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BRANCHINGS AND PARTITIONS

L. CARLITZ[1] AND RICHARD P. STANLEY[2]

ABSTRACT. A generating function is obtained for the number of partitions corresponding to a complete branching on a nonincreasing sequence of n integers. Complete branchings are shown to be related to certain types of plane partitions.

Given a sequence $k_1 \geq k_2 \geq k_3 \geq \cdots \geq k_n$ of integers, a *branching* is a sequence of integers $k'_1, k'_2, \ldots, k'_{n-1}$ such that $k_i \geq k'_i \geq k_{i+1}$ $(i = 1, 2, \ldots, n-1)$. Successively branching $n-1$ times, one obtains a single integer. Such a sequence of $n-1$ successive branchings is called a *complete branching*. S. Gelbart [3] proposed as a Monthly problem that the number of distinct complete branchings of the sequence $k_1, k_2, \ldots, k_n$ is equal to

$$\prod_{1 \leq i < j \leq n} [(k_i - k_j + j - i)/(j - i)]. \tag{1}$$

A complete branching may be indicated by the triangular array

$$T_n: \quad \begin{array}{ccccccc} k_1 & k_2 & k_3 & & \cdots & & k_n \\ & k'_1 & k'_2 & \cdots & k'_{n-1} & & \\ & & k''_1 & \cdots & k''_{n-2} & & \\ & & & \cdots & & & \\ & & & k_1^{(n-1)} & & & \end{array}$$

where $k_i^{(0)} = k_i$ and

$$k_i^{(j)} \geq k_i^{(j+1)} \geq k_{i+1}^{(j)} \qquad (i = 1, 2, \ldots, n-j-1). \tag{2}$$

For a given integer m and fixed sequence $k_1 \geq k_2 \geq \cdots \geq k_n$, let $P_n(m) = P(m; k_1, \ldots, k_n)$ denote the number of arrays T_n satisfying (2) and whose entries sum to m, i.e.,

$$m = \sum_{j=0}^{n-1} \sum_{i=1}^{n-j} k_i^{(j)}.$$

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We shall be interested in evaluating the generating function

$$N(k_1, \ldots, k_n; x) = N(x) = \sum_{m=-\infty}^{\infty} P_n(m)x^m. \tag{3}$$

Note that the sum (3) is actually finite.

There is no loss of generality in supposing that the k_i's are all positive, for this amounts to adding some constant c to each entry of T_n and therefore multiplying (3) by $x^{c\binom{n+1}{2}}$. Assuming therefore that each $k_i > 0$, rewrite T_n as

$$\begin{array}{cccc} k_1 & k_1' & k_1'' & \cdots & k_1^{(n-1)} \\ & k_2 & k_2' & \cdots & k_2^{(n-2)} \\ & & & \cdots & \\ & & & & k_n \end{array}$$

The ith row satisfies $k_i \geq k_i' \geq \cdots \geq k_i^{(n-i)} > 0$ and therefore may be regarded as a partition of $k_i + k_i' + \cdots + k_i^{(n-i)}$. Take the conjugate partition [4, p. 274] of each row and left justify, giving a new array

$$\begin{array}{ll} j_1 & j_1' \cdots \\ j_2 & j_2' \cdots \\ \cdots & \\ j_n & j_n' \end{array}$$

where $(j_i, j_i', \ldots)$ is the conjugate partition to $(k_i, k_i', \ldots)$.

The resulting array of j's is always a column-strict plane partition of shape $(k_1, k_2, \ldots, k_n)$ and largest part n, as defined in [7, §1]. The sum of the j's is equal to the sum of the k's. This correspondence between the array of k's (complete branchings with first row $k_1, k_2, \ldots, k_n$ and sum m) and j's (column-strict plane partitions of shape $\lambda = (k_1, k_2, \ldots, k_n)$ with largest part n and sum m) is easily seen to be a bijection. Hence $P_n(m)$ is equal to the number of column-strict plane partitions of shape λ with largest part n and sum m. In this context, the generating function (3) has been determined (implicitly) by D. E. Littlewood [5, p. 124, Theorem I] and more explicitly in [7, Theorem 15.3]. The result may be stated as follows.

Theorem 1. *We have*

$$N(x) = x^a \prod \frac{(\mathrm{n} + \mathrm{j} - \mathrm{i})}{(\mathrm{h}_{\mathrm{ij}})}, \tag{4}$$

where $(\mathrm{d}) = 1 - x^d$, $a = \sum i k_i$, *and* $h_{ij} = k_i + \overline{k}_j - i - j + 1$. *Here* $(\overline{k}_1, \overline{k}_2, \ldots)$

is the conjugate partition to $(k_1, k_2, \dots)$, *and the product is over all* $k_1 + k_2 + \cdots + k_n$ *pairs* (i, j) *such that* $k_i > 0$ *and* $\bar{k}_j > 0$.

Some elementary manipulation shows that the right-hand side of (4) can be written in the form

$$N(x) = \prod_{1 \le i < j \le n} (x^{k_j} - x^{k_i + j - i}) \Big/ (1)!\,(2)! \cdots (n-1)! \tag{5}$$

where $(i)! = (1)(2) \cdots (i)$. If we put $x = 1$ in (5) we obtain Gelbart's result (1). In the context of plane partitions this result was first given by MacMahon [6].

We shall give a brief indication of how (5) can be proved directly. Clearly one has

$$N(k_1, \dots, k_{n+1}; x) = \sum x^{k_1' + \cdots + k_n'} N(k_1', \dots, k_n'; x), \tag{6}$$

where the summation is over all $k_1', \dots, k_n'$ such that $k_1 \ge k_1' \ge k_2 \ge \cdots \ge k_n' \ge k_{n+1}$. But the $n \times n$ determinant

$$D(k_1, \dots, k_n; x) = (-1)^{\binom{n}{2}} x^{n^3 - n} \left| \binom{k_j - j - 1}{i - 1} \right|, \tag{7}$$

where $\binom{i}{j} = (i)!/(j)!(i-j)!$, also satisfies the recursion (6) and equals $N(k_1, \dots, k_n; x)$ in the case $n = 1$. It follows that $N(k_1, \dots, k_n; x) = D(k_1, \dots, k_n; x)$. The determinant in (7) may be evaluated by Vandermonde's theorem or otherwise, yielding (5).

We remark that an evaluation of a determinant equivalent to (7) appears in [1]. See also [7, §15] for a direct proof that this determinant (actually a related but equivalent determinant) is given by (4). When $x = 1$, the determinant (7) was first evaluated by Frobenius [2].

It follows from (5) that $N(x) = x^A N(1/x)$, where $A = (n+1)\sum_{i=1}^{n} k_i$. From this we obtain the symmetry relation $P_n(m) = P_n(A - m)$. It would be interesting to know, whether for fixed n, $k_1, \dots, k_n$, the sequence $\{P_n(m)\}$ is unimodal, that is, whether there exists a C such that $P_n(m-1) \le P_n(m)$ if $m \le C$ and $P_n(m) \ge P_n(m+1)$ if $m \ge C$.

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DEPARTMENT OF MATHEMATICS, DUKE UNIVERSITY, DURHAM, NORTH CAROLINA 27706

DEPARTMENT OF MATHEMATICS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY, CAMBRIDGE, MASSACHUSETTS 02139