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COMBINATORICS AND INVARIANT THEORY

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ABSTRACT. Let $\rho: G\to GL(V)$ be a representation of a compact Lie group on a finite-dimensional complex vector space V. The action of G extends to the polynomial ring $R = \mathbb{C}[V]$. If χ is an irreducible character of G, then let R_χ^G denote the module of χ-invariants of G over the ring R^G of absolute invariants. Combinatorial techniques are used to investigate the Molien series (or Poincaré series) of R_χ^G and conditions for R_χ^G to be Cohen-Macaulay.

1. PRELIMINARIES. Invariant theory is for the main part concerned with the following situation. Let G be a group, and let $\rho: G\to GL(V)$ be a representation of G as a group of linear transformations of a vector space V of dimension $m<\infty$ over a field k. When $g\in G$ and $v\in V$, we write $g\cdot v$ for $\rho(g)(v)$. Let $x_1,\ldots,x_m$ be a basis for V. Then G acts on the polynomial ring $R = k[x_1,\ldots,x_m]$ by $g\cdot f(x_1,\ldots,x_m) = f(g\cdot x_1,\ldots,g\cdot x_m)$. A polynomial f in R is an <u>invariant</u> of G (or more precisely, of the pair (G,ρ)) if $g\cdot f = f$ for all $g\in G$. The set of all invariants forms a subalgebra R^G of R called the <u>ring of invariants</u> of G. We are concerned with the problem of "determining" R^G, or at least saying as much as possible about the structure of R^G.

In order to say anything nice one must put additional restrictions on G and ρ. Here we will assume that $k = \mathbb{C}$, that G is a <u>compact Lie group</u>, and that ρ is continuous. (This latter assumption on ρ will be automatically assumed without further comment.) Under these assumptions, the representation of G on R breaks up (uniquely) into irreducible representations, so we have a vector space direct sum $R = \coprod Q_i$, where each Q_i is an irreducible G-invariant subspace. Each irreducible representation of G is determined by its character $\chi: G\to \mathbb{C}$. Let R_χ^G denote the direct sum of those Q_i which correspond to the irreducible character χ. Note that if χ is the trivial character ($\chi(g) = 1$ for all $g\in G$), then $R^G = R_\chi^G$. It follows from Schur's lemma that each R_χ^G

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is a module over the ring R^G, called the module of <u>invariants relative to χ</u> or the module of <u>χ-invariants</u>. Note that $R = \coprod_\chi R^G_\chi$, where χ ranges over all irreducible characters of G. Note also that if χ is a linear character of G (i.e., a homomorphism $G \to \mathbb{C}^* = \mathbb{C} - \{0\}$), then $R^G_\chi = \{f \in R\colon g \cdot f = \chi(g) f \text{ for all } g \in G\}$.

Let R_n denote the vector space of all polynomials in R (including 0) which are homogeneous of degree n. Then R becomes a <u>graded algebra</u>, i.e., $R = \coprod_{n \geq 0} R_n$, $R_i R_j \subset R_{i+j}$, $R_0 = \mathbb{C}$. Since R_n is clearly a G-invariant subspace, it follows that R^G has the structure of a graded algebra (viz., $R^G_n = R^G \cap R_n$) and that each R^G_χ has the structure of a <u>graded R^G module</u>, i.e., $R^G_\chi = \coprod_{n \geq 0} (R^G_\chi)_n$, $R^G_i (R^G_\chi)_j \subset (R^G_\chi)_{i+j}$. The <u>Molien series</u> (also called the Poincaré series, Hilbert series, or generating function) of R^G_χ is the formal power series

$$F_\chi(G,\lambda) = (\deg \chi) \sum_{n=0}^{\infty} c_\chi(n) \lambda^n$$

where $c_\chi(n)$ is the multiplicity of χ in the action of G on R_n. Thus $\dim_{\mathbb{C}} (R_\chi)_n = (\deg \chi) c_\chi(n)$. When χ is trivial we write $F(G,\lambda)$ for $F_\chi(G,\lambda)$. Note that

$$\sum_\chi F_\chi(G,\lambda) = (1-\lambda)^{-m} .$$

A theorem of Molien [9][3,§227] gives an expression for $F_\chi(G,\lambda)$ when G is finite. This result generalizes immediately to compact groups once the rudiments of the representation theory of such groups is known.

1.1 THEOREM. Let G be a compact Lie group acting on V, and let χ be an irreducible character of G. Then

$$F_\chi(G,\lambda) = (\deg \chi) \int_{g \in G} \frac{\bar{\chi}(g)\, dg}{\det(1-\lambda g)} , \tag{1}$$

where the integral is the Haar integral and the bar denotes complex conjugation.

For instance, when G is finite then (1) reduces to

$$F_\chi(G,\lambda) = \frac{\deg \chi}{|G|} \sum_{g \in G} \frac{\bar{\chi}(g)}{\det(1-\lambda g)} . \tag{2}$$

A fundamental result of invariant theory states that when G is compact, R^G is a finitely-generated $\mathbb{C}$-algebra (see [10] for a brief history of this problem). The same techniques can be used to show that R^G_χ is a finitely-generated R^G-module. It follows from a standard result of commutative algebra [2,Ch.11] that $F_\chi(G,\lambda)$ is a rational function of λ. The <u>Krull dimension</u> of R^G_χ, denoted $\dim R^G_\chi$, is defined to be the order of the pole of $F_\chi(G,\lambda)$ at $\lambda = 1$. It follows from a well-known property of Krull dimension that $\dim R_\chi^G = \dim(R^G/\mathrm{Ann}\, R_\chi^G)$, where $\mathrm{Ann}\, R_\chi^G = \{f\varepsilon R\colon fR_\chi^G = 0\}$. Clearly $\mathrm{Ann}\, R_\chi^G = 0$ if R_χ^G is non-void, so we have

$$\dim R_\chi^G = \dim R^G, \text{ if } R_\chi^G \neq \emptyset . \tag{3}$$

Let $d = \dim R_\chi^G$. It follows from the Noether normalization lemma that there exist homogeneous elements $\theta_1,\ldots,\theta_d \ \varepsilon\ R^G$, necessarily algebraically independent over $\mathbb{C}$, such that R_χ^G is a finitely-generated module over the polynomial ring $\mathbb{C}[\theta_1,\ldots,\theta_d]$. The polynomials $\theta_1,\ldots,\theta_d$ are called a <u>homogeneous system of parameters</u> (h.s.o.p.) for R_χ^G. From (3) it follows that if $R_\chi^G \neq \emptyset$, then $\theta_1,\ldots,\theta_d$ is an h.s.o.p. for R^G if and only if $\theta_1,\ldots,\theta_d$ is an h.s.o.p. for R_χ^G. A basic result of commutative algebra [11,p.IV-20, Thm.2] states that R_χ^G is a <u>free</u> $\mathbb{C}[\theta_1,\ldots,\theta_d]$-module for <u>some</u> h.s.o.p. $\theta_1,\ldots,\theta_d$ if and only if R_χ^G is a free $\mathbb{C}[\theta_1,\ldots,\theta_d]$-module for <u>every</u> h.s.o.p. $\theta_1,\ldots,\theta_d$. If R_χ^G is indeed a free $\mathbb{C}[\theta_1,\ldots,\theta_d]$-module, then R_χ^G is called a <u>Cohen-Macaulay module</u>. (If R^G itself is a free $\mathbb{C}[\theta_1,\ldots,\theta_d]$-module, then R^G is called a <u>Cohen-Macaulay ring</u>.) Suppose that R_χ^G is Cohen-Macaulay, that $\theta_1,\ldots,\theta_d$ is an h.s.o.p., and that $\eta_1,\ldots,\eta_t$ is a homogeneous basis for R_χ^G as a $\mathbb{C}[\theta_1,\ldots,\theta_d]$-module. This may be written symbolically as

$$R_\chi^G = \coprod_{i=1}^{t} \mathbb{C}[\theta_1,\ldots,\theta_d]\,\eta_i . \tag{4}$$

One reason that it is nice for (4) to hold is that every element f of R_χ^G can be put in a simple canonical form, viz., $f = \Sigma\, p_i(\theta_1,\ldots,\theta_d)\eta_i$, where $p_i \ \varepsilon\ \mathbb{C}[\theta_1,\ldots,\theta_d]$. Let $c_i = \deg \theta_i$, $e_i = \deg \eta_i$. A simple combinatorial argument shows that

$$F_\chi(G,\lambda) = \left[\sum_{1}^{t} \lambda^{e_i}\right] \Big/ \prod_{1}^{d} (1-\lambda^{c_j}) . \tag{5}$$

1.2 EXAMPLE. Let G be the group of order 2 generated by $g = \mathrm{diag}(-1,-1)$ (with respect to some basis x,y for V). Let χ be defined by $\chi(g) = -1$. Then we have

$$R^G = \mathbb{C}[x^2,y^2](1 \oplus xy)$$
$$R_\chi^G = \mathbb{C}[x^2,y^2](x \oplus y).$$

Hence R^G and R_χ^G are Cohen-Macaulay, and we have $F(G,\lambda)=(1+\lambda^2)/(1-\lambda^2)^2$, $F_\chi(G,\lambda)=2\lambda/(1-\lambda^2)^2$. Note that $F(G,\lambda)+F_\chi(G,\lambda)=1/(1-\lambda)^2$.

1.3 EXAMPLE. Let G be the one-dimensional torus $G = \{g(u) = \mathrm{diag}(u,u,u^{-1},u^{-1})\colon \ |u| = 1\}$. For $i \ \varepsilon\ \mathbb{Z}$, let χ_i be defined by $\chi_i(g(u)) = u^i$. Then it can be shown that $R_{\chi_i}^G$ is Cohen-Macaulay if and only if $i = -1, 0,$ or 1. The "if" part follows from Theorem 3.5 below.

It is not hard to compute that $F_{\chi_2}(G,\lambda) = F_{\chi_{-2}}(G,\lambda) = (3\lambda^2-\lambda^4)/(1-\lambda^2)^3$. There is no way to write this in the form (5) (e.g., the numerator will always have the positive root $\sqrt{3}$). Hence $R^G_{\chi_2}$ and $R^G_{\chi_{-2}}$ are not Cohen-Macaulay. We omit the proof that $R^G_{\chi_i}$ is not Cohen-Macaulay for $|i| \geq 3$.

Suppose that R^G is Cohen-Macaulay, so that it has a decomposition $R^G_\chi = \coprod_{i=1}^{t} \mathbb{C}[\theta_1,\ldots,\theta_d]\eta_i$. There is then a useful and important R^G-module $\Omega(R^G_\chi)$ associated with R^G_χ which is a kind of "dual" module. The simplest description of $\Omega(R^G_\chi)$ is the following. Let $B = \mathbb{C}[\theta_1,\ldots,\theta_d]$, and let $\Omega(R^G_\chi) = \mathrm{Hom}_B(R^G_\chi, B)$. This defines $\Omega(R^G_\chi)$ as a B-module. The R^G-module structure is given by $(f\phi)(g) = \phi(fg)$ where $f \in R^G$, $\phi \in \Omega(R^G_\chi)$, $g \in R^G_\chi$. It turns out that $\Omega(R^G_\chi)$, considered as an R^G-module, is independent of the choice of the h.s.o.p. $\theta_1,\ldots,\theta_d$. When χ is trivial so $R^G_\chi = R^G$, one calls $\Omega(R^G)$ the <u>canonical module</u> of R^G. See, e.g., [5][15,§7] for further information. A basic combinatorial property of $\Omega(R^G_\chi)$ which follows from the techniques of [14] is that $\Omega(R^G_\chi)$ has a natural grading such that its Poincaré series $\bar{F}_\chi(G,\lambda)$ is given by

$$\bar{F}_\chi(G,\lambda) = (-1)^d \lambda^q F_\chi(G,1/\lambda) \tag{6}$$

for some $q \in \mathbb{Z}$, where $d = \dim R^G_\chi$. If the module $\Omega(R^G)$ is a free R^G-module of rank one (i.e., isomorphic to R^G as an R^G-module), then R^G is called a <u>Gorenstein ring</u>. It follows from (6) that if R^G is Gorenstein then $F(G,1/\lambda) = (-1)^d\lambda^q F(G,\lambda)$ for some $q \in \mathbb{Z}$. It follows from [14, Thm. 4.4] and Theorem 4.1 below that the converse is true:

1.4 THEOREM. A necessary and sufficient condition for R^G to be Gorenstein is that $F(G,1/\lambda) = (-1)^d\lambda^q F(G,\lambda)$ for some $q \in \mathbb{Z}$.

2. FINITE GROUPS. We wish to describe two interesting properties of the modules R^G_χ when G is finite. When G is not finite these properties need not hold, and in Sections 3 and 4 we will discuss some combinatorial techniques for verifying these properties in special cases.

Consider the following two properties of the pair (G,χ), where G is a compact Lie group acting on V, and where χ is an irreducible character of G.

<u>Property 1.</u> The module R^G_χ is Cohen-Macaulay, and the "dual module" $\Omega(R^G_\chi)$ is isomorphic to R^G_ψ, where ψ is the character defined by $\psi(g) = \bar{\chi}(g)(\overline{\det}\, g)$, the bars denoting complex conjugation. (By det g, we mean the determinant of the action of g on V.)

<u>Property 2</u>. Let $d = \dim R^G_\chi$, $m = \dim_{\mathbb{C}} V$, and let ψ be as above. Then

$$F_\chi(G,1/\lambda) = (-1)^d\lambda^m F_\psi(G,\lambda).$$

Note that Property 2 yields an expression for the degree of $F_\chi(G,\lambda)$ (as a rational function), viz., $\deg F_\chi(\lambda) = -(m+\ell)$, where ℓ is the least degree of a ψ-invariant of G. This in turn implies that the largest e_i in (5) is equal to $c_1+\ldots+c_d-m-\ell$. For instance, if $\rho: G\to SL(V)$ then $\deg F(G,\lambda) = -m$ when Property 2 holds.

It follows from (6) and an examination of the way in which $\Omega(R_\chi^G)$ is graded that Property 1 implies that $F_\chi(G,1/\lambda) = (-1)^d\lambda^q F_\psi(G,\lambda)$ for <u>some</u> integer q. However, in general it need not follow that $q = m$ (see Example 3.7). When Property 2 holds, this yields strong evidence for Property 1.

For finite groups it is relatively easy to verify Properties 1 and 2.

2.1 THEOREM. If G is finite, then all the pairs (G,χ) satisfy Properties 1 and 2.

<u>Proof</u>. Hochster and Eagon [6,Prop.13] showed that R^G is Cohen-Macaulay. For a relatively self-contained proof, see [15,§3]. The same techniques show R_χ^G is Cohen-Macaulay. Alternatively, we can use Lemma 3.2 below, together with the finite sum decomposition $R = \coprod R_\chi^G$, to show that R_χ^G is Cohen-Macaulay. The computation of $\Omega(R_\chi^G)$ follows from the techniques of Watanabe [16] or by a direct argument shown to me by David Eisenbud. It remains to prove Property 2. In view of (2) this is a formal calculation. One immediately sees that

$$1/\det(1-\lambda^{-1}g) = (-1)^m\lambda^m(\det g^{-1})/\det(1-\lambda g^{-1}).$$

Summing on g^{-1} instead of g gives

$$F_\chi(G,1/\lambda) = (\deg\chi)(-1)^m\lambda^m \sum_{g\in G} \frac{\chi(g)(\det g)}{\det(1-\lambda g)} = (-1)^m\lambda^m F_\psi(G,\lambda).$$

Since it is easy to see that $\dim R_G^\chi = m$ or $F_\chi(\lambda) = F_\psi(\lambda) = 0$, and since $\deg\chi = \deg\psi$, the proof is complete.

3. TORUS GROUPS. We now turn to the case where G is an s-dimensional torus, i.e., isomorphic to the group $T=T_s=\{g(u)=\mathrm{diag}(u_1,u_2,\ldots,u_s): |u_i| = 1\}$. Every continuous representation of T of degree m may be described (after a suitable choice of basis for V) by m vectors $\alpha_i \in \mathbb{Z}^s$, $1\le i\le s$. If $\omega = (\omega_1,\ldots,\omega_s) \in \mathbb{Z}^s$, then write $u^\omega = u_1^{\omega_1}\cdots u_s^{\omega_s}$. Then the representation $\rho = \rho(\alpha_1,\ldots,\alpha_m)$ is defined by $g(u) \mapsto \mathrm{diag}(u^{\alpha_1},\ldots,u^{\alpha_m})$. The representation ρ is faithful if and only if the greatest common divisor of the $s\times s$ minors of the matrix $\begin{bmatrix}\alpha_1\\ \vdots \\ \alpha_m\end{bmatrix}$ is equal to one. We denote the image of T in $GL(V)$ by $T(\alpha) = T(\alpha_1,\ldots,\alpha_m)$. Every irreducible character χ of T_s is linear and may be described by a vector $\beta = (\beta_1,\ldots,\beta_s) \in \mathbb{Z}^s$. The character $\chi = \chi_\beta$ is given by $\chi(g(u)) = u^\beta$. Conversely, if ρ is faithful then any such β defines a character. The

module $R^T_{\chi_\beta}$ has as a vector space basis all monomials $x_1^{a_1}\ldots x_m^{a_m}$ such that $a_1\alpha_1+\ldots+a_m\alpha_m = \beta$.

3.1 EXAMPLE. Let $s = 2$, $m = 3$, $\alpha_1 = (1,1)$, $\alpha_2 = (1,-1)$, $\alpha_3 = (-1,0)$, $\beta = (1,0)$, so $T(\alpha) = \{\mathrm{diag}\ (uv, uv^{-1}, u^{-1}):|u| = |v| = 1\}$. The module $R^T_{\chi_\beta}$ is spanned (as a vector space) by all monomials $x^ay^bz^c$ with $a+b-c=1$ and $a-b=0$. Hence $R^T_{\chi_\beta} = \mathbb{C}[xyz^2]xyz$.

We have seen in Example 1.3 that $R^T_{\chi_\beta}$ need not be Cohen-Macaulay. A result of Hochster [7] states that R^T is always Cohen-Macaulay. This result is generalized in [8] and also proved in [4]. We will use Hochster's result to give a sufficient (but not necessary) condition for $R^T_{\chi_\beta}$ to be Cohen-Macaulay. First we require a simple result from commutative algebra, whose proof is omitted.

3.2 LEMMA. Let $A = A_0 \oplus A_1 \oplus \ldots$ be a Cohen-Macaulay graded algebra over the field $k = A_0$. Suppose that $A = M_0 \oplus M_1 \oplus \ldots \oplus M_{r-1}$, where $M_0 = B$ is a graded subalgebra of A and each M_i is a graded B-module. Then each M_i is a Cohen-Macaulay B-module.

3.3 EXAMPLE. Let r be a positive integer and let $0\le i<r$. Given $A = A_0 \oplus A_1 \oplus \ldots$, let $M_i = A_i \oplus A_{r+i} \oplus A_{2r+i} \oplus \ldots$. The assumptions of Lemma 3.2 clearly hold, so each M_i is Cohen-Macaulay if A is Cohen-Macaulay. The subalgebra $B = M_0$ is called a <u>Veronese subalgebra</u> of A, and we may call M_i a <u>Veronese module</u>.

3.4 DEFINITION. Let $T(\alpha)$ denote the torus $\mathrm{diag}(u^{\alpha_1},\ldots,u^{\alpha_m})$. Define a character χ_β of $T(\alpha)$ to be <u>critical</u> if the system of linear equations $z_1\alpha_1+\ldots+z_m\alpha_m = \beta$ has a <u>rational</u> solution $(z_1,\ldots,z_m) = (a_1,\ldots,a_m)$ with the following two properties:

(i) $a_i\le 0$ for $1\le i\le m$

(ii) If $(b_1,\ldots,b_m)$ is an <u>integer</u> solution to $z_1\alpha_1+\ldots+z_m\alpha_m = \beta$ such that $b_i\ge a_i$ for all i, then $b_i\ge 0$ for all i.

We now come to the main result of this section.

3.5 THEOREM. If χ_β is a critical character of T, then $R^T_{\chi_\beta}$ is Cohen-Macaulay.

<u>Proof</u>. Suppose in the notation of Definition 3.4 that $a_i=-p_i/q_i$ for integers $p_i\ge 0$ and $q_i>0$. Let μ be the least common multiple of $q_1,q_2,\ldots,q_m$, and define $\alpha_i' = (\mu/q_i)\alpha_i$. Let T' be the torus $T' = T(\alpha_1',\ldots,\alpha_m')$. For any vector $\nu = (\nu_1,\ldots,\nu_m)$ of integers satisfying $0\le\nu_i<q_i$, let $R'(\nu)$ be the subspace of $R^{T'}$ spanned by all monomials $x_1^{c_1}\ldots x_m^{c_m}$ such that $c_i \equiv \nu_i \pmod{q_i}$. Clearly $R'(0,0,\ldots,0) = B$ is a subalgebra of $R^{T'}$, each

$R'(\nu)$ is a B-module, and $R^{T'} = \coprod_{\nu} R'(\nu)$. (The modules $R'(\nu)$ are "generalized Veronese modules.") Let σ_i be the least non-negative residue of p_i modulo q_i, and let $\sigma = (\sigma_1,\ldots,\sigma_m)$. We claim that $B \cong R^T$ and $R'(\sigma) \cong R^T_{\chi_\beta}$. Since $R^{T'}$ is Cohen-Macaulay by Hochster's result [7], it will follow from Lemma 3.2 that $R^T_{\chi_\beta}$ is a Cohen-Macaulay R^T-module.

To prove the claim, first note that $B \cong R^T$ is clear, since this follows from the statement that $(z_1,\ldots,z_m)$ is a solution to $\Sigma z_i\alpha_i = 0$ in non-negative integers z_i if and only if $(q_1z_1,\ldots,q_mz_m)$ is a solution to $\Sigma y_i\alpha_i' = 0$ in non-negative integers y_i with $y_i \equiv 0 \pmod{q_i}$. Now suppose $(q_1z_1 + \sigma_1,\ldots, q_mz_m + \sigma_m)$ is a solution to $\Sigma y_i\alpha_i' = 0$ in non-negative integers. Hence $\Sigma(q_iz_i + \sigma_i)(\mu/q_i)\alpha_i = 0$, which is equivalent to

$$\Sigma\left(z_i - \frac{p_i-\sigma_i}{q_i}\right)\alpha_i = \beta \quad .$$

Now $z_i - \frac{p_i-\sigma_i}{q_i} \geq -(p_i/q_i) = a_i$. Hence by assumption $z_i - \frac{p_i-\sigma_i}{q_i} \geq 0$. It follows that the linear transformation $R'(\sigma) \to R^T_{\chi_\beta}$ defined by $x_1^{c_1}\cdots x_m^{c_m} \longmapsto x_1^{(c_1/q_1)+a_1}\cdots x_m^{(c_m/q_m)+a_m}$ is an isomorphism of R^T-modules, so the proof is complete.

Note that the condition on $(a_1,\ldots,a_m)$ in Definition 3.4 is automatically satisfied if $-1<a_i\leq 0$. In other words, χ_β is critical if β lies in the interior of the convex polytope (actually a zonotope) $\Delta_T = \{\Sigma\lambda_i\alpha_i\colon -1\leq\lambda_i \leq 0\}$, called the <u>critical zonotope</u> of T. For instance, if T is the one-dimensional torus $\{\mathrm{diag}(u^{\alpha_1},\ldots,u^{\alpha_r},u^{-\beta_1},\ldots,u^{-\beta_s})\}$ with $\alpha_i > 0$ and $\beta_j > 0$, then an integer β belongs to Δ_T if and only if $-\Sigma\alpha_i < \beta < \Sigma\beta_j$. In general, it can be shown that the number of integer vectors β in the interior of Δ_T is given by $\sum_X (-1)^{s-|X|}h(X)$, where X ranges over all linearly independent subsets of $\{\alpha_1,\ldots,\alpha_m\}$ and where $h(X)$ is equal to the greatest common divisor of the $s\times s$ minors of the matrix whose rows are the elements of X. For instance, if $s=1$, $\alpha_1=1$, $\alpha_2=1$, $\alpha_3=-1$, $\alpha_4=-1$ (as in Example 1.3), then we obtain $1+1+1+1-1 = 3$. If $s=2$, $\alpha_1=(1,0)$, $\alpha_2=(-1,1)$, $\alpha_3=(-2,-4)$, $\alpha_4=(0,1)$, then we obtain $1+4+1+6+1+2-1-1-2-1+1=11$.

REMARK. It should be possible to prove Theorem 3.5 using the techniques of [18], but the proof we have given is certainly more elementary.

3.6 EXAMPLE. Let $s=1$, $\alpha_1=6$, $\alpha_2=-2$, $\alpha_3=-3$, $\beta=6$. Then β is critical (e.g., let $(a_1,a_2,a_3) = (0,-\frac{3}{2},-1)$) but $\beta \notin \Delta_T$.

3.7 EXAMPLE. Let $s=1$, $\alpha_1=1$, $\alpha_2=-1$, $\beta = 1$. Then (T,χ_β) satisfies Property 1 but not Property 2.

3.9 EXAMPLE. [13, Ex.8.6] can be used to produce a pair (T,χ_β) which satisfies Property 2 but for which $R^T_{\chi_\beta}$ is not Cohen-Macaulay. For this example one has $s = \dim T = 7$, $\dim R^T_\chi = 4$, and $m = 11$.

We now state a strengthening of Theorem 3.5.

3.10 THEOREM. Let χ_β be a critical character of the torus T. Then (T,χ_β) satisfies Properties 1 and 2.

Sketch of proof. Property 2 can be deduced from [13, Thm. 10.2]. Property 1 follows from Theorem 3.5 and the techniques used to prove [14, Thm. 6.7].

4. COMPACT GROUPS. We now turn to the consideration of arbitrary compact groups. First we state what is perhaps the deepest known result in invariant theory.

4.1 THEOREM (Hochster and Roberts [8]). Let the compact group G act on a finite-dimensional vector space V. Then R^G is a Cohen-Macaulay ring.

REMARK. Hochster and Roberts state their result for linearly reductive linear algebraic groups, but this easily yields the result for compact groups.

As we did for toruses we can ask for a generalization of Theorem 4.1 to χ-invariants. We do not know how to prove an analogue of Theorem 3.5 for arbitrary compact G, but by combinatorial reasoning we can give a plausible conjecture. To do so we now consider Property 2. Recall (Theorem 2.1) that Property 2 was verified for finite groups simply by substituting $1/\lambda$ for λ in (2). Unfortunately the same proof does not work for arbitrary compact groups because the operation of substituting $1/\lambda$ for λ does not commute with the integral in (1). In fact, we know such a proof cannot work because Property 2 need not hold (Examples 1.3 and 3.7). We can use Theorem 3.10, however, to give a sufficient condition for Property 2 to hold. First we need to review some facts concerning integration on compact groups.

Let G be compact and connected, and let T be a maximal torus of G. Thus T is isomorphic to $\{\mathrm{diag}\,(u_1,u_2,\ldots,u_s)\colon \ |u_1| =\ldots=|u_s| = 1\}$. Suppose we have an action $\rho\colon\ G \to GL(V)$. With respect to a suitable basis for V, the image $\rho(T)$ will be of the form $T(\alpha) = T(\alpha_1,\ldots,\alpha_m)$, with $\alpha_1,\ldots,\alpha_m \in \mathbb{Z}^s$. (The vectors $\alpha_1,\ldots,\alpha_m$ are the "weights" of ρ with respect to an appropriate basis.) Then there exist non-zero vectors $\beta_1,\ldots,\beta_r \in \mathbb{Z}^s$ depending only on G and not on ρ (the "roots" of G with respect to an appropriate basis - the roots are the non-zero weights of the adjoint representation of G) such that for any irreducible character χ of G we have

$$F_\chi(G,\lambda) = \frac{\deg\chi}{|W|}\int_{g\in T(\alpha)} \frac{(1-\chi_{\beta_1}(g))\ldots(1-\chi_{\beta_r}(g))\overline{\chi}(g)\,dg}{\det(1-\lambda g)} \qquad (7)$$

where W is the Weyl group of G. Equation (7) is an immediate consequence of the Weyl integration formula [17, Thm. 7.4.D.] [1, Thm. 6.1] and Theorem 1.1. For an example of the use of (7) in computing $F_\chi(G,\lambda)$, see [12, Appendix]. If G is not connected then there is a straightforward generalization of (7) involving a sum over the components of G. For simplicity's sake we will assume henceforth that G is connected, though our results can be extended to arbitrary compact G .

The character χ when restricted to T breaks up into irreducible characters of T , say

$$\chi(g) = \chi_{\gamma_1}(g)+\dots+\chi_{\gamma_t}(g).$$

We now define χ to be a <u>critical character</u> of the representation ρ: $G \to GL(V)$ if for all $1 \le i \le t$ and all subsets S of $\{\beta_1,\dots,\beta_r\}$, the character χ_ω of the torus $T(\alpha)$ defined by $\omega = \gamma_i - \sum_{j\in S} \beta_j$ is a critical character of $T(\alpha)$.

4.2 EXAMPLE. Let $G = SU(2,\mathbb{C})$. Then $s=1$, $r=2$, $\beta_1=-2$, $\beta_2=2$. For each positive integer m there is a unique irreducible representation ρ_m of degree m, and $\alpha_1=-m+1, \alpha_2=-m+3$, $\alpha_3=-m+5,\dots$, $\alpha_m=m-1$. Take, for instance, the case $m=6$ and let χ have degree 8. Then $\alpha_1=-5$, $\alpha_2=-3$, $\alpha_3=-1$, $\alpha_4=1$, $\alpha_5=3$, $\alpha_6=5$, $\gamma_1=-7$, $\gamma_2=-5$, $\gamma_3=-3$, $\gamma_4=-1$, $\gamma_5=1$, $\gamma_6=3$, $\gamma_7=5$, $\gamma_8=7$. Let $\omega = 7-(-2) = 9$. Now χ_ω is not a critical character of the torus diag $(u^{-5},u^{-3},u^{-1},u,u^3,u^5)$, so χ is not a critical character of ρ_6. However, any character χ of $SU(2,\mathbb{C})$ of degree <8 is a critical character of ρ_6. More generally, any irreducible character χ of $SU(2,\mathbb{C})$ of degree $\le \frac{1}{4}m^2-2$ is a critical character of ρ_m.

4.3 THEOREM. Let ρ: $G\to GL(V)$ be a representation of the compact connected Lie group G on V. If χ is a critical character of ρ , then (G,χ) has Property 2.

<u>Proof</u>. Write the numerator $(1-\chi_{\beta_1}(g))\dots(1-\chi_{\beta_r}(g))\overline{\chi}(g)$ of the integrand of (7) as a linear combination of characters of T. Thus $F_\chi(G,\lambda)$ is a linear combination of terms of the form

$$F_\gamma(\lambda) = \int_{T(\alpha)} \frac{\overline{\chi}_\gamma(g)\,dg}{\det(1-\lambda g)} .$$

The definition of critical character insures that each χ_γ is a critical character of $T(\alpha)$. Hence by Theorem 3.10 we have

$$F_\gamma(1/\lambda) = (-1)^{d_\gamma}\lambda^m \int_{T(\alpha)} \frac{\chi_\gamma(g)(\det g)\,dg}{\det(1-\lambda g)} ,$$

where $d_\gamma = \dim R^T_{\chi_\gamma}$. Since by (3) the numbers d_γ are all equal (say to d) whenever $R^T_{\chi_\gamma}$ is non-void, we get

$$F_\chi(G,1/\lambda)=(\deg \chi)(-1)^d\lambda^m \int_{T(\alpha)} \frac{(1-\overline{\chi}_{\beta_1}(g))\dots(1-\overline{\chi}_{\beta_r}(g))\chi(g)(\det g)dg}{\det(1-\lambda g)}$$

$$= (-1)^d\lambda^m F_\psi(G,\lambda) ,$$

since a well-known property of roots states that β is a root if and only if $-\beta$ is a root. Since $\dim R^G_\chi = \dim R^G_\psi$ (the above formula showing that R^G_χ is void if and only if R^G_ψ is void), it follows from our original definition of Krull dimension that $(-1)^d = (-1)^{\dim R^G_\chi}$, completing the proof. (Remark: It is not necessarily true that $d = \dim R^G_\chi$, but when χ is critical we have shown that $d \equiv \dim R^G_\chi \pmod 2$. If χ is not critical then this congruence need not hold, e.g., for the adjoint representation of $SU(2,\mathbb{C})$ when χ is trivial).

Theorem 4.3, together with (6), suggest the following conjecture.

CONJECTURE. Let ρ be as in Theorem 4.3, and let χ be a critical character of ρ. Then (G,χ) satisfies Property 1.

There is one special case for which we can verify the above conjecture. This is when χ is trivial and $\rho: G \to SL(V)$. In this case, $\chi = \psi$ and Theorem 4.3 states that when χ is critical we have

$$F(G,1/\lambda) = \pm \lambda^m F(G,\lambda). \tag{8}$$

Then by Theorem 1.4 R^G is Gorenstein, so $\Omega(R^G) \cong R^G$ and R^G has Property 1. This writer and independently M. Hochster conjecture that for any $\rho: G \to SL(V)$, the ring R^G is Gorenstein. Theorems 1.4 and 4.3 establish this for finite G and toroidal G. In these cases independent algebraic proofs can be given [4] [16]. Hochster and Roberts [8, Cor.1.9] show that R^G is Gorenstein when G is semisimple, although (8) need not hold. For instance, for the representations ρ_m of $SU(2,\mathbb{C})$ defined above, we have (8) for $m \geq 4$ by Theorem 4.3. However, when $m=2$ we have $F(G,1/\lambda) = F(G,\lambda)$, and when $m=3$ we have $F(G,1/\lambda) = -\lambda^2 F(G,\lambda)$.

Presumably algebraic techniques will be required to resolve the above conjecture, but at least combinatorial reasoning has led to its formulation and enhanced its believability.

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