Annals of Discrete Mathematics 6 (1980) 333–342

DECOMPOSITIONS OF RATIONAL CONVEX POLYTOPES

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1. Decompositions

Let $\mathcal{P}$ be a d-dimensional convex polytope (or d-polytope) in $\mathbb{R}^m$ whose vertices have rational coordinates. We call $\mathcal{P}$ a *rational* convex polytope. Let $\partial\mathcal{P}$ denote the boundary of $\mathcal{P}$. If n is a positive integer, define $i(\mathcal{P}, n)$ (resp. $j(\mathcal{P}, n)$) to be the number of points $\alpha \in \mathcal{P}$ (resp. $\alpha \in \mathcal{P} - \partial\mathcal{P}$) such that $n\alpha \in \mathbb{Z}^m$. These functions have been studied by Ehrhart[1] [3], Macdonald [8, 9], McMullen [10, 11], and others. Here we will develop some new properties of $j(\mathcal{P}, n)$ and $i(\mathcal{P}, n)$. Proofs for the most part will be omitted.

A word on notation: $\mathbb{N}$ denotes the non-negative integers, $\mathbb{P}$ the positive integers, $[n]$ the set $\{1, 2, \ldots, n\}$ where $n \in \mathbb{N}$. $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ have their usual meanings. If $\alpha \in \mathbb{Q}^m$, then $l(\alpha)$ denotes the least positive integer t for which $t\alpha \in \mathbb{Z}^m$. In particular, if $\alpha \in \mathbb{Q}$, then $l(\alpha)$ is the denominator of α when written in lowest terms.

It will be more convenient to work not with $j(\mathcal{P}, n)$ and $i(\mathcal{P}, n)$ themselves, but with the formal power series

$$J(\mathcal{P}, \lambda) = 1 + \sum_{n=1}^{\infty} i(\mathcal{P}, n)\lambda^n,$$

$$\bar{J}(\mathcal{P}, \lambda) = \sum_{n=1}^{\infty} j(\mathcal{P}, n)\lambda^n.$$

Let $V = V(\mathcal{P})$ denote the set of vertices of $\mathcal{P}$, and let $\omega = (\alpha_1, \ldots, \alpha_\nu)$ be a linear ordering of the elements of V. For each non-void face F of $\mathcal{P}$, define $\delta(F) = \delta_\omega(F)$ to be that vertex α_i of F for which i is minimal. For instance, $\delta(\mathcal{P}) = \alpha_1$. Now let $\Phi = (F_0, F_1, \ldots, F_d)$ be a *flag* of faces of $\mathcal{P}$, i.e., F_i is an i-dimensional face (or i-face) of $\mathcal{P}$ and $F_0 \subset F_1 \subset \cdots \subset F_d = \mathcal{P}$. Call Φ a *full flag* if $\delta(F_i)$ is not a vertex of F_{i-1} for $1 \leq i \leq d$. If Φ is a full flag, then define $\Delta(\Phi)$ to be the d-simplex with vertices $F_0, \delta(F_1), \delta(F_2), \ldots, \delta(F_d)$.

* Partially supported by NSF Grant MCS 7701947.

[1] Our only reference to the work of Ehrhart will be [3], which is an exposition of his work over a period of many years. Further references may be found in [3].

Lemma 1.1. *The set of simplices $\Delta(\Phi)$, as Φ ranges over all full flags of $\mathcal{P}$, form the maximal faces of a triangulation Γ_ω of $\mathcal{P}$.*

An elementary Euler characteristic argument, basically that given in [9, Section 1], expresses $J(\mathcal{P}, \lambda)$ and $\bar{J}(\mathcal{P}, \lambda)$ in terms of simpler generating functions, as follows.

Proposition 1.2. *Let ω be an ordering of $V(\mathcal{P})$ with corresponding triangulation $\Gamma = \Gamma_\omega$. Let $\partial\Gamma$ denote the set of those faces $\mathcal{S} \in \Gamma$ which lie on $\partial\mathcal{P}$. Then*

$$J(\mathcal{P}, \lambda) = \sum_{\mathcal{S} \in \Gamma - \partial\Gamma} (-1)^{d - \dim \mathcal{S}} J(\mathcal{S}, \lambda) = \sum_{\mathcal{S} \in \Gamma} \bar{J}(\mathcal{S}, \lambda),$$

$$\bar{J}(\mathcal{P}, \lambda) = \sum_{\mathcal{S} \in \Gamma} (-1)^{d - \dim \mathcal{S}} \bar{J}(\mathcal{S}, \lambda) = \sum_{\mathcal{S} \in \Gamma - \partial\Gamma} J(\mathcal{S}, \lambda).$$

In order to obtain explicit expressions for $J(\mathcal{P}, \lambda)$ and $\bar{J}(\mathcal{P}, \lambda)$, it remains to evaluate $J(\mathcal{S}, \lambda)$ when $\mathcal{S}$ is a simplex. This is a result of Ehrhart, briefly discussed in [3, p. 54] (see also [8, Section 1]). Suppose that $\mathcal{S}$ is a k-simplex in $\mathbb{R}^m$ with rational vertices $\beta_0, \ldots, \beta_k$. Let $G \subset \mathbb{Z}^{m+1}$ be the abelian group generated by the set

$$S = \{(\beta, r) : \beta \in \mathbb{Z}^m,\ r \in \mathbb{P}, \text{ and } \beta/r \in \mathcal{S}\}.$$

Let H be the subgroup of G generated by the $k+1$ vectors $\gamma_i = (l(\beta_i)\beta_i, l(\beta_i))$. Then H is a subgroup of G of finite index t; in fact, t is equal to the greatest common divisor of all $(k+1) \times (k+1)$ minors of the matrix whose rows are the vectors γ_i. Now define the "half-open" parallelopipeds

$$\Delta_H = \{a_1\gamma_1 + \cdots + a_{k+1}\gamma_{k+1} : 0 \leqslant a_i < 1\},$$

$$\bar{\Delta}_H = \{a_1\gamma_1 + \cdots + a_{k+1}\gamma_{k+1} : 0 < a_i \leqslant 1\}.$$

The sets $\Delta_H \cap \mathbb{Z}^{m+1}$ and $\bar{\Delta}_H \cap \mathbb{Z}^{m+1}$ are each a set of coset representatives for H in G. Let

$$\Delta_H \cap \mathbb{Z}^{m+1} = \{(\varepsilon_i, r_i) : 0 \leqslant i < t\}, \qquad \bar{\Delta}_H \cap \mathbb{Z}^{m+1} = \{(\zeta_i, s_i) : 0 \leqslant i < t\}.$$

For instance, if we choose (ε_0, r_0) and (ζ_0, r_0) to be in H, then we have $(\varepsilon_0, r_0) = 0$ and $(\zeta_0, s_0) = \gamma_1 + \cdots + \gamma_{k+1}$.

Theorem 1.3. *With the above notation, we have*

$$J(\mathcal{S}, \lambda) = \left(\sum_{i=0}^{t-1} \lambda^{r_i}\right) \prod_{j=0}^{k} (1 - \lambda^{l(\beta_j)})^{-1},$$

$$\bar{J}(\mathcal{S}, \lambda) = \left(\sum_{i=0}^{t-1} \lambda^{s_i}\right) \prod_{j=0}^{k} (1 - \lambda^{l(\beta_j)})^{-1}.$$

Moreover, the s_i's *are just the numbers* $\sum l(\beta_j)-r_i$ *in some order, so that*

$$\bar{J}(\mathcal{S}, \lambda)=(-1)^{k+1}J(\mathcal{S}, 1/\lambda).$$

Example 1.4. Let $\mathcal{S}$ have vertices $(0,0)$, $(1,\frac{1}{2})$, $(1,\frac{3}{2})$. Then $G=\mathbb{Z}^3$, $\gamma_1=(0,0,1)$, $\gamma_2=(2,1,2)$, $\gamma_3=(2,3,2)$, $t=4$, $(\varepsilon_0, r_0)=(0,0,0)$, $(\varepsilon_1, r_1)=(1,1,1)$, $(\varepsilon_2, r_2)=(2,2,2)$, $(\varepsilon_3, r_3)=(3,3,3)$, $(\zeta_0, s_0)=(4,4,5)$, $(\zeta_1, s_1)=(1,1,2)$, $(\zeta_2, s_2)=(2,2,3)$, $(\zeta_3, s_3)=(3,3,4)$, $J(\mathcal{S},\lambda)=(1+\lambda+\lambda^2+\lambda^3)/(1-\lambda)(1-\lambda^2)^2$, $\bar{J}(\mathcal{S},\lambda)=(\lambda^2+\lambda^3+\lambda^4+\lambda^5)/(1-\lambda)(1-\lambda^2)^2$.

Remark. Following Ehrhart [3, p. 47], we say that the rational convex d-polytope $\mathcal{P}$ in $\mathbb{R}^m$ is *reticular* if the lattice $\mathbb{Z}^m \cap A$ of integer points of the affine space A spanned by $\mathcal{P}$ has rank d, i.e., $\mathbb{Z}^m \cap A \cong \mathbb{Z}^d$. This is equivalent to the statement that $\mathbb{Z}^m \cap A \neq \emptyset$ (see [3, Proposition 25]). If $\mathcal{P}$ is reticular, then let $\phi: A \to \mathbb{R}^d$ be an affine transformation which is a bijection between $A \cap \mathbb{Z}^m$ and $\mathbb{Z}^d$. (Such a ϕ clearly exists.) The image $\phi(\mathcal{P})$ of $\mathcal{P}$ is a d-polytope in $\mathbb{R}^d$, and hence has a positive Euclidean volume $\psi(\mathcal{P})$, called the *relative volume* of $\mathcal{P}$. It is easy to see that $\psi(\mathcal{P})$ depends only on $\mathcal{P}$, not on ϕ. If $d=m$, then $\psi(\mathcal{P})$ is just the volume of $\mathcal{P}$. As an example, let $\mathcal{P}$ be the line segement connecting $(0,0)$ to $(1,1)$. Define $\phi(a,a)=a$. Then $\phi(\mathcal{P})=[0,1]$, so $\psi(\mathcal{P})=1$. Now let $\mathcal{S}$ be a k-simplex as in Theorem 1.3, and assume that $\mathcal{S}$ is reticular. It is then easily seen that the integer $t=[G:H]$ appearing in Theorem 1.3 is given by

$$t=k!\,l(\beta_0)l(\beta_1)\cdots l(\beta_k)\psi(\mathcal{S}).$$

It then follows from Proposition 1.2 that the Laurent expansion of $J(\mathcal{P},\lambda)$ about $\lambda=1$ begins

$$J(\mathcal{P},\lambda)=d!\,\psi(\mathcal{P})(1-\lambda)^{-d-1}+\cdots. \tag{1}$$

It is also easy to give a direct proof of (1).

Combining Proposition 1.2 and Theorem 1.3, we obtain an explicit expression for $J(\mathcal{P},\lambda)$ and $\bar{J}(\mathcal{P},\lambda)$, showing that they are rational functions of λ with denominator

$$\prod_{\beta\in V(\mathcal{P})}(1-\lambda^{l(\beta)}),$$

a result due originally to Ehrhart [3, p. 53]. An immediate corollary of Proposition 1.2 and Theorem 1.3 is the following "reciprocity theorem", due essentially to Ehrhart [3, p. 30] and Macdonald [9, Theorem 4.6] (see also [14]).

Theorem 1.5. *As rational functions we have*

$$\bar{J}(\mathcal{P},\lambda)=(-1)^d J(\mathcal{P}, 1/\lambda).$$

Because of Theorem 1.5, we will now confine our attention to $J(\mathcal{P},\lambda)$. While we now have an explicit expression for this generating function, we would like a more informative expression. Toward this end, suppose that the maximal faces of the triangulation Γ_ω of $\mathcal{P}$ can be ordered $G_1, G_2, \ldots, G_s$ so that for each $2 \leq i \leq s$, the set $G_i \cap (G_1 \cup G_2 \cup \cdots \cup G_{i-1})$ is a union of facets $(=(d-1)$-faces) of G_i. Then $(G_1, G_2, \ldots, G_s)$ is called a *shelling* of Γ_ω, and Γ_ω is called *shellable.*

Theorem 1.6. *Let ω be an ordering of $V(\mathcal{P})$, and suppose that Γ_ω has a shelling $(G_1, G_2, \ldots, G_s)$. Then*

$$J(\mathcal{P}, \lambda) = \sum_{i=1}^{s} \left(\sum_{j=0}^{t_i-1} \lambda^{r_{ij}+q_{ij}} \right) \prod_{k=0}^{d} (1 - \lambda^{l(\beta_{ik})})^{-1}, \tag{2}$$

where $\beta_{i0}, \beta_{i1}, \ldots, \beta_{id}$ are the vertices of G_i, where

$$J(G_i, \lambda) = \left(\sum_{j=0}^{t_i-1} \lambda^{r_{ij}} \right) \prod_{k=0}^{d} (1 - \lambda^{l(\beta_{ik})})^{-1}$$

(as in Theorem 1.3), and where q_{ij} is the sum of certain of the numbers $l(\beta_{ik})$, $0 \leq k \leq d$. The precise definition of q_{ij} is not important here, but we note that

$$q_{i0} = \sum_{\beta \in V(F)} l(\beta)$$

where F is the unique minimal face of G_i whose interior is disjoint from $G_i \cap (G_1 \cup G_2 \cup \cdots \cup G_{i-1})$. In particular, $q_{10} = 0$.

In order for Theorem 1.6 to be of any value, we must show that the triangulation Γ_ω of $\mathcal{P}$ is shellable. It is easy to see that the triangulation Γ_ω has the following alternative description. Let $\omega = (\alpha_1, \ldots, \alpha_\nu)$. Pull the vertices of $\mathcal{P}$ (as defined in [6, p. 80] or [12, pp. 116–117]) in the order $\alpha_1, \ldots, \alpha_\nu$. This yields a triangulation Λ of $\partial\mathcal{P}$. Then the maximal faces of Γ_ω are those of the form $\{\alpha_1\} \cup F$, where F is a maximal face of Λ not containing α_1. In particular, $\partial\Gamma_\omega = \Lambda$.

Using [1] and the above description of Γ_ω, it is easy to deduce the following result.

Theorem 1.7. *If ω is an ordering of $V(\mathcal{P})$, then Γ_ω is shellable. Hence $J(\mathcal{P}, \lambda)$ can always be written in the form* (2).

2. Consequences

We now use the results of Section 1 to examine the generating function $J(\mathcal{P}, \lambda)$ in more detail. To save space we will not state our results in the fullest possible generality.

Theorem 2.1. *Suppose every vertex of $\mathcal{P}$ has integer coordinates. Then*

$$J(\mathcal{P}, \lambda) = \frac{W(\mathcal{P}, \lambda)}{(1-\lambda)^{d+1}}, \tag{3}$$

where $W(\mathcal{P}, \lambda)$ is a polynomial of degree at most d with non-negative integer coefficients.

Proof. Each $l(\beta_{ik}) = 1$ in (2), from which the proof follows.

Theorem 2.1 implies that $i(\mathcal{P}, n)$ is a polynomial in n of degree d, as was shown by Ehrhart [3, p. 50] (repeated in [8]) and McMullen [10, 11]. What these persons had not shown was that the coefficients of $W(\mathcal{P}, \lambda)$ are non-negative. This fact was first established in [16, Proposition 4.5] by algebraic techniques (Cohen–Macaulay rings), but now we have a more geometric proof. We remark that it follows from (1) that the leading coefficient of $i(\mathcal{P}, n)$ is $\psi(\mathcal{P})$. For further ramifications of this fact, see [3, 8, 14, pp. 209–211].

It is natural to ask whether the polynomial $W(\mathcal{P}, \lambda)$ has a more combinatorial interpretation than afforded by (2). In certain cases we can indeed say more.

Definition 2.2. Let $\mathcal{P}$ be a d-polytope in $\mathbb{R}^m$ with integer vertices, and let ω be an ordering of $V(\mathcal{P})$. We say that ω is *compressed* if the following property holds: If $\mathcal{S}$ is any face (equivalently, any maximal face) of the triangulation Γ_ω, then the relative volume $\psi(\mathcal{S})$ of $\mathcal{S}$ is equal to $1/(\dim \mathcal{S})!$. We say that $\mathcal{P}$ itself is *compressed* if every ordering ω is compressed.

Theorem 2.3. *Let $\mathcal{P}$ be a d-polytope in $\mathbb{R}^m$ with integer vertices. Let ω be an ordering of $V(\mathcal{P})$, and let $\delta = \delta_\omega$ be defined as in Section 1. Suppose that for any rational point α in the relative interior of some face F of $\mathcal{P}$, if c is the unique positive number (necessarily rational) such that $(\alpha - c\delta(F))/(1-c)$ lies on ∂F, then $l(\alpha)c \in \mathbb{Z}$. Then ω is compressed.*

Example 2.4. (a) Suppose that one of the vertices of $\mathcal{P}$ is the origin O, and that the matrix whose rows are the vertices of $\mathcal{P}$ is totally unimodular. Let $\omega = (\alpha_1, \ldots, \alpha_\nu)$ be any ordering of $V(\mathcal{P})$ for which $\alpha_1 = \mathrm{O}$. Then ω is compressed. (The proof is omitted.) However, other orderings ω need not be compressed. For instance, if

$$V(\mathcal{P}) = \{(0,0,0), (0,0,1), (0,1,0), (1,0,0), (0,1,1), (1,1,0), (1,1,1)\},$$

then any ordering ω with $\alpha_1 = (0,1,0)$ is not compressed.

(b) Let Ω_n be the convex polytope of all $n \times n$ doubly stochastic matrices. Then Ω_n is compressed. For let M be a rational matrix in Ω_n. Then $l(M)M$ is a sum of permutation matrices. Let k be the greatest integer for which $l(M)M - k\delta(F)$ has non-negative entries. Then $l(M)c = k \in \mathbb{Z}$.

(c) Let $\mathcal{P}$ be the 3-simplex with vertices $(0,0,0)$, $(1,1,0)$, $(1,0,1)$, $(0,1,1)$. Then *no* ordering ω is compressed.

Corollary 2.5. *Let $\mathcal{P}$ be as in Definition 2.2, and let ω be a compressed ordering of*

$V(\mathcal{P})$. Let f_i be the number of i-faces of Γ_ω. Then

$$i(\mathcal{P}, n) = \sum_{i=0}^{d} f_i \binom{n-1}{i}. \tag{4}$$

It follows that if $W(\mathcal{P}, \lambda) = h_0 + h_1\lambda + \cdots + h_d\lambda^d$ (as defined in (3)), then $(h_0, h_1, \ldots, h_d)$ is the h-vector of Γ_ω, as defined in [15, *p.* 136].

Note that it follows from the definition of Γ_ω that the numbers f_i in the above corollary depend only on ω and the lattice $L(\mathcal{P})$ of faces of $\mathcal{P}$ (even if ω is not compressed). Hence, if ω is compressed we can compute $i(\mathcal{P}, n)$ from ω and $L(\mathcal{P})$ alone. In particular, if $\mathcal{P}$ is compressed then $L(\mathcal{P})$ determines $i(\mathcal{P}, n)$.

Corollary 2.6. *Preserve the notation of Corollary 2.5. Let $(G_1, G_2, \ldots, G_s)$ be a shelling of Γ_ω. Then h_i is equal to the number of integers $j \in [r]$ for which $G_j \cap (G_1 \cup \cdots \cup G_{j-1})$ is a union of i facets of G_j.*

Corollary 2.6 provides a kind of combinatorial interpretation to the coefficients of $W(\mathcal{P}, \lambda)$ when $\mathcal{P}$ possesses a compressed ordering ω, since by Theorem 1.6 the triangulation Γ_ω is always shellable.

Corollary 2.7. *If $\mathcal{P}$ is compressed and ω is an ordering of $V(\mathcal{P})$, then the f-vector $(f_0, f_1, \ldots, f_d)$ of Γ_ω (with f_i as in Corollary 2.5) depends only on $\mathcal{P}$, not on ω.*

Proof. Immediate from (4), since $i(\mathcal{P}, n)$ has a unique representation in the form (4).

Corollary 2.7 is a purely combinatorial statement (i.e., involving only the structure of the lattice of faces) about $\mathcal{P}$. Thus for instance from Example 2.4(b) we get restrictions on the facial structure of the polytopes Ω_n. These polytopes have been studied by various authors (see, e.g., [5] and the references given there), and it may be interesting to see how Corollary 2.7 ties in with previously known results.

Remark. Triangulations Γ of rational polytopes $\mathcal{P}$ whose simplices have small volume have been considered elsewhere, such as [4] and [7]. (In the latter reference, see especially Theorem 12* on p. 95 and Chapter III. It would be interesting to interpret the theory developed in [7] in a purely combinatorial manner.) These writers, however, do not consider the restriction that the vertices of Γ must coincide with the vertices of $\mathcal{P}$. Such restrictions have apparently been considered in some unpublished work of Lovasz, but this work is not available to this author.

Let us now briefly consider polytopes $\mathcal{P}$ whose vertices don't necessarily have integer coordinates, but only rational coordinates. Let M be the least common multiple of the numbers $l(\alpha)$, $\alpha \in V(\mathcal{P})$. Let $\zeta = e^{2\pi i/M}$. It follows from Proposition 1.2 and Theorem 1.3 (or from the stronger Theorems 1.6 and 1.7) that there are polynomials $P_0, P_1, \ldots, P_{M-1}$ such that

$$i(\mathcal{P}, n) = \sum_{k=0}^{M-1} \zeta^{kn} P_k(n).$$

Define

$$\gamma(\mathcal{P}) = \max_{1 \leq i \leq M-1} (1 + \deg P_i),$$

where we set $\deg 0 = -1$. The number $\gamma(\mathcal{P})$ measures how close $i(\mathcal{P}, n)$ is to being a polynomial. We have $\gamma(\mathcal{P}) = 0$ if and only if $i(\mathcal{P}, n)$ is a polynomial. (The number $\gamma(\mathcal{P})$ is called by Ehrhart [3, p. 12] the *grade* of $i(\mathcal{P}, n)$.) The next result establishes a conjecture of Ehrhart [3, p. 53].

Theorem 2.8. *If every j-face of $\mathcal{P}$ is reticular, then $\gamma(\mathcal{P}) \leq j$.*

Sketch of proof. Fix j. If $\mathcal{R}$ is a rational convex polytope such that every j-face of $\mathcal{R}$ is reticular, then define $\eta(\mathcal{R})$ to be the number of j-faces of R which have no integer vertices. The proof is by induction on $\eta(\mathcal{P})$. First suppose $\eta(\mathcal{P}) = 0$, i.e., every j-face has an integer vertex. Choose an ordering ω of $V(\mathcal{P})$ such that all the integer vertices come first. It follows easily from Theorem 1.5 that $\gamma(\mathcal{P}) \leq j$.

Now let $\eta(\mathcal{P}) > 0$ be arbitrary, and assume the theorem for all $\mathcal{R}$ with $\eta(\mathcal{R}) < \eta(\mathcal{P})$. Let F be a j-face of $\mathcal{P}$ with no integer vertex. Let β be an integer point on the affine span A of F but not on F. (Such a point exists since $\mathbb{Z}^m \cap A \cong \mathbb{Z}^j$. Let $\mathcal{Z}$ be the convex hull of $\mathcal{P}$ and β. Every j-face of $\mathcal{Z}$ is either a j-face of $\mathcal{P}$ or has β as a vertex. Moreover, F itself is a *proper* subset of a j-face of $\mathcal{Z}$. Hence $\eta(\mathcal{Z}) < \eta(\mathcal{P})$. By the induction hypothesis, $\gamma(\mathcal{Z}) \leq j$.

Now $\mathcal{Z}$ has an obvious cellular decomposition whose maximal cells consist of $\mathcal{P}$ together with the convex hull of β with each facet of $\mathcal{P}$ "visible" from β. Denoting these maximal cells different from $\mathcal{P}$ by $\mathcal{Z}_1, \ldots, \mathcal{Z}_t$, it is easily seen as above that $\eta(G) < \eta(\mathcal{P})$ for any face G of any $\mathcal{Z}_i$ (including $G = \mathcal{Z}_i$). Now an Euler characteristic argument (as in Proposition 1.2) expresses $J(\mathcal{P}, \lambda)$ as a linear combination of $J(\mathcal{Z}, \lambda)$ and the $J(G, \lambda)$'s where G is a face of some $\mathcal{Z}_i$. Since $\gamma(\mathcal{Z}) \leq j$ and each $\gamma(G) \leq j$, there follows $\gamma(\mathcal{P}) \leq j$, as was to be proved.

Remark. Theorem 2.8 does not determine $\gamma(\mathcal{P})$, in the sense that $\mathcal{P}$ may have a j-face which is not reticular, yet $\gamma(\mathcal{P}) = j$. For instance, let $\mathcal{P}$ be the convex 3-polytope in $\mathbb{R}^3$ with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(1,1,0)$, and $(\frac{1}{2}, 0, \frac{1}{2})$. Thus $\mathcal{P}$ has a 0-face (vertex) which is not reticular, yet $i(\mathcal{P}, n) = \binom{n+3}{3}$ so $\gamma(\mathcal{P}) = 0$.

Remark. Using Theorem 2.8, one can give non-algebraic proofs of some further results in [16], viz., Theorems 3.3 and 5.5.

3. Examples

For what classes of polytopes $\mathcal{P}$ can $i(\mathcal{P}, n)$ or $J(\mathcal{P}, \lambda)$ be explicitly computed? This interesting question has not been systematically investigated. Here we give two examples to show that further work may be worthwhile. We know of several other classes besides the two below, but for brevity's sake they will not be discussed.

Example 3.1. Let $\gamma_1, \ldots, \gamma_t \in \mathbb{Z}^m$. Let

$$\mathcal{P} = \mathcal{P}(\gamma_1, \ldots, \gamma_t) = \{a_1\gamma_1 + \cdots + a_t\gamma_t : 0 \leq a_i \leq 1\}.$$

Thus by definition $\mathcal{P}$ is a vector sum of line segments and therefore a *zonotope*. Using the techniques of [13, Section 5], it can be shown that

$$i(\mathcal{P}, n) = \sum_X f(X) n^{|X|}, \tag{5}$$

where X ranges over all linearly independent subsets of $\{\gamma_1, \ldots, \gamma_t\}$ and where $f(X)$ denotes the greatest common divisor of all minors of size $|X|$ of the matrix whose rows are the elements of X. For instance, if the matrix whose rows are $\gamma_1, \ldots, \gamma_t$ is *totally unimodular* (every minor equals 0 or ± 1), then the coefficient of n^i in $i(\mathcal{P}, n)$ is equal to the number of i-element linearly independent subsets of $\{\gamma_1, \ldots, \gamma_t\}$. In other words, these coefficients are the "independent set numbers" of the geometry or matroid (in the sense of [2] or [17]) determined by $\gamma_1, \ldots, \gamma_t$. An especially interesting case occurs when $\mathcal{P}$ is the convex hull of all $m!$ vectors $(b_1, \ldots, b_m)$ whose entries are the distinct integers $1, 2, \ldots, m$ in some order. (Although $\mathcal{P}$ is not of the form $\{a_1\gamma_1 + \cdots + a_t\gamma_t : 0 \leq a_i \leq 1\}$, there is an integral unimodular affine transformation which transforms $\mathcal{P}$ into such a polytope.) $\mathcal{P}$ is known as a *permutohedron*, and it can be deduced from (5) that

$$i(\mathcal{P}, n) = \sum_{i=0}^{m-1} g_i n^i$$

where g_i is the number of forests on m vertices with i edges. In particular, $g_{m-1} = m^{m-2}$, so the relative volume of $\mathcal{P}$ is m^{m-2}. On the other hand, Zaslavsky has shown (private communication) that $i(\mathcal{P}, 1)$ is equal to the number of distinct *ordered* score vectors $(s_1, s_2, \ldots, s_m)$ of an m-vertex tournament, i.e., if T is a tournament on the vertex set $\{v_1, \ldots, v_m\}$, then s_i is the outdegree of v_i. Hence the number of ordered score vectors of length m is equal to the number of forests on m vertices. More generally, it can be shown that if G is any (undirected) graph, then the number of distinct outdegree sequences of orientations of G is equal to the number of spanning forests of G. It would be interesting to prove this result by finding a simple one-to-one correspondence. (We can give a messy, inductive one-to-one correspondence.)

Example 3.2. Let $\mathcal{P}_d$ be the convex d-polytope in $\mathbb{R}^d$ of all points $(x_1, \ldots, x_d)$ satisfying

$$x_i \geq 0, \qquad 1 \leq i \leq d,$$

$$x_i + x_{i+1} \leq 1, \qquad 1 \leq i \leq d-1.$$

The following facts can be proved. The generating functions $J(\mathcal{P}_d, \lambda)$ (denoted $J_d(\lambda)$ for short) are determined by

$$J_0(\lambda) = 1/(1-\lambda), \qquad J_1(\lambda) = 1/(1-\lambda-\lambda^2),$$

$$J_d(\lambda) = 1/(2-\lambda^2 - J_{d-2}(\lambda)), \quad d \geq 2.$$

We have the explicit formulas

$$J_{2d}(\lambda) = \frac{(1-\lambda-\bar{\phi})\phi^{d-1} + (-1+\lambda+\phi)\bar{\phi}^{d-1}}{(1-\lambda-\bar{\phi})\phi^d + (-1+\lambda+\phi)\phi^{-d}},$$

$$J_{2d-1}(\lambda) = \frac{(1-\lambda-\lambda^2-\bar{\phi})\phi^{d-1} + (-1+\lambda+\lambda^2+\phi)\bar{\phi}^{d-1}}{(1-\lambda-\lambda^2-\bar{\phi})\phi^d + (-1+\lambda+\lambda^2+\phi)\bar{\phi}^d}$$

where

$$\phi = \tfrac{1}{2}(2-\lambda^2+\lambda(\lambda^2-4)^{\frac{1}{2}}), \qquad \bar{\phi} = \tfrac{1}{2}(2-\lambda^2-\lambda(\lambda^2-4)^{\frac{1}{2}}).$$

Moreover, the coefficient of λ^j in $W(\mathcal{P}_d, \lambda)$ is equal to the number of permutations $(b_1, b_2, \ldots, b_d)$ of $[d]$ satisfying:

(i) if $d = 2e$, then i and $i+1$ must precede $i+e$ for $1 \leq i \leq e-1$, while e must precede $2e$;

(ii) if $d = 2e+1$, then i and $i+1$ must precede $i+e+1$ for $1 \leq i \leq e$;

(iii) the number of integers $i \in [d-1]$ for which $b_i > b_{i+1}$ is equal to j.

From this one can show that the relative volume (in this case, the actual volume) $\psi_d = \psi(\mathcal{P}_d)$ is given by

$$\sum_{d=0}^{\infty} \psi_d x^d = \sec x + \tan x.$$

Note added in proof

(1) An independent proof of Theorem 2.8 appears in P. McMullen, Lattice invariant valuations on rational polytopes, Arch. Math. (Basel) 31 (1978/79) 509–516.

(2) The one-to-one correspondence asked for at the end of Example 3.1 has been found by D. Kleitman and K. Winston, Forest and score vectors, to appear.

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