

(ii) if b is any other maximal chain in $[x,y]$ then $\lambda^c(b) >_L \lambda^c(a_{x,y,c})$.

The poset P is said to be *chain-wise lexicographically shellable*, or *CL-shellable*, if it admits a CL-labeling.

This definition is illustrated in the following picture which shows a rank 3 poset and a CL-labeling of its maximal chains.



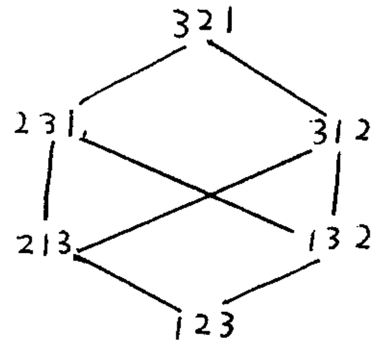
(11)

It is clear that an EL-labeling of P induces a CL-labeling, hence EL-shellable implies CL-shellable. The reason for calling these posets "lexicographically shellable" is that the lexicographic order of the labels gives a shelling order to the maximal chains. For a definition of "shelling order" and the general notion of "shellable posets" see [Bj_1].

The notion of *descent set* generalizes immediately from EL-labelings to CL-labelings: For a maximal chain m in P and a CL-labeling λ, $\mathcal{D}(m) = \{i\varepsilon[d] \mid \lambda_i(m) > \lambda_{i+1}(m)\}$. Theorem 2.2 remains true for CL-shellable posets. Also, a CL-labeling of P restricts to a CL-labeling of any rooted interval $[x,y]_c$, so Corollary 2.3 suitably reformulated also remains true.

2.8 EXAMPLE: Bruhat order. Let (W,S) be a *Coxeter group* (see [Bo, Ch. 4]). For instance, W could be the symmetry group of a regular polytope or the Weyl group of a root system, and S the set of reflections across the walls of a fundamental chamber. *Bruhat order* is a partial ordering of W which appears repeatedly in the geometry and representation theory of algebraic groups and Lie algebras. For instance, let G be a connected complex semisimple algebraic group and B a Borel subgroup. The Schubert varieties $\bar{C}_w$ in the irreducible complex projective variety G/B are indexed by the corresponding Weyl group W, and Bruhat order describes the disposition of Schubert varieties: $w \leq w'$ if and only if $\bar{C}_w \subseteq \bar{C}_{w'}$. We will here explicitly formulate Bruhat order only for the symmetric groups S_n and refer to [De] or [BW_1] for the general definition.

Let S_n be the group of all permutations of [n], and let $\pi = a_1 a_2 \cdots a_n \varepsilon S_n$. The elements covering π in Bruhat order are precisely those permutations π' obtained from π by interchanging a pair of elements a_i and a_j with $i < j$, satisfying: (i) $a_i < a_j$, and (ii) if $i < k < j$ then $a_k < a_i$ or $a_k > a_j$. It follows that the poset rank of π equals its inversion number, i.e., the number of pairs (i,j) with $i < j$ and $a_i > a_j$. Here is the Bruhat order of S_3:



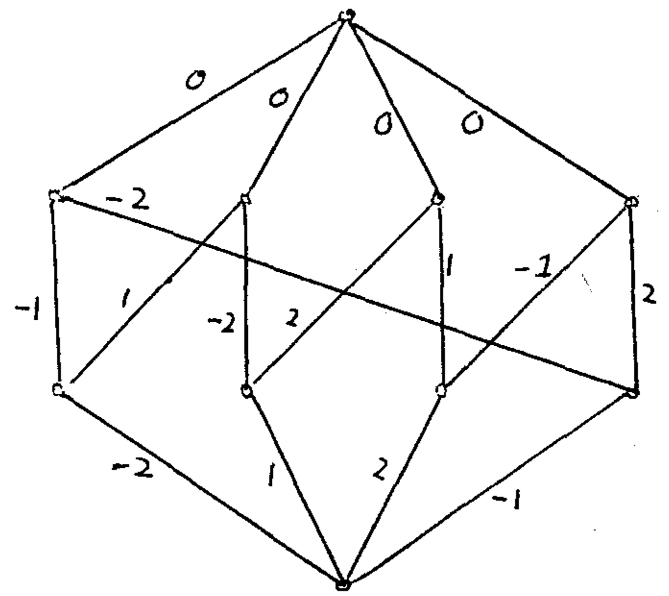
(12)

Let (W,S) be a Coxeter group (not necessarily finite). Each interval [w,w'] in the Bruhat order of W is a finite graded poset, which can be shown to be CL-shellable [BW_1]. For instance, the labeling algorithm applied to S_3 produces the labeling of maximal chains in picture (11). A slightly stronger formulation is possible. For $J \subseteq S$, let W^J denote the set of minimal coset representatives modulo the parabolic subgroup W_J (as defined e.g. in [Bo, p. 19]). Then every interval [w,w'] in the Bruhat order of W^J is CL-shellable.

It is not known whether Bruhat order is in general EL-shellable. However, EL-labelings have been found (cf. [Ed_2],[Pr]) for the classical finite Weyl groups corresponding to root systems of type A, B, C and D.

2.9 EXAMPLE: Face lattices. Let Δ be a finite simplicial or polyhedral complex with all maximal faces of the same dimension. The faces of Δ ordered by inclusion form a graded lattice $L(\Delta)$ (the improper faces ϕ and Δ included), the *face-lattice* of Δ. The face-lattices (or their duals) of the following complexes are known to be CL-shellable (cf. [BW_2]): (1) The boundary complex of a convex polytope, (2) the Coxeter complex of a finite Coxeter group, (3) the Tits building of a finite group with BN-pair, (4) the independent set complex of a matroid, (5) the broken circuit complex of a matroid. Face-lattices of simplices, cubes and cross-polytopes of all dimensions are known to be EL-shellable [Bj_1, p. 174]; picture (13) shows an EL-labeling of the face-lattice of the 2-cube. It is not known whether face-lattices of convex polytopes are in general El-shellable.

2.10 EXAMPLE: Totally semimodular posets. A finite poset P is said to be (upper) *semimodular* if whenever two distinct elements u,v both cover $t \varepsilon P$ there exists a $z \varepsilon P$ which covers both u and v (cf. [Bi, p. 39]). P is said to be *totally semimodular* if it has $\hat{0}$ and $\hat{1}$ and all intervals [x,y] are semimodular. Totally semimodular posets are CL-shellable [BW_2]



(13)

3. ORDER HOMOLOGY

Let P be a finite poset, and let k be a field or the ring of integers $\mathbb{Z}$. For each $i \in \mathbb{Z}$ let $C_i(P,k)$ denote the free k-module on the basis of the i-chains $x_0 < x_1 < \cdots < x_i$ of P. We consider ϕ to be a (-1)-chain, so $C_{-1}(P,k) \cong k$. Also, $C_i(P,k) = 0$ for $i < -1$ and $i > \text{length}(P)$. Define a map $d_i\colon C_i(P,k) \to C_{i-1}(P,k)$ by linearly extending

$$d_i(x_0 < x_1 < \cdots < x_i)$$

$$= \sum_{j=0}^{i} (-1)^j (x_0 < x_1 < \cdots < x_{j-1} < x_{j+1} < \cdots < x_i)$$

when i-chains exist, and setting $d_i = 0$ otherwise. Then $d_i d_{i+1} = 0$; that is, $\operatorname{Im} d_{i+1} \subseteq \ker d_i$. Define, for $i \in \mathbb{Z}$,

$$\tilde{H}_i(P,k) = \ker d_i / \operatorname{Im} d_{i+1} . \tag{14}$$

The k-modules $\tilde{H}_i(P,k)$ are the *order homology groups of P with coefficients in* k. Clearly, $\tilde{H}_i(P,k) = 0$ for $i < -1$ and $i > \ell(P)$. Also, $\tilde{H}_{-1}(P,k) = 0$ if and only if $P \neq \phi$, and $\tilde{H}_0(P,k) = 0$ if and only if P is connected (as a poset). In fact, $\tilde{H}_0(P,k)$ is a free k-module of rank one less than the number of connected components of P.

Let $c_i(P)$ denote the number of i-chains of P. For k a field we have the *Euler-Poincaré formula*

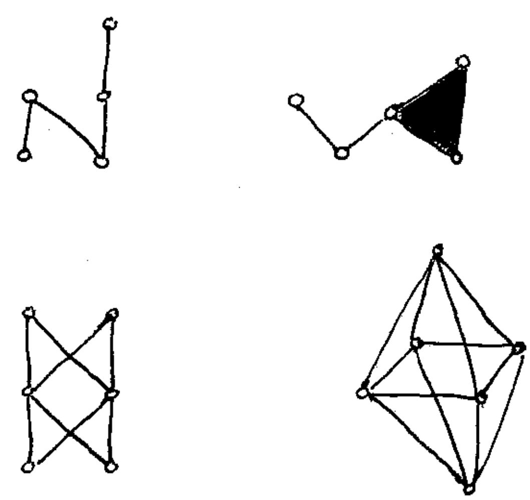
$$\sum_{i=-1}^{\ell(P)} (-1)^i c_i(P) = \sum_{i=-1}^{\ell(P)} (-1)^i \dim_k \tilde{H}_i(P,k) . \tag{15}$$

As was mentioned in the Introduction the left side of (15) equals $\mu(\hat{P})$. Hence, the Möbius function $\mu(\hat{P})$ is the *Euler characteristic* of order homology:

$$\mu(\hat{P}) = \sum_{i=-1}^{\ell(P)} (-1)^i \dim_k \tilde{H}_i(P,k) . \tag{16}$$

The numbers $\dim_{\mathbb{Q}} \tilde{H}_i(P,\mathbb{Q})$ are called the *Betti numbers* of P.

Define a simplicial complex $\Delta(P)$ in the following way: the vertices of $\Delta(P)$ are the elements of P and the i-faces of $\Delta(P)$ are the i-chains $x_0 < x_1 < \cdots < x_i$. The order homology of P defined in (14) is then the (reduced) simplicial homology of $\Delta(P)$, as familiar from combinatorial topology. In fact, the properties stated above are just a recapitulation in poset terminology of some basic facts from simplicial topology. As a finite simplicial complex, the *order complex* $\Delta(P)$ determines a compact topological space $|\Delta(P)|$, the *geometric realization* of P. Thus one may speak of topological type, homotopy type, etc., of posets. The following picture shows two posets and next to them their geometric realizations:



(17)

To obtain a poset with preassigned homology groups or topological type, one can take a simplicial complex τ with the desired properties and look at the poset P_τ of faces of τ ordered by inclusion. Then $|\Delta(P_\tau)|$ and $|\tau|$ are homeomorphic (the former being the barycentric subdivision of the latter). In particular, any compact triangulable space is the geometric realization of a finite poset (in fact, of the proper part $\bar{L}$ of a finite lattice L).

Suppose now that P is a lexicographically shellable poset of rank $d+1$. The order homology of $\bar{P}$ can easily be computed by standard methods of algebraic topology (the Mayer-Vietoris exact sequence): $\tilde{H}_i(P,k) = 0$ for $i \neq d-1$, and $\tilde{H}_{d-1}(P,k) \cong k^{|\mu(P)|}$. Since each interval $[x,y]$ in P is lexicographically shellable, we have found that for each open interval (x,y) in P "homology vanishes" below the top dimension. This brings us to the key definition.

3.1 DEFINITION: Let P be a graded poset. We say that P is *Cohen-Macaulay over* k (CM/k, for short) if for each open interval (x,y) in P: $\tilde{H}_i((x,y),k) = 0$ for $i \neq \delta(x,y)$, where $\delta(x,y) = r(x,y)-2$ is the dimension of $\Delta((x,y))$.

In particular,

3.2 THEOREM: *If* P *is lexicographically shellable, then* P *is Cohen-Macaulay over* $\mathbb{Z}$ *and all fields* k.

In a lexicographically shellable poset the Möbius function $\mu(x,y)$ has a combinatorial interpretation (Corollary 2.3) which shows that $(-1)^{r(x,y)}\mu(x,y) \geq 0$. More generally, in a Cohen-Macaulay poset by (16): $(-1)^{r(x,y)}\mu(x,y) = \dim_k H_{\delta(x,y)}((x,y),k) \geq 0$.

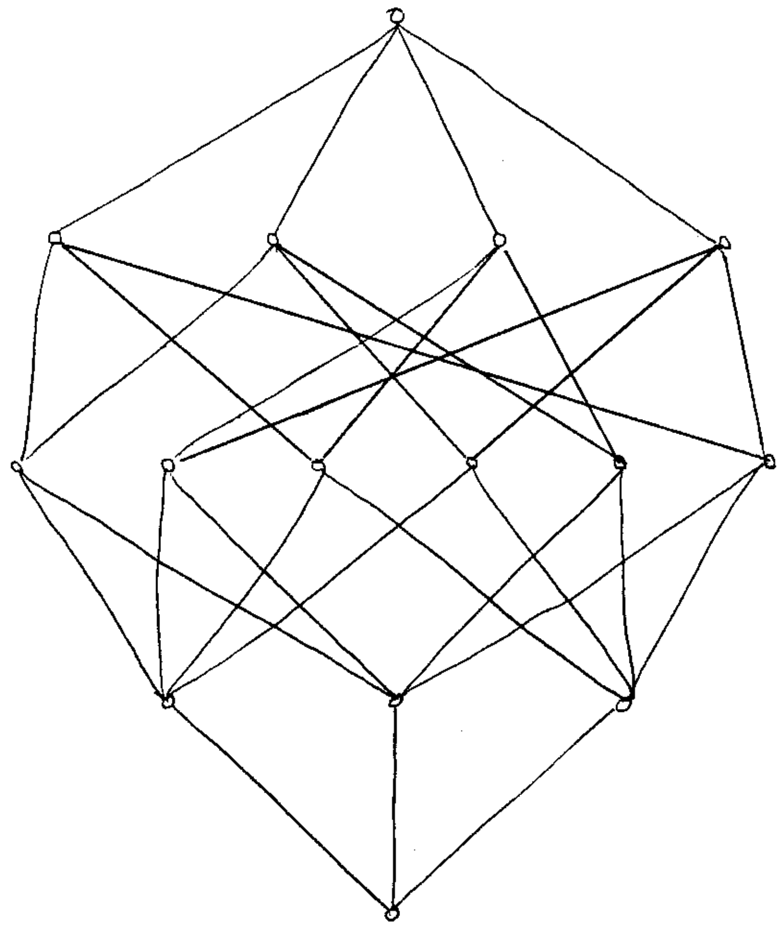
In a similar way the nonnegativity of the numbers $\beta(S)$ (cf. Theorem 2.2) extends to any Cohen-Macaulay poset P (cf. Theorem 5.2).

3.3 THEOREM: *For each* $S \subseteq [d]$, *the top homology group of* $\bar{P}_S$ *is free over* k *of rank* $\beta(S)$.

The definition of Cohen-Macaulayness given here applies only to *graded* posets. However, it is possible to define the notion in a similar manner for any finite poset P. It can be shown that this definition implies that P is graded ($\hat{0}$ and $\hat{1}$ may have to be added), so there is in fact no loss of generality.

The property of being Cohen-Macaulay depends on the ring k as well as the poset. The universal coefficient theorem for homology shows that CM-ness over $\mathbb{Z}$ is the strongest property and CM-ness over $\mathbb{Q}$ the weakest. More precisely, CM/$\mathbb{Z}$ implies CM/k for *all* fields k, while CM/k for *some* field k implies CM/$\mathbb{Q}$.

All graded posets of rank one and two are CM. A graded poset P of rank 3 is CM if and only if the proper part $\bar{P}$ is connected. From rank 4 on CM-ness cannot be as easily characterized. Consider, for instance, the following rank 4 poset P.



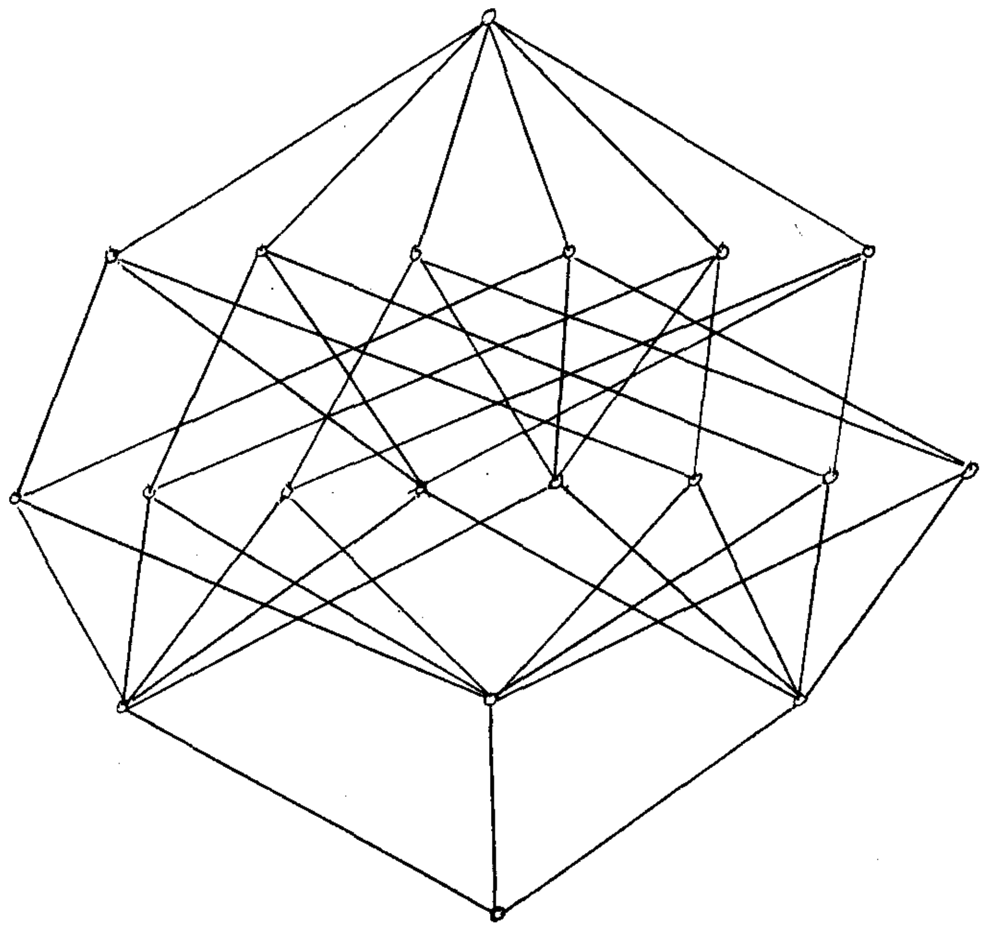
(18)

It can be seen by inspection that all rank 3 intervals are connected, so to test for CM-ness one must only compute the homology of the proper part $\bar{P}$. Now, $|\Delta(\bar{P})|$ is the real projective plane, so

$$\tilde{H}_i(\bar{P}, \mathbb{Z}) = \begin{cases} \mathbb{Z}_2, & i = 1 \\ 0, & i \neq 1 \end{cases}$$

Hence, P is not CM over $\mathbb{Z}$ or over $\mathbb{Z}_2$, but P is CM over fields of characteristic $\neq 2$.

For another example, consider the following poset.



(19)

Here $\tilde{H}_i(\bar{P}, \mathbb{Z}) = 0$ for all i (the geometric realization of $\bar{P}$ is the dunce hat), so P is CM/$\mathbb{Z}$. However, P is not (lexicographically) shellable.

Let us finally mention the remarkable fact that the CM-ness of a poset is a topological property: If P_1 and P_2 are two finite graded posets and $|\Delta(\bar{P}_1)|$ is homeomorphic to $|\Delta(\bar{P}_2)|$, then P is CM/k if and only if P_2 is CM/k.

4. THE POSET RING

With any finite poset P we will associate a commutative ring R_P whose structure closely reflects the combinatorial and topological properties of P. In this way we gain new insight into the significance of Cohen-Macaulay posets. Assume that P has a $\hat{0}$ and $\hat{1}$, and let k be any field (which for our purposes can be taken to be the rational numbers $\mathbb{Q}$). (Much of the theory goes through for an arbitrary commutative ring k, but for simplicity we take k to be a field throughout this and the next section.) If $\bar{P} = \{x_1, \ldots, x_n\}$, then form the polynomial ring $R = k[x_1, \ldots, x_n]$, where the elements of $\bar{P}$ are regarded as independent indeterminates.

Let I_P be the ideal of R generated by all products $x_i x_j$ where x_i and x_j are incomparable elements of $\bar{P}$. Set $R_P = R/I_P$. We call R_P the *poset ring* corresponding to P. (In some papers it is not required that P have a $\hat{0}$ and $\hat{1}$, and our R_P would be denoted $R_{\bar{P}}$.) More generally, one can define a ring R_Δ associated with an arbitrary simplicial complex Δ [St5,6] [Re], so that the ring $R_{\Delta(\bar{P})}$ associated with the order complex of $\bar{P}$ coincides with the poset ring R_P. In this general setup R_Δ is called the "Stanley-Reisner ring" of Δ. Here, however, we will be concerned exclusively with posets.

The ring R_P is an algebra over the field k. A k-basis for R_P consists of all monomials of the form $y_1^{a_1} y_2^{a_2} \cdots y_s^{a_s}$, where $a_i > 0$ and $y_1 < y_2 < \cdots < y_s$. We can identify this monomial with the multichain (= chain with repeated elements)

$$y_1^{a_1} < y_2^{a_2} < \cdots < y_s^{a_s}$$

of P, where the notation indicates that y_i is repeated a_i times. Thus we can regard elements of R_P as linear combinations of multichains.

In general, a k-algebra A is said to be $\mathbb{N}^m$-*graded*, where $m \in \mathbb{P}$, if it is given a vector space direct sum decomposition $A = \coprod_{\alpha \in \mathbb{N}^m} A_\alpha$ satisfying $A_\alpha A_\beta \subseteq A_{\alpha+\beta}$. We then call an element $x \in A_\alpha$ *homogeneous of degree* α, denoted $\deg x = \alpha$. If A is finitely-generated as a k-algebra, then each A_α is a finite-dimensional vector space over k. In this case, the *Hilbert series* of A is defined to be the formal power series

$$F(A,t) = \sum_{\alpha \in \mathbb{N}^m} (\dim_k A_\alpha) t^\alpha \quad ,$$

where $t = (t_1, \ldots, t_m)$ and $t^\alpha = t_1^{\alpha_1} \cdots t_m^{\alpha_m}$ for $\alpha = (\alpha_1, \ldots, \alpha_m)$. For poset rings there are several possible ways to define an $\mathbb{N}^m$-grading, of which only one will concern us here. Namely, if P is graded of rank $d+1$ and $x \in P_i$, $1 \le i \le d$, then we define $\deg x$ to be the i^{th} unit coordinate vector e_i in $\mathbb{N}^d$. This makes R_P into an $\mathbb{N}^d$-graded k-algebra where $(R_P)_\alpha$ has a k-basis consisting of all multichains containing α_i elements of rank i, where $\alpha = (\alpha_1, \ldots, \alpha_d)$. Let us compute the Hilbert series $F(R_P, t)$ with respect to this grading. By grouping multichains in P according to their supports (i.e., the chain of elements of P which appear at least once in the multichain), we see that

$$F(R_P,t) = \sum_{S\subseteq[d]} \alpha(S) \prod_{i\in S} \frac{t_i}{1-t_i} .$$

When put over the common denominator $\prod_{i=1}^{d}(1-t_i)$, the coefficient of $\prod_{i\in S} t_i$ in the numerator becomes $\sum_{T\subseteq S} (-1)^{|S-T|}\alpha(T) = \beta(S)$. There follows:

4.1 THEOREM: *We have*

$$F(R_P,t) = \Big(\sum_{S\subseteq[d]} \beta(S) \prod_{i\in S} t_i\Big) \prod_{i=1}^{d} (1-t_i)^{-1} .$$

If A is any finitely-generated $\mathbb{N}^m$-graded k-algebra and $x\varepsilon A_\alpha$, then it is easily seen that

$$F(A,t) \leq \frac{F(A/x,t)}{1-t^\alpha} \qquad (\text{termwise} \leq) ,$$

with equality if and only if x is a non-zero-divisor in A [St_7, Theorem 3.1]. Define the *rank-level parameters* $\theta_1\ldots,\theta_d$ of R_P by

$$\theta_i = \sum_{x\varepsilon P_i} x \qquad ,$$

so θ_i is homogeneous of degree e_i. The quotient ring $Q_P = R_P/(\theta_1,\ldots,\theta_d)$ inherits an $\mathbb{N}^d$-grading from R_P. The following result is now an immediate consequence of Theorem 4.1.

4.2 THEOREM: *Let* $\theta_1,\ldots,\theta_d$ *be the rank-level parameters of the graded poset* P *of rank* $d+1$. *Let* $Q_P = R_P/(\theta_1,\ldots,\theta_d)$. *Then*

$$F(Q_P,t) = \sum_{S\subseteq[d]} \beta(S) \prod_{i\in S} t_i \tag{20}$$

if and only if θ_i *is not a zero-divisor in the ring* $R_P/(\theta_1,\ldots,\theta_{i-1})$, $1 \leq i \leq d$.

We define the ring R_P to be *Cohen-Macaulay* if it satisfies either of the two equivalent conditions of Theorem 4.2. Note that our definition is only applicable to *graded* posets. It is possible to define what is meant for any finitely-generated graded algebra to be Cohen-Macaulay. One can then prove that if P is any finite poset with $\hat{0}$ and $\hat{1}$ such that R_P is Cohen-Macaulay, then P is indeed graded. Thus it costs us nothing to restrict our attention to graded posets from the beginning.

An important property of a Cohen-Macaulay poset ring R_P is that there is a simple canonical form for its elements. First note that if $x \in P_i$ then $x\theta_i = x^2$ in R_P. Hence $x^2 = 0$ in Q_P, so Q_P is spanned by all *chains* (or squarefree monomials) in P. (This also follows from (20)). Choose a k-basis $\mathcal{B}$ for Q_P consisting of chains of P. We call $\mathcal{B}$ a set of *separators* for R_P. By (20), the number of chains $0 < y_1 < \cdots < y_s < 1$ in $\mathcal{B}$ satisfying $S = \{r(y_1),\ldots,r(y_s)\}$ is equal to $\beta(S)$. The next result is a consequence of the previous theorem.

4.3 COROLLARY: *Let R_P be Cohen-Macaulay, and let $\mathcal{B}$ be a set of separators. Then every element f of R_P can be written uniquely in the form*

$$f = \sum_{\eta \in \mathcal{B}} \eta \cdot p_\eta(\theta_1,\ldots,\theta_d), \tag{21}$$

where p_η is a polynomial in $\theta_1,\ldots,\theta_d$ (with coefficients in k).

It is an interesting problem to decide efficiently when a poset ring is Cohen-Macaulay and to find a set of separators when this is the case. The following result appears in [Ga_2], together with an algorithm for testing CM-ness.

4.4 THEOREM: *Let $\mathcal{B}$ be a collection of chains of P, such that $\beta_P(S)$ elements of $\mathcal{B}$ have rank set S, for all $S \subseteq [d]$. Let Φ be the incidence matrix between $\mathcal{B}$ and the set M of maximal chains of P, i.e., if $c \in \mathcal{B}$ and $m \in M$ then*

$$\Phi_{cm} = \begin{cases} 1, & \text{if } c \subseteq m \\ 0, & \text{if } c \nsubseteq m \end{cases}.$$

The following two conditions are equivalent:
(i) R_P is CM and $\mathcal{B}$ is a set of separators,
(ii) the matrix Φ is invertible (over k).

Now suppose P is lexicographically shellable, and let

$$m: \hat{0} = y_0 \to y_1 \to \cdots \to y_d \to y_{d+1} = \hat{1}$$

be a maximal chain of P. Define the *restriction*

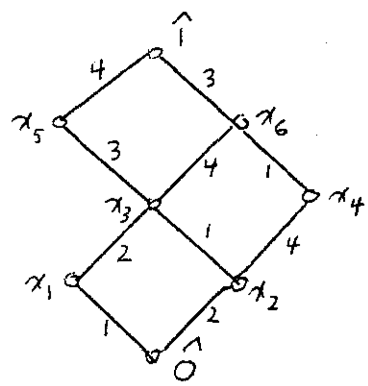
$$\mathcal{R}(m) = \{y_i : \lambda(y_{i-1} \to y_i) > \lambda(y_i \to y_{i+1})\},$$

and let $\mathcal{B} = \{\mathcal{R}(m) : m \in M\}$. If one orders M lexicographically and orders $\mathcal{B}$ correspondingly, then the matrix Φ of Theorem 4.4 is upper triangular with 1's on the main diagonal. Hence from Theorem 4.4 we deduce:

4.5 COROLLARY: *Let* P *be lexicographically shellable, and let* $B = \{R(m) : m \in M\}$. *Then* R_P *is Cohen-Macaulay (over any field* k*), and* B *is a set of separators.*

Corollary 4.5 can be formulated for arbitrary shellable posets [Ga2, Thm. 4.2], and can be generalized to arbitrary shellable simplicial complexes [KK].

4.6 EXAMPLE: Let P be the distributive lattice



(22)

with EL-labeling as shown (cf. (8)). We then have the following table.

maximal chain m	$R(m)$
$\hat{0} \to x_1 \to x_3 \to x_5 \to \hat{1}$	ϕ
$\hat{0} \to x_1 \to x_3 \to x_6 \to \hat{1}$	x_6
$\hat{0} \to x_2 \to x_3 \to x_5 \to \hat{1}$	x_2
$\hat{0} \to x_2 \to x_3 \to x_6 \to \hat{1}$	$x_2 < x_6$
$\hat{0} \to x_2 \to x_4 \to x_6 \to \hat{1}$	x_4

It follows that every element f of R_P can be written uniquely in the form

$$f = p_1 + x_6 p_2 + x_2 p_3 + x_2 x_6 p_4 + x_4 p_5 ,$$

where p_i is a polynomial in the variables $\theta_1 = x_1 + x_2$, $\theta_2 = x_3 + x_4$, $\theta_3 = x_5 + x_6$.

5. REISNER'S THEOREM

Perhaps the most important result in the theory of Cohen-Macaulay posets is the following. While we state it for the case of a field k, the theorem and its attendant background carry over straightforwardly to more general commutative rings k, in particular to the important case $k = \mathbb{Z}$.

5.1 THEOREM: *Fix a field* k *and let* P *be a finite graded poset. Then* P *is Cohen-Macaulay (in the topological sense of Definition 3.1) if and only if the poset ring* R_P *is Cohen-Macaulay.*

There is no really simple proof known of this fundamental result. The original proof of Reisner [Re] (which dealt with arbitrary simplicial complexes) used two tools from commutative algebra: (i) the theory of local cohomology first developed by Grothendieck, and (ii) the theory of the "purity of Frobenius" developed by Hochster and Roberts which enables one to compute local cohomology in characteristic p. Subsequently a proof was given by Hochster [Hoc_2] which replaced local cohomology by properties of the Koszul complex. Hochster's proof yields a homological criterion for R_P to be Cohen-Macaulay which requires considerable topological arguments [Mu] to show is equivalent to Definition 3.1. A proof similar to Hochster's, but more elementary, was given by Baclawski and Garsia [BG, Prop. 6.2]. Hochster also gave a further proof using local cohomology but avoiding purity of Frobenius. This proof is unpublished by Hochster but reproduced in [St_{10}]. For readers familiar with homological algebra it is the shortest and most elegant proof of Reisner's theorem to date. Finally, at this very Symposium on Ordered Sets, Stanley and Walker succeeded in finding a proof devoid of all but the simplest ring theory and using nothing from topology beyond the Mayer-Vietoris sequence.

Reisner's theorem is merely the first step in a beautiful theory connecting the combinatorics and topology of P with the algebraic properties of R_P. We will give the briefest glimpse of what lies beyond Reisner's theorem in Section 6e. For the conclusion of this section we will content ourselves with an illustration of the usefulness of Reisner's theorem in proving purely combinatorial statements concerning P.

Let P be Cohen-Macaulay of rank $d+1$, and let $S\subseteq[d]$. Let I be the ideal of R_P generated by $\bigcup_{i\in[d]-S} P_i$. Then $R_P/I = R_{P_S}$, where P_S is the S-rank-selected subposet of P. Moreover, $Q_P/\bar{I} = Q_{P_S}$, where $\bar{I}$ denotes the image of I in Q_P. But since

$$F(Q_P,t) = \sum_{T\subseteq[d]} \beta_P(T) \prod_{i\in T} t_i \quad ,$$

there follows

$$F(Q_P/\bar{I},t) = \sum_{T\subseteq S} \beta_P(T) \prod_{i\in T} t_i \quad .$$

Hence R_{P_S} is Cohen-Macaulay, and we have proved the result alluded to in Section 3.

5.2 THEOREM: *If* P *is Cohen-Macaulay of rank* $d+1$, *then so is* P_S *for all* $S \subseteq [d]$.

While it is possible to give a purely topological proof of this result, the details are rather messy [Mu][Ba4]. The ring-theoretic approach yields the above almost trivial proof, which first appeared in [St8].

6. FURTHER DEVELOPMENTS

The main purpose of preceding sections was to introduce the reader to the notion of a Cohen-Macaulay poset starting from scratch. In this section we will briefly mention some areas in which there is current research activity. For a fuller treatment see our exposition in [BGS].

(a) *Applications to ring theory.* An elaborate theory now exists showing how properties of a poset ring R_P can be "transferred" to other rings of interest in invariant theory, representation theory and algebraic geometry. The idea to put the poset ring to such use seems to be due independently to DeConcini and Garsia. Thus DeConcini and collaborators [DEP2], [DL], [DP], [Ei] have developed the notion of "Hodge algebras," while Garsia's ideas have been further developed by him and Baclawski [Ga1], [Ga2], [BG], [Ba5], [GS] leading to the notion of "lexicographic rings." Preambles to these developments appear in the work of Hodge (see below) and of Rota and his colleagues [DRS], [DKR] on "straightening laws." This latter work in turn has its roots in the work of Alfred Young.

The interest to algebraic geometry in these developments derives from the fact that certain rings related to algebraic varieties, in particular the homogeneous coordinate rings of certain projective varieties, are closely related to the poset rings of Bruhat order and other related posets. The archetypal example, due to Hodge [Hod], [HP, p. 378], concerns the Grassmann variety G_{rn} of r-planes in $\mathbb{C}^n$. Recent work of DeConcini, Lakshmibai, Musili and Seshadri [DL], [LMS] extends the range of the method to wide classes of "generalized Schubert varieties." It is a noteworthy development that the Cohen-Macaulayness of the homogeneous coordinate rings of these varieties is deduced from the Cohen-Macaulayness of the corresponding Bruhat poset rings, and hence is made to ultimately depend on the pure combinatorics of lexicographic shellability.

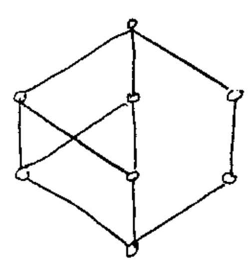
(b) *Group actions.* Let G be a finite group of automorphisms of a Cohen-Macaulay poset P of rank d+1. For each $S \subseteq [d]$, G permutes the chains of rank set S. Call the character of this permutation representation $\underset{\sim}{\alpha}_S$. Thus, $\underset{\sim}{\alpha}_S(g)$ is the number of chains of rank set S fixed by $g \in G$. Also, the action of G on P_S induces an action on the homology $\tilde{H}_{|S|-1}(\bar{P}_S, \mathbb{C})$. Call the character of this complex representation $\underset{\sim}{\beta}_S$. The following character relations can be deduced from the Hopf trace formula:

$$\begin{aligned} \underset{\sim}{\alpha}_S(g) &= \sum_{T \subseteq S} \underset{\sim}{\beta}_T(g) \\ \underset{\sim}{\beta}_S(g) &= \sum_{T \subseteq S} (-1)^{|S-T|} \underset{\sim}{\alpha}_T(g). \end{aligned} \tag{23}$$

Note that these formulas evaluated at the identity $e \in G$ reduce to the fundamental relationship (1)-(2) between the numbers $\alpha(S)$ and $\beta(S)$ mentioned in the Introduction. The formulas (23) were anticipated by Solomon [So_1, p. 389] and first proved by Stanley [St_9].

The typical example of the situation described above is that of the symmetric group S_n acting on the Boolean algebra B_n of all subsets of an n-element set. Here the characters $\underset{\sim}{\alpha}_S$ were first considered by Frobenius and $\underset{\sim}{\beta}_S$ by Solomon [So_2, § 6]. There is an explicit formula for decomposing $\underset{\sim}{\beta}_S$ into irreducible characters, which is due to Solomon [So_2] (see also [St_9, Thm. 4.37]). Another example is that of the general linear group $GL_{d+1}(q)$ acting on the lattice of subspaces of a $(d+1)$-dimensional vector space over $GF(q)$. Here the characters $\underset{\sim}{\alpha}_S$ were first considered by Steinberg [Ste], and $\underset{\sim}{\beta}_{[d]}$ is now known as the Steinberg character. For other examples, see [St_9] and [Bj_4].

(c) *Alternating Möbius function.* Let us call a graded poset P of rank d+1 *alternating* if $\beta(S) = (-1)^{|S|-1}\mu(P_S) \geq 0$ for all $S \subseteq [d]$. Cohen-Macaulay posets are alternating (Theorem 3.3), but the converse is not necessarily true, as the following example shows.



(24)

A simple condition which assures that a graded poset P is alternating is the following: P is said to be *ER* (also called *partitionable*) if for each maximal chain m there is a subchain $R(m) \subseteq m$, the *restriction* of m, such that $C = \biguplus_{m \in M} [R(m), m]$. This condition says that for each chain c there is exactly one maximal chain m such that $R(m) \subseteq c \subseteq m$. The poset above is ER. Also, all lexicographically shellable posets are ER; for $R(m)$ simply take the subchain of m where the descents in the label occur (cf. Corollary 4.5). It is not known whether all Cohen-Macaulay posets are ER.

The nonnegativity of the numbers $\beta(S)$ for a Cohen-Macaulay poset has several interpretations. Thus, we have seen that $\beta(S)$ occur as Betti numbers for the homology of rank-selected subposets and determine the Hilbert series of the poset ring. The following purely combinatorial interpretation appears in [St_8]: If P is a Cohen-Macaulay poset of rank $d+1$ then there exists a subcomplex Λ of the order complex $\Delta(\bar{P})$ such that for any $S \subseteq [d]$ the number of faces of Λ having rank set S is equal to $\beta(S)$. It follows that if $\beta(S) \neq 0$ and $T \subseteq S$, then $\beta(T) \neq 0$. The proof is based on manipulations with the poset ring. This result can be viewed as giving a necessary condition that a collection of integers $\beta(S)$, $S \subseteq [d]$, are the numbers $\beta(S)$ of a Cohen-Macaulay poset. The condition is not quite sufficient for posets, but if one allows slightly more general objects (so called "Cohen-Macaulay completely balanced complexes") then the condition is both necessary and sufficient [BS].

(d) *Further topological developments.* The topological definition 3.1 of a Cohen-Macaulay poset imposes a condition on the homology of each open interval. By sharpening the attention to homotopy type we get a related notion: A graded poset P is said to be *homotopy Cohen-Macaulay* if for every open interval (x,y) in P the homotopy groups $\pi_i(|\Delta((x,y))|)$, $i < \delta(x,y)$, are trivial. Thus P is homotopy CM if and only if P is CM/$\mathbb{Z}$ and when $\delta(x,y) \geq 2$, $|\Delta((x,y))|$ is simply connected. This notion was introduced by Quillen [Qu] for the purpose of studying certain posets of subgroups of a group G. For instance, he showed that if $G = GL_n(k)$, where k is a field, and if $p \neq \text{char}\, k$ and k has a primitive p-th root of unity, then the poset of elementary Abelian p-subgroups of G is homotopy Cohen-Macaulay. For the full subgroup lattice there is the following characterization which follows from results of Björner, Iwasawa and Stanley [Bj_1, p. 167]: The subgroup lattice L(G) of a finite group G is homotopy Cohen-Macaulay if and only if G is a supersolvable group. A lexicographically shellable poset is homotopy CM and a homotopy CM poset is CM/$\mathbb{Z}$, but neither of the converse implications are true. For instance, the poset of figure (19) is homotopy CM but not lexicographically shellable.

For a variety of topological developments related to the Cohen-Macaulay property of posets we refer the reader to [Ba_{1-4}] [$Bj_{2,3}$], [BWr], [Fa], [Mu], [Qu] and [Wr].

(e) *Further ring-theoretical developments:* With every finite poset P one can associate its poset ring R_P, and as a very general program one might ask in what way various ring-theoretic properties of R_P influence the structure of P. In this paper we have been concerned with merely one instance of this, viz., the Cohen-Macaulay property. Even for the Cohen-Macaulay case one can seek to probe much further into the ring-theoretic structure, asking for explicit minimal free resolutions, description of the canonical modules, etc. ... . Let us here merely examplify by mentioning Gorenstein posets. A poset ring R_P (or poset P) is said to be *Gorenstein* (with respect to k) if the ideal generated by the rank-level parameters is *irreducible*, i.e., cannot be expressed as an intersection of two strictly larger ideals. When is a graded poset P Gorenstein? There is a characterization of this property in terms of the homology of open intervals of P which is analogous to Reisner's theorem, [Hoc_2, § 6], [St_6, § 8]. This can be formulated as follows: P is Gorenstein if and only if P is CM and the Möbius function of nuc P satisfies $\mu(x,y) = (-1)^{r(x,y)}$. Here "nuc P" denotes P minus those elements $\neq \hat{0},\hat{1}$ which are comparable to all $x \in P$. Somewhat surprisingly, this condition for Gorensteinness is equivalent to the apparently weaker condition that P is CM and the Möbius function of nuc P satisfies $\mu(x,y) = 1$ only for intervals $[x,y]$ of length 2, together with the additional requirement that $\mu(\hat{0},\hat{1}) \neq 0$.

Consider, for instance, the two distributive lattices

(25)

The first one is Gorenstein but the second is not (the open circles denote elements of nuc P). Nevertheless, the order complexes of their proper parts are homeomorphic. On the other hand, if $\bar{P}$ and $\bar{Q}$ are *nonacyclic* (i.e., their order homology is non-zero) and if $|\Delta(\bar{P})|$ and $|\Delta(\bar{Q})|$ are homeomorphic, then P is Gorenstein if and only if Q is Gorenstein. It follows from the above characterization of Gorenstein posets that Boolean algebras, (full) Bruhat order, and face lattices of convex polytopes (cf. Section 2) are Gorenstein.

The poset ring R_P of a Gorenstein poset P satisfies a certain "self-duality" property. While we cannot enter into the details here, we can mention one simple combinatorial manifestation of this fact. Suppose P is Gorenstein of rank $d+1$ and that $P = \mathrm{nuc}\, P$.

If S is any subset of [d], then $\beta(S) = \beta([d]-S)$. More generally [Sta_9, Prop. 2.2], if Q is any (induced) subposet of $\bar{P}$ then $\hat{\mu}(\hat{Q}) = (-1)^d \mu(P-Q)$. This result extends to the situation of group actions as follows [Sta_9, Thm. 2.4]. Let P be as above, and let $G \subset \mathrm{Aut}\,P$. Then the β characters of Section 6b satisfy $\underset{\sim}{\beta}_S = \underset{\sim}{\beta}_{[d]}\underset{\sim}{\beta}_{[d]-S}$.

Further ring-theoretic aspects of CM posets are studied in [$Ba_{5,6}$], [BG], [$DEP_{1,2}$], [$Ga_{1,2}$], [Ho_2], and [$St_{6,8,10}$].

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DIAGRAM OF IMPLICATIONS

The relationships among most of the poset properties discussed in this paper are summarized by the diagram below. While we have included all implications of which we are aware, we have not indicated which reverse implications are open or are known to be false. For instance, it is unknown whether CL-shellable implies EL-shellable, and whether shellable implies CL-shellable. On the other hand, it follows from a recent deep result of R. Edwards that spheres (~~i.e., the poset of faces of a triangulation of a sphere, with a $\hat{1}$ adjoined~~) need not be homotopy CM.

i.e., a poset P with $\hat{0}$ and $\hat{1}$ such that $|\Delta(P-\{\hat{0},\hat{1}\})|$ is a sphere.