Exponential Structures

By Richard P. Stanley

Generating functions of the type $\exp\sum_1^\infty f(n)x^n/n!M(n)$ occur throughout enumerative combinatorics. We give a general setting for obtaining such generating functions in which the numbers $M(n)$ have an explicit combinatorial meaning. Applications are given to the computation of Möbius functions and related invariants of certain posets.

1. Introduction

In the course of his investigation of Ising ferromagnets, Garrett Sylvester [10] was led to consider the poset (partially ordered set) Q_n of all *even* partitions π of the set $[2n]=\{1,2,\ldots,2n\}$, i.e., partitions π of $[2n]$ such that every block of π has even cardinality. The elements of Q_n are ordered in the usual way by refinement, so that Q_n is a subposet (actually a sub-join-semilattice) of the well-known lattice Π_{2n} of all partitions of $[2n]$. Let $\hat{Q}_n$ denote Q_n with a unique minimal element $\hat{0}$ adjoined, and let μ denote the Möbius function (in the sense of [7]) of $\hat{Q}_n$. Let $\hat{1}$ denote the unique maximal element of $\hat{Q}_n$ (i.e., the partition of $[2n]$ into one block), and set $\mu_n=\mu(\hat{0},\hat{1})$. For instance, $\mu_1=-1$ and $\mu_2=2$. Sylvester [10, p. 145] showed that

$$-\sum_{n=1}^{\infty}\frac{\mu_n y^{2n-1}}{(2n-1)!}=\tanh y.$$

or equivalently

$$-\sum_{n=1}^{\infty}\frac{\mu_n y^{2n}}{(2n)!}=\log\cosh y. \tag{1}$$

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An equivalent result, though not stated in terms of posets and Möbius functions, appears in [6, Lemma 3].

It is natural to ask for a general theory which includes Sylvester's result as a special case. What abstract properties of the posets Q_n cause results like (1) to be valid? In this paper we construct a general class of posets, called *exponential structures*, which behave analogously to Sylvester's posets Q_n. With each exponential structure is associated an "exponential formula" and more generally a "convolutional formula", which is an analogue of the well-known exponential formula of enumerative combinatorics, as described for instance in [4] or [9, Sec. 6]. The convolutional formula allows us to derive generating function identities such as (1) in a mechanical way. We also obtain the result that certain sequences of polynomials related to exponential structures in a natural way form a sequence of binomial type in the sense of [5] and [8]. Some of the basic invariants associated with sequences of binomial type have a simple combinatorial interpretation in this setting.

Our exponential structures are closely related to the exponential prefabs of Bender and Goldman [1]. Exponential structures and exponential prefabs are basically two ways of looking at the same phenomenon: the former from the viewpoint of posets and the latter from the viewpoint of abstract algebra. In the context of posets certain concepts become more natural, such as the Möbius function and the fundamental invariants $M(n)$ described below. Our work may be regarded as a direct generalization of [2, Sec. 5.2], which sets up an isomorphism between the algebra of multiplicative functions in the incidence algebra of Π_n and the algebra of formal power series under composition. For the sake of simplicity, however, we will not use the language of incidence algebras. The reader familiar with incidence algebras will have little difficulty in translating our results into that context.

Throughout this paper we let **P** denote the positive integers and **C** the complex numbers, and we let $[n]=\{1,2,\ldots,n\}$ for $n\in\mathbf{P}$.

2. Definitions and examples

An *exponential structure* is a sequence $\mathbf{Q}=(Q_1,Q_2,\ldots)$ of posets satisfying the following three axioms:

(i) For each positive integer n, Q_n is finite and has a unique maximal element $\hat{1}_n$ (denoted simply by $\hat{1}$), and every maximal chain of Q_n has n elements (or length $n-1$).

(ii) If $\pi\in Q_n$, then the interval $[\pi,\hat{1}]$ is isomorphic to Π_k (the lattice of partitions of a k-element set) for some k. We then write $|\pi|=k$. Thus if $|\pi|=k$, then every maximal chain from π to $\hat{1}$ has k elements.

(iii) Suppose $\pi\in Q_n$ and ρ is a minimal element of Q_n satisfying $\rho\leqslant\pi$. Then by (i) and (ii), $[\rho,\hat{1}]\cong\Pi_n$.

Hence by the well-known properties of Π_n (e.g., [7, p. 359], [2, Sec. 5.2]) we have $[\rho,\pi]\cong\Pi_1^{a_1}\times\Pi_2^{a_2}\times\cdots\times\Pi_n^{a_n}$ for unique non-negative integers $a_1,a_2,\ldots,a_n$ satisfying $\sum ia_i=n$ (and $\sum a_i=|\pi|$). We require that the subposet $\{\sigma\in Q_n:\sigma\leqslant\pi\}$ be isomorphic to $Q_1^{a_1}\times Q_2^{a_2}\times\cdots\times Q_n^{a_n}$. In particular, if ρ' is another minimal element of Q_n satisfying $\rho'\leqslant\pi$, then $[\rho,\pi]\cong[\rho',\pi]$. We call $(a_1,a_2,\ldots,a_n)$ the *type* of π.

Intuitively, one should think of Q_n as forming a set of "decompositions" of some structure S_n of "size" n into "pieces" which are smaller S_i's. Then (ii) states that given a decomposition of S_n, one can take any partition of the pieces of the decomposition and join together the pieces in each block in a unique way to obtain a coarser decomposition. Moreover, (iii) states that each piece can be decomposed independently to form a finer decomposition.

If $\mathbf{Q}=(Q_1,Q_2,\dots)$ is an exponential structure, let $M(n)$ denote the number of minimal elements of Q_n. As will be seen below, all the basic combinatorial properties of $\mathbf{Q}$ can be deduced from the numbers $M(n)$. We call the sequence $\mathbf{M}=(M(1),M(2),\dots)$ the *denominator sequence* of $\mathbf{Q}$.

We now proceed to some examples of exponential structures.

Example 2.1. The prototypical example of an exponential structure is given by $Q_n=\Pi_n$, the lattice of partitions of an n-element set. In this case we have $M(n)=1$.

Example 2.2. Let V_n be an n-dimensional vector space over the finite field GF(q). Let Q_n consist of all collections $\{W_1,W_2,\dots,W_k\}$ of subspaces of V_n such that $\dim W_i>0$ for all i, and such that $V_n=W_1\oplus W_2\oplus\cdots\oplus W_k$ (direct sum). An element of Q_n is called a *direct sum decomposition* of V_n. We order Q_n in the obvious way by refinement, i.e., $\{W_1,W_2,\dots,W_k\}\leqslant\{W_1',W_2',\dots,W_j'\}$ if each W_r is contained in some W_s'. We call Q_n the poset of *direct sum decompositions* of V_n. It is easily seen that $(Q_1,Q_2,\dots)$ is an exponential structure with $M(n)=q^{\binom{n}{2}}(1+q)(1+q+q^2)\cdots(1+q+q^2+\cdots+q^{n-1})/n!$. We may regard Q_n as a "q-analogue" of the partition lattice Π_n, since putting $q=1$ in the above expression for $M(n)$ yields $M(n)=1$. Although Q_n is a q-analogue of Π_n, it is not a very satisfactory one. Ideally, a q-analogue of Π_n would be a geometric lattice which somehow "contains" (perhaps as a sublattice) the lattice $L(V_n)$ of subspaces of V_n. But Q_n is not even a lattice; and it contains in a natural way copies of the lattice Π_n, not $L(V_n)$. For most purposes the Dowling lattices [3] are better q-analogues of Π_n, though they are actually G-analogues, where G is a finite group. On the other hand, the exponential structure $(Q_1,Q_2,\dots)$ is well suited for studying direct sum decompositions of V_n.

Example 2.3. Let G_n be a graph (without loops or multiple edges) on the vertex set $[n]=\{1,2,\dots,n\}$, and let π be a partition of $[n]$ such that each block B of π is independent in G (i.e., no two vertices in B are connected by an edge of G). The pair (G_n,π) may be regarded as a *colored graph* (with unlabeled colors) on the vertex set $[n]$. Define a partial order on the set Q_n of all colored graphs (G_n,π) on $[n]$ by setting $(G_n,\pi)\leqslant(H_n,\sigma)$ if (i) $\pi\leqslant\sigma$ in Π_n, and (ii) if i and j are two vertices which do not belong to the same block of σ; then i and j are connected by an edge in G_n if and only if they are connected in H_n. Then $\mathbf{Q}=(Q_1,Q_2,\dots)$ is an exponential structure with $M(n)=2^{\binom{n}{2}}$. An easy modification yields for any integers $q\geqslant 1$ and $k\geqslant 2$ an exponential structure with $M(n)=q^{\binom{n}{k}}$. Even greater generality is possible, e.g., $M(n)=q_1^{\binom{n}{2}}q_2^{\binom{n}{3}}\cdots q_k^{\binom{n}{k}}$ for any fixed positive integers q_i. Such generalizations seem rather contrived and not of much value for solving problems of enumerative combinatorics.

Example 2.4. Let $\mathbf{Q}=(Q_1,Q_2,\dots)$ be an exponential structure with denominator sequence $\mathbf{M}=(M(1),M(2),\dots)$. Fix a positive integer r, and define $Q_n^{(r)}$ to be the subposet of Q_{rn} consisting of all $\pi\in Q_{rn}$ of type $(a_1,a_2,\dots,a_{rn})$, where $a_i=0$ unless r divides i. Then $\mathbf{Q}^{(r)}=(Q_1^{(r)},Q_2^{(r)},\dots)$ is an exponential structure with denominator sequence $(M_r(1),M_r(2),\dots)$ given by

$$M_r(n)=\frac{M(rn)(rn)!}{M(r)^n n!r!^n}. \tag{2}$$

In particular, if $Q_n=\Pi_n$, then $Q_n^{(2)}$ is the poset considered by Sylvester.

Example 2.5. Let r be a positive integer, and let S be an n-element set. Define an *r-partition of* S to be a set

$$\pi=\{(B_{11},B_{12},\dots,B_{1r}),(B_{21},B_{22},\dots,B_{2r}),\dots,(B_{k1},B_{k2},\dots,B_{kr})\} \tag{3}$$

satisfying:

(i) For each $j\in[r]$, the set $\pi_j=\{B_{1j},B_{2j},\dots,B_{kj}\}$ forms a partition of S into k blocks, and

(ii) For fixed i, $|B_{i1}|=|B_{i2}|=\dots=|B_{ir}|$.

The set Q_n of all r-partitions of S has an obvious partial ordering by refinement which makes $(Q_1,Q_2,\dots)$ into an exponential structure with $M(n)=n!^{r-1}$. The type $(a_1,a_2,\dots,a_n)$ of $\pi\in Q_n$ is equal to the type of any of the partitions π_j. [By (ii), all the π_j's have the same type.] An application of r-partitions to counting matrices with equal row and column sums appears in [9, Example 6.11].

The poset Q_n of all r-partitions of S is a kind of product of r copies of Π_n. One can devise similar "products" of various other exponential structures. If $\mathbf{Q}$ and $\mathbf{Q}'$ have the denominator sequences $\mathbf{M}$ and $\mathbf{M}'$, then the "product structure" analogous to 2-partitions will have a denominator sequence $\mathbf{N}$ satisfying $N(n)=M(n)M'(n)n!$. Further details are left to the reader.

3. The convolutional formula and the exponential formula

The basic combinatorial properties of an exponential structure will be obtained from the following lemma.

LEMMA 3.1. *Let* $\mathbf{Q}=(Q_1,Q_2,\dots)$ *be an exponential structure with denominator sequence* $(M(1),M(2),\dots)$. *Then the number of* $\pi\in Q_n$ *of type* $(a_1,a_2,\dots,a_n)$ *is equal to*

$$\frac{n!M(n)}{1!^{a_1}\cdots n!^{a_n}a_1!\cdots a_n!M(1)^{a_1}\cdots M(n)^{a_n}}.$$

Proof: Let $N=N(a_1,\dots,a_n)$ be the number of pairs (σ,π) where σ is a minimal element of Q_n such that $\sigma\le\pi$ and π has type $(a_1,\dots,a_n)$. On the one hand we can pick σ in $M(n)$ ways, and then pick $\pi\ge\sigma$. The number of choices for π is the number of elements of Π_n of type $(a_1,\dots,a_n)$, which is well known to equal

$n!/1!^{a_1}\cdots n!^{a_n}a_1!\cdots a_n!$. Hence

$$N=\frac{n!M(n)}{1!^{a_1}\cdots n!^{a_n}a_1!\cdots a_n!}. \tag{4}$$

On the other hand, if K is the desired number of $\pi\in Q_n$ of type $(a_1,\dots,a_n)$, then we can pick π in K ways and then choose $\sigma\leqslant\pi$. Since Q_n has $M(n)$ minimal elements, the poset $Q_1^{a_1}\times Q_2^{a_2}\times\cdots\times Q_n^{a_n}$ has $M(1)^{a_1}M(2)^{a_2}\cdots M(n)^{a_n}$ minimal elements. Hence there are $M(1)^{a_1}M(2)^{a_2}\cdots M(n)^{a_n}$ choices for σ, so

$$N=K\cdot M(1)^{a_1}\cdots M(n)^{a_n}. \tag{5}$$

The proof follows from (4) and (5). □

We are now in a position to state and prove the basic combinatorial property of exponential structures.

THEOREM 3.2 (The convolutional formula). *Let $(Q_1,Q_2,\dots)$ be an exponential structure with denominator sequence $(M(1),M(2),\dots)$. Given functions $f:\mathbf{P}\to\mathbf{C}$ and $g:\mathbf{P}\to\mathbf{C}$, define a new function $h:\mathbf{P}\to\mathbf{C}$ by*

$$h(n)=\sum_{\pi\in Q_n} f(1)^{a_1}f(2)^{a_2}\cdots f(n)^{a_n}g(|\pi|),$$

where π has type $(a_1,a_2,\dots,a_n)$ and where $|\pi|=a_1+a_2+\cdots+a_n$. Define a formal power series $F(x)$, $G(x)$, $H(x)$ with complex coefficients by

$$F(x)=\sum_1^\infty \frac{f(n)x^n}{n!M(n)},$$

$$G(x)=\sum_1^\infty \frac{g(n)x^n}{n!},$$

$$H(x)=\sum_1^\infty \frac{h(n)x^n}{n!M(n)}.$$

Then $H(x)=G(F(x))$.

Proof: We have

$$G(F(x))=\sum_{k=1}^\infty \frac{g(k)}{k!}\left[\sum_{i=1}^\infty \frac{f(i)x^i}{i!M(i)}\right]^k$$

$$=\sum_{k=1}^\infty \frac{g(k)}{k!}\sum \frac{f(b_1)\cdots f(b_k)}{b_1!\cdots b_k!M(b_1)\cdots M(b_k)}x^{b_1+\cdots+b_k},$$

where the inner sum is over all k-tuples $(b_1,\ldots,b_k)\in\mathbf{P}^k$. Let a_i be the number of b_j's which are equal to i, so that $k=\Sigma a_i$; and let $n=\Sigma b_i=\Sigma ia_i$. We obtain

$$G(F(x))$$
$$=\sum_{n=1}^{\infty}\frac{x^n}{n!M(n)}\sum\frac{n!M(n)\alpha(a_1,\ldots,a_n)}{1!^{a_1}\cdots n!^{a_n}M(1)^{a_1}\cdots M(n)^{a_n}k!}f(1)^{a_1}\cdots f(n)^{a_n}g(k),$$

where the inner sum is over all solutions in non-negative integers a_i to $n=\Sigma ia_i$, where $k=\Sigma a_i$, and where $\alpha(a_1,\ldots,a_n)$ is the number of distinct k-tuples $(b_1,\ldots,b_k)$ with exactly a_i of the b_j's equal to i. Clearly $\alpha(a_1,\ldots,a_n)$ is just the multinomial coefficient $k!/a_1!a_2!\cdots a_n!$. Hence

$$G(F(x))$$
$$=\sum_{n=1}^{\infty}\frac{x^n}{n!M(n)}\sum\frac{n!M(n)}{1!^{a_1}\cdots n!^{a_n}a_1!\cdots a_n!M(1)^{a_1}\cdots M(n)^{a_n}}f(1)^{a_1}\cdots f(n)^{a_n}g(k).$$

By the previous lemma, we conclude

$$G(F(x))=\sum_{n=1}^{\infty}\frac{x^n}{n!M(n)}\sum_{\pi\in Q_n}f(1)^{a_1}\cdots f(n)^{a_n}g(k),$$

and the proof follows. □

COROLLARY 3.3 (The exponential formula). *In Theorem* 3.2 *let* $g(n)=1$ *for all* $n\in\mathbf{P}$. *Then*

$$1+H(x)=e^{F(x)}.$$

4. Applications

We will now apply Theorem 3.2 to certain counting problems associated with exponential structures in general. For some further applications which deal with *specific* exponential structures (viz., r-partitions) see [9, Sec. 6]. The simplest application is that of counting the total number $|Q_n|$ of elements of the poset Q_n. Since

$$|Q_n|=\sum_{\pi\in Q_n}1,$$

we simply substitute $f(n)=1$, $g(n)=1$, $h(n)=|Q_n|$ in Theorem 3.2 (or Corollary 3.3) to obtain

$$1+\sum_{n=1}^{\infty}\frac{|Q_n|x^n}{n!M(n)}=\exp\sum_{n=1}^{\infty}\frac{x^n}{n!M(n)}.$$

More generally, we can ask for the number S_{nk} of $\pi \in Q_n$ satisfying $|\pi| = k$. Define a polynomial

$$W_n(\lambda) = \sum_{\pi \in Q_n} \lambda^{|\pi|} = \sum_{k=1}^{n} S_{nk}\lambda^k.$$

Now by Theorem 3.2, if we put $f(n) = \lambda$ and $g(n) = 1$ [or equivalently $f(n) = 1$ and $g(n) = \lambda^n$], then we have $h(n) = W_n(\lambda)$ and

$$1 + \sum_{1}^{\infty} \frac{W_n(\lambda)x^n}{n!M(n)} = \exp\left[\lambda \sum_{1}^{\infty} \frac{x^n}{n!M(n)}\right].$$

This shows that the polynomials $W_n(\lambda)/M(n)$ form a polynomial sequence of *binomial type* in the sense of [5] and [8]. Moreover, the corresponding delta operator $q(D)$ satisfies

$$q^{\langle -1 \rangle}(x) = \sum_{1}^{\infty} \frac{x^n}{n!M(n)},$$

where $q^{\langle -1 \rangle}(x)$ denotes the inverse formal power series to $q(x)$ [5, Corollary 2, p. 189; 8, Corollary 3, p. 693].

Now let $\hat{Q}_n$ denote Q_n with a unique minimal element $\hat{0}$ adjoined, and let μ denote the Möbius function [7] of $\hat{Q}_n$. We wish to compute the integer $\mu_n = \mu(\hat{0}, \hat{1})$. More generally, define a polynomial

$$w_n(\lambda) = -\sum_{\pi \in Q_n} \mu(\hat{0}, \pi)\lambda^{|\pi|},$$

so that $-\mu_n$ is the coefficient of λ. The defining recursion [7, p. 344]

$$\mu(\hat{0}, x) = -\sum_{\hat{0} < y \leq x} \mu(y, x), \qquad x > \hat{0}$$

for the Möbius function yields

$$w_n(\lambda) = \sum_{\pi \in Q_n} \sum_{\substack{\sigma \in Q_n \\ \sigma \leq \pi}} \mu(\sigma, \pi)\lambda^{|\pi|}$$

$$= \sum_{\sigma} \sum_{\pi \geq \sigma} \mu(\sigma, \pi)\lambda^{|\pi|}.$$

Now for fixed $\sigma \in Q_n$, the poset of all $\pi \geq \sigma$ is isomorphic to the partition lattice $\Pi_{|\sigma|}$. Hence by a well-known property of partition lattices (e.g., [7, Example 1, p. 362]) the inner sum above is given by $(\lambda)_{|\sigma|}$, where $(\lambda)_k = \lambda(\lambda - 1) \cdots (\lambda - k + 1)$.

Hence

$$w_n(\lambda)=\sum_{\sigma\in Q_n}(\lambda)_{|\sigma|},$$

or in the language of the calculus of finite differences,

$$\Delta^k w_n(0)=k!\,S_{nk}, \tag{6}$$

with S_{nk} as above. We now put $f(n)=1$ and $g(n)=(\lambda)_n$ in Theorem 3.2 to obtain $h(n)=w_n(\lambda)$ and

$$\begin{aligned}1+\sum_1^\infty \frac{w_n(\lambda)x^n}{n!M(n)}&=\sum_0^\infty \frac{(\lambda)_n F(x)^n}{n!}\\&=\sum_0^\infty \binom{\lambda}{n}F(x)^n=[1+F(x)]^\lambda\\&=\left[1+\sum_1^\infty \frac{x^n}{n!M(n)}\right]^\lambda.\end{aligned} \tag{7}$$

In particular, the polynomials $w_n(\lambda)/M(n)$ also form a sequence of binomial type. Since $-\mu_n=(d/d\lambda)w_n(0)$, we obtain

$$\begin{aligned}-\sum_1^\infty \frac{\mu_n x^n}{n!M(n)}&=\frac{d}{d\lambda}\left[1+\sum_1^\infty \frac{x^n}{n!M(n)}\right]^\lambda\Bigg|_{\lambda=0}\\&=\log\left[1+\sum_1^\infty \frac{x^n}{n!M(n)}\right].\end{aligned} \tag{8}$$

Hence the delta operator $q(D)$ for $w_n(\lambda)$ is given by

$$q^{\langle -1\rangle}(x)=-\sum_1^\infty \frac{\mu_n x^n}{n!M(n)}.$$

As a special case, suppose $Q_n=\Pi_n^{(2)}$, the lattice of even partitions studied by Sylvester. Then by (2), $M(n)=(2n)!/n!2^n$, so

$$-\sum_1^\infty \frac{\mu_n 2^n x^n}{(2n)!}=\log\left[1+\sum_1^\infty \frac{2^n x^n}{(2n)!}\right].$$

Put $2x=y^2$ to obtain

$$-\sum_{1}^{\infty}\frac{\mu_n y^{2n}}{(2n)!}=\log\cosh y,$$

which is (1). Note also that in the case $Q_n=\Pi_n$ [so $M(n)=1$], (8) reduces to

$$-\sum_{1}^{\infty}\frac{\mu_n x^n}{n!}=x.$$

This is because Π_n already has a unique minimal element, so when we adjoin $\hat{0}$ the Möbius function $\mu_n=\mu(\hat{0},\hat{1})$ will vanish unless $\hat{Q}_n$ has length one, i.e., $n=1$.

There is a general result which includes our results on $w_n(\lambda)$ and $W_n(\lambda)$. Let ζ denote the zeta function of $\hat{Q}_n$ (as defined in [7]), and define for any integer r the polynomial

$$P_n(r,\lambda)=\sum_{\pi\in Q_n}\left[\zeta^r(\hat{0},\pi)-\zeta^{r-1}(\hat{0},\pi)\right]\lambda^{|\pi|}.$$

Since $\zeta^0(\hat{0},\pi)=0$ for all $\pi\in Q_n$, we have $P_n(0,\lambda)=w_n(\lambda)$ and $P_n(1,\lambda)=W_n(\lambda)$. Moreover, when $r\in\mathbf{P}$ we have that $\zeta^r(\hat{0},\pi)-\zeta^{r-1}(\hat{0},\pi)$ is the number of chains $\pi_1\leqslant\pi_2\leqslant\cdots\leqslant\pi_r=\pi$ in Q_n, so the coefficient of λ^k in $P_n(r,\lambda)$ is equal to the number of chains $\pi_1\leqslant\pi_2\leqslant\cdots\leqslant\pi_r$ with $|\pi_r|=k$.

In the same way that (7) was proved it can be shown that for any integer r,

$$1+\sum_{1}^{\infty}\frac{P_n(r,\lambda)x^n}{n!M(n)}=\exp\left[\lambda\sum_{1}^{\infty}\frac{P_n(r-1,1)x^n}{n!M(n)}\right]. \tag{9}$$

Hence for fixed r the polynomials $P_n(r,\lambda)/M(n)$ are of binomial type and their inverse delta operators $q^{\langle -1\rangle}(D)$ are determined by (9). One can write down such daunting formulas as

$$1+\sum_{1}^{\infty}\frac{P_n(4,\lambda)x^n}{n!M(n)}=\exp\left\{\lambda\left[\exp\left(e^{\sum_1^\infty x^n/n!M(n)}-1\right)-1\right]\right\}$$

and

$$1+\sum_{1}^{\infty}\frac{P_n(-2,\lambda)x^n}{n!M(n)}=\left\{1+\log\left[1+\log\left[1+\sum_{1}^{\infty}\frac{x^n}{n!M(n)}\right]\right]\right\}^{\lambda}.$$

Moreover, Eq. (6) generalizes to

$$\sum_k \left[\Delta^k P_n(r,0)\right]\frac{\lambda^k}{k!} = P_n(r+1,\lambda),$$

where Δ operates on the variable λ.

Finally we remark that for fixed λ and n, it is easy to see that $P_n(r,\lambda)$ is a polynomial function of r. For instance,

$$P_1(r,\lambda) = \lambda,$$

$$P_2(r,\lambda) = M(2)\lambda r + \lambda\left[M(2)\lambda - M(2) + 1\right].$$

Hence there is a natural way to define $P_n(r,\lambda)$ for any $r \in \mathbf{C}$, and Eq. (9) continues to remain valid with this definition.

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