Hyperplane Arrangements, Parking Functions and Tree Inversions

Richard P. Stanley

Dedicated to Gian-Carlo Rota on the occasion of his sixty-fourth birthday, for twenty-nine years of encouragement, support, and friendship

July 15, 1996

Introduction. A *(real) hyperplane arrangement* is a discrete set of hyperplanes in $\mathbb{R}^n$. We will be concerned with hyperplane arrangements that "interpolate" between two well-known arrangements: (1) the set $\mathcal{B}_n$ of hyperplanes $x_i = x_j$, for $1 \leq i < j \leq n$, and (2) the set $\tilde{\mathcal{B}}_n$ of hyperplanes $x_i - x_j = m$, for $1 \leq i < j \leq n$ and $m \in \mathbb{Z}$. The arrangement $\mathcal{B}_n$ is known as the *braid arrangement* or the *reflection arrangement of type* A_{n-1} (i.e., the set of reflecting hyperplanes of the symmetric group $\mathfrak{S}_n$, which is the Coxeter group of type A_{n-1}). Similarly, $\tilde{\mathcal{B}}_n$ is the *affine braid arrangement* or *reflection arrangement of type* $\tilde{A}_n$, i.e., the set of reflecting hyperplanes of the affine Weyl group $\tilde{\mathfrak{S}}_n$ of type $\tilde{A}_n$.

The class of arrangements we will discuss is the following. For $k \geq 1$ define the *extended Shi arrangement* $\mathcal{S}_n^k$ to be the collection of hyperplanes

$$x_i - x_j = -k+1, -k+2, \ldots, k, \text{ for } 1 \leq i < j \leq n.$$

The arrangement $\mathcal{S}_n^1 = \mathcal{S}_n$ is known as the *Shi arrangement* or *sandwich arrangement*, and was first considered by Shi [15, Ch. 7][16] and later Headley [9, Ch. VI][10, §5]. Some properties of $\mathcal{S}_n$ are stated without proof in [20, §5]. In this paper we extend these results to $\mathcal{S}_n^k$ and provide the proofs. For some additional arrangements related to $\mathcal{B}_n$, see [20] and [14].

The main property of $\mathcal{S}_n^k$ to concern us here will be the number of regions R separated from a "natural" base region R_0 by a given number r of hyperplanes in the arrangement. Let us make this notion more precise. If we remove the union of the hyperplanes of an arrangement $\mathcal{A}$ from $\mathbb{R}^n$, then we obtain a disjoint union of open cells, called the *regions* of $\mathcal{A}$. Fix a region R_0 of $\mathcal{A}$, called the *base region.* Given a region R of $\mathcal{A}$, let $d(R)$ denote the number of hyperplanes H of $\mathcal{A}$ which separate R_0 from R, i.e., R_0 and R lie on different sides of H. (This number will always be finite since $\mathcal{A}$ is discrete.) For instance, $d(R_0) = 0$. Think of $d(R)$ as the "distance" of R from R_0. Define the *distance enumerator* of $\mathcal{A}$ (with respect to R_0) to be the generating function

$$D_{\mathcal{A}}(q) = \sum_R q^{d(R)},$$

where R ranges over all regions of $\mathcal{A}$. Thus $D_{\mathcal{A}}$ is a formal power series, which becomes a polynomial if $\mathcal{A}$ is finite.

Let us first consider the braid arrangement $\mathcal{B}_n$. It is most natural for us to let R_0 be defined by the conditions $x_1 > x_2 > \cdots > x_n$. There is a canonical way to label the regions R by the elements w of $\mathfrak{S}_n$, namely, $\mathfrak{S}_n$ acts on $\mathbb{R}^n$ as a group generated by reflections in the hyperplanes of $\mathcal{B}_n$. This action permutes the regions, and for any region R there is a unique $w \in \mathfrak{S}_n$ for which $w(R_0) = R$. Label by w this region $w(R_0)$. (The transitivity of $\mathfrak{S}_n$ on the regions shows that $D_{\mathcal{B}_n}$ is independent of the choice of R_0.) Equivalently, the label of region R is the unique permutation w such that for $i < j$ we have $w(i) > w(j)$ if and only if the hyperplane $x_i = x_j$ separates R_0 from R. It follows that $d(R)$ is the number $\ell(w)$ of inversions of w, i.e., the number of pairs $i < j$ for which $w(i) > w(j)$. This number $\ell(w)$ is also the *length* of w in the Coxeter group sense, i.e., the minimum number p such that w can be written as a product of p adjacent transpositions. It is then well-known, either from combinatorics [19, Cor. 1.3.10] or Coxeter group theory [1, Cor. 4.7][3, Exercise 10(a), pp. 230–231], that

$$D_{\mathcal{B}_n}(q) = (1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1}), \tag{1}$$

the standard q-analogue of $n!$.

There is another way of labeling the regions and obtaining the formula (1). Let $\mathbb{N} = \{0, 1, 2, \ldots\}$. We will label each region with an n-tuple $\lambda(R) = (a_1, \ldots, a_n) \in \mathbb{N}^n$

as follows. Let $e_i \in \mathbb{N}^n$ denote the vector with a 1 in the ith coordinate and 0's elsewhere. First label the region R_0 by $\lambda(R_0) = (0, 0, \ldots, 0)$. Suppose now that R has been labelled, and that R' is an unlabelled region which is separated from R by a unique hyperplane $x_i = x_j$, where $i < j$. Then define $\lambda(R') = \lambda(R) + e_i$. It is easy to see that this labeling is independent of the order in which the regions are labelled. In fact, if $R = w(R_0)$ (i.e., R corresponds to the permutation $w \in \mathfrak{S}_n$) and $\lambda(R) = (a_1, \ldots, a_n)$, then

$$a_i = \#\{j \mid j > i \text{ and } w(j) < w(i)\}.$$

Thus $\lambda(R)$ is essentially the *inversion table* or *code* of w, as defined in [19, p. 21]. Moreover,

$$d(R) = a_1 + a_2 + \cdots + a_n.$$

The codes of permutations $w \in \mathfrak{S}_n$ are precisely those sequences $(a_1, \ldots, a_n) \in \mathbb{N}^n$ satisfying $a_i \leq n - i$. These observations make equation (1) obvious.

Similar results hold for $\tilde{\mathcal{B}}_n$. Define R_0 to be the region given by $x_1 > x_2 > \cdots > x_n > x_1 - 1$. For any region R, there is a unique element $w \in \tilde{\mathcal{S}}_n$ such that $w(R_0) = R$, and $d(R) = \ell(w)$, the length of w as an element of the Coxeter group $\tilde{\mathfrak{S}}_n$. By e.g. [3, Exercise 10(b), p. 231] we have

$$D_{\tilde{\mathcal{B}}_n}(q) = \sum_{w \in \tilde{\mathfrak{S}}_n} q^{\ell(w)} = \frac{1 + q + \cdots + q^{n-1}}{(1-q)^{n-1}}. \tag{2}$$

We can also ask if there is a labeling of the regions R by n-tuples $\lambda(R) \in \mathbb{N}^n$, similar to what was done for $\mathcal{B}_n$. Later we will describe such a labeling as a limiting case of a labeling of the regions of $\mathcal{S}_n^k$.

2. Labeling the extended Shi arrangements. We will define a labeling $\lambda(R) \in \mathbb{N}^n$ of the regions R of the extended Shi arrangement $\mathcal{S}_n^k$ similar to what is described above for the braid arrangement $\mathcal{B}_n$. For the Shi arrangement itself ($k = 1$), this method of labeling was suggested by I. Pak and is described in [20, §5]. Some similarities between the Shi arrangement and the extended Shi arrangements were pointed out by A. Postnikov[1] after which it was straightforward to extend Pak's method of labeling. (However, there remained the problem of actually proving that Pak's labeling and its extension to $\mathcal{S}_n^k$ had the desired properties.)

Define the base region R_0 of $\mathcal{S}_n^k$ by

$$R_0 : \quad x_1 > x_2 > \cdots > x_n > x_1 - 1,$$

the same as for $\tilde{\mathcal{B}}_n$. First label the region R_0 by $\lambda(R_0) = (0, 0, \ldots, 0) \in \mathbb{N}^n$. Suppose now that R has been labelled, and that R' is an unlabelled region which is separated from R by a unique hyperplane $x_i - x_j = m$, where $i < j$ and $m \leq 0$. Then define $\lambda(R') = \lambda(R) + e_j$. On the other hand, if instead $m > 0$, then define $\lambda(R') = \lambda(R) + e_i$. It is easy to see that this labeling is well-defined (i.e., is independent of the order in which the regions are labelled), since $\lambda(R)$ depends only on the set of hyperplanes separating R from R_0.

From the definition of λ we see immediately that if $\lambda(R) = (a_1, \ldots, a_n)$, then

$$d(R) = a_1 + \cdots + a_n. \tag{3}$$

[1]For instance, the characteristic polynomial (as defined e.g. in [20]) of $\mathcal{S}_n^k$ is equal to $q(q - kn)^{n-1}$.

In order to describe the labels that occur, we define a *k-parking function of length n* to be a sequence $\alpha = (a_1, \ldots, a_n) \in \mathbb{N}^n$ satisfying the following condition: if $b_1 \leq b_2 \leq \cdots \leq b_n$ is the monotonic rearrangement of the terms of α, then $b_i \leq k(i-1)$. A 1-parking function is called simply a *parking function.* Parking functions (defined slightly differently, but equivalent to our definition) were first considered by Konheim and Weiss [11]. Other references include [4][5][12]. (In [12] a sequence $(n+1-a_1, \ldots, n+1-a_n)$, where $(a_1, \ldots, a_n)$ is a parking function, is called a "suite majeure.") See for example [11][4, p. 10][20, p. 2625] for the reason for the terminology "parking function."

The main theorem on the arrangements $\mathcal{S}_n^k$ is the following.

2.1 Theorem. *The labels $\lambda(R)$ of the extended Shi arrangement $\mathcal{S}_n^k$ are just the k-parking functions of length n, each occuring exactly once.*

Proof. For simplicity we will assume here that $k = 1$. A region R of the Shi arrangement $\mathcal{S}_n$ may be thought of as a pair (w, I), where $w \in \mathfrak{S}_n$ and I is a collection of sets $[w(i), w(j)] := \{w(i), w(i+1), \ldots, w(j)\}$ with the following properties: (1) if $[w(i), w(j)] \in I$ then $1 \leq i < j \leq n$ and $w(i) < w(j)$, and (2) the elements of I, ordered by inclusion, form an *antichain* , i.e., no element of I is a subset of another element of I. We regard such a pair (w, I) as defining the region

$$
\begin{aligned}
& x_{w(1)} > x_{w(2)} > \cdots > x_{w(n)}, \\
& x_{w(r)} - x_{w(s)} < 1 \text{ if } [w(r), w(s)] \in I, \\
& x_{w(r)} - x_{w(s)} > 1 \text{ if } r < s, w(r) < w(s), \text{ and no set} \\
& \qquad [w(i), w(j)] \in I \text{ satisfies } i \leq r < s \leq j.
\end{aligned}
$$

In general, define a *valid pair* or *valid t-pair* to be an ordered pair (v, J) where $v = v(1), \ldots, v(t)$ is a permutation of some t-element subset of $\{1, 2, \ldots, n\}$ and J is an antichain of subsets of the form $\{v(i), v(i+1), \ldots, v(j)\}$, where $i < j$. We call the elements ι of J *intervals*, and say that ι is an *interval of* J. If $i < j$, $v(i) < v(j)$, and no interval of J contains both $v(i)$ and $v(j)$, then we say that the pair $(v(i), v(j))$ is *separated.* Similarly if $i < j$ and $v(i) > v(j)$, then we say that the pair $(v(i), v(j))$ is an *inversion.* If (v, J) is a valid t-pair and $1 \leq i \leq t$, then define

$$
\begin{aligned}
F(v, J, i) &= \{j : (i, j) \text{ is an inversion}\} \cup \{j : (i, j) \text{ is separated}\} \\
f(v, J, i) &= \#F(v, J, i).
\end{aligned}
$$

If (w, I) corresponds to the region R, then $(w(i), w(j))$ is an inversion if and only if the hyperplane $x_{w(j)} - x_{w(i)} = 0$ separates R from R_0, while $(w(i), w(j))$ is separated if and only if $x_{w(i)} - x_{w(j)} = 1$ separates R from R_0. There follows

$$
\lambda(R) = (f(w, I, 1), f(w, I, 2), \ldots, f(w, I, n)). \tag{4}
$$

It is easy to see that $\lambda(R)$ is a parking function. Indeed, $f(w, I, w(i))$ cannot exceed $n - i$, the number of elements in w to the right of $w(i)$.

The essence of the proof of the theorem is to show that for every k-parking function α there is a unique region R for which $\lambda(R) = \alpha$. The following lemma on the structure of valid pairs will be of crucial importance.

Lemma. Let (v, J) be a valid pair. Suppose that $i < j$, and that either $(v(i), v(j))$ is an inversion or $(v(i), v(j))$ is separated. Then $f(v, J, v(i)) > f(v, J, v(j))$.

Proof of lemma. Suppose $(v(i), v(j))$ is an inversion. If $h > j$ then $(v(i), v(h))$ is an inversion whenever $(v(j), v(h))$ is an inversion (since $(v(i) > v(j))$. Suppose now that $(v(j), v(h))$ is separated. If $v(h) < v(i)$ then $(v(i), v(h))$ is an inversion. On the other hand, if $v(h) > v(i)$ then $(v(i), v(h))$ is separated (since any interval containing $v(i)$ and $v(h)$ would also contain $v(j)$). Hence $f(v, J, v(i)) \geq f(v, J, v(j))$. But since $(v(i), v(j))$ is an extra inversion not yet taken into account, we have strict inequality.

A similar argument works when $(v(i), v(j))$ is separated. If $h > j$ then $(v(i), v(h))$ is separated whenever $(v(j), v(h))$ is separated. Suppose now that $(v(j), v(h))$ is an inversion. If $v(i) > v(h)$ then $(v(i), v(h))$ is an inversion, while if $v(i) < v(h)$ then $(v(i), v(h))$ is separated. Thus $f(v, J, v(i)) \geq f(v, J, v(j))$, and we get strict inequality since $(v(i), v(j))$ is separated. This completes the proof of the lemma.

Now consider a parking function $\alpha = (a_1, \ldots, a_n)$, such as

$$\alpha = (2, 3, 0, 0, 7, 2, 3, 0, 3). \tag{5}$$

We will build up the pair (w, I) corresponding to the region R satisfying $\lambda(R) = \alpha$ one step at a time. After the mth step we will have a valid m-pair (w^m, I^m). Let $b_1, b_2, \ldots, b_n$ be the permutation of $1, 2, \ldots, n$ obtained by listing the indices (coordinates) of the smallest terms of α from right-to-left, then the indices of the next smallest terms from right-to-left, etc. For α given by (5) we have $b_1, \ldots, b_9 = 8, 4, 3, 6, 1, 9, 7, 2, 5$. Then w^m will be a permutation of $b_1, \ldots, b_m$, obtained by inserting b_m into a certain position of w^{m-1}, while I^m will be obtained from I^{m-1} by adjoining a certain interval (possibly empty) $[b_m, c_m]$ and removing any interval properly contained in another (so that I^m remains an antichain).

If $\lambda(R) = \alpha$, then by (4) we need that $f(w, I, i) = a_i$ for all i. It follows from the lemma that we must insert b_m into w^{m-1} so that $f(w^{m-1}, I^{m-1}, h) = f(w^m, I^m, h)$ for all terms h of w^{m-1}. This means that b_m cannot be inserted to the right of a larger element, and cannot be inserted to the right of a smaller element c unless there is some $d > c$ to the right of b such that (c, d) is not separated. Moreover, the interval $[b_m, c_m]$ cannot contain two terms that are separated in (w^{m-1}, I^{m-1}). (I.e., separated pairs stay separated.) We claim that there is exactly one way to insert b_m and to choose I^m according to these rules, so that $f(w^m, I^m, b_m) = a_i$.

First note that once we decide where to insert b_m, say after $w^{m-1}(p)$ (or at the beginning, in which case we set $p = 0$), then the interval $[b_m, c_m]$ in uniquely determined (if it exists at all) by the condition $f(w^m, I^m, b_m) = a_i$. Thus if there are two ways to insert b_m, then we must insert b_m into different places of w^{m-1}, say after $w^{m-1}(p)$ and $w^{m-1}(j)$, where $p < j$, to get permutations w^m and $\bar{w}^m$, respectively. Let $[b_m, c_m]$ be the interval corresponding to the insertion of b_m after $w^{m-1}(p)$, and similarly $[b_m, d_m]$ for $w^{m-1}(j)$. By the lemma, we have that $w^{m-1}(j-1) < b_m$. Thus $c_m < d_m$, so (b_m, d_m) is separated in w^m. Therefore $(w(j-1), d_m)$ is separated in w^m, so also in w^{m-1}. But then by the lemma $(w(j-1), d_m)$ must remain separated in $\bar{w}^m$, a contradiction. Hence there is at most one choice of w^m and I^m for each m, and in particular at most one choice of the pair $(w, I) = (w^n, I^n)$.

The above argument shows that the map $R \mapsto \lambda(R)$ from regions to parking functions is injective. Since the number of regions of $\mathcal{S}_n$ is known to equal $(n+1)^{n-1}$ [15, Thm. 7.3.1], and similarly for the number of parking functions of length n [4][12], the proof follows for the case $k = 1$.

For general k, the proof is analogous but more complicated. The regions of $\mathcal{S}_n^k$ are specified by a $(k+1)$-tuple $(w, I_1, \ldots, I_k)$, where $w \in \mathfrak{S}_n$ and $I_1, \ldots, I_k$ are antichains of subsets of $\{1, 2, \ldots, n\}$ of the form $\{w(i), w(i+1), \ldots, w(j)\}$. The permutation w specifies the order of the coordinates (as in the case $k = 1$), and the antichains I_m specify which coordinates are with distance m of each other. There are certain compatibility conditions which w and the I_m's must satisfy. Given a k-parking function $\alpha = (a_1, \ldots, a_n)$, we build up $(w, I_1, \ldots, I_k)$ one step at a time as before, inserting elements in the same order as for $k = 1$, i.e, first the coordinates in descending order of the smallest terms of α, then the coordinates in descending order of the next smallest terms of α, etc. There will always be a unique choice with the necessary properties. The details are tedious and will be omitted. $\square$

Example. For the example (5), the successive valid pairs (w^m, I^m) are as follows (beginning with $m = 3$):

$$\begin{array}{rl} 348, & \{[3,8]\} \\ 6348, & \{[6,8]\} \\ 16348, & \{[1,3],[6,8]\} \\ 169348, & \{[1,3],[6,8]\} \\ 1769348, & \{[1,3],[7,8]\} \\ 21769348, & \{[2,3],[7,8]\} \\ 521769348, & \{[5,7],[2,3],[7,8]\}. \end{array}$$

Combining equation (3) and Theorem 2.1, we obtain the following corollary.

2.2 Corollary. *The distance enumerator of the extended Shi arrangement $\mathcal{S}_n^k$ is given by*

$$D_{\mathcal{S}_k^n}(q) = \sum_{(a_1,\ldots,a_n)} q^{a_1+\cdots+a_n},$$

where $(a_1, \ldots, a_n)$ ranges over all k-parking functions of length n.

If we let $k \to \infty$ in our labeling λ of the regions of $\mathcal{S}_n^k$, then we obtain a labeling of the regions of the affine braid arrangement $\tilde{\mathcal{B}}_n$ by vectors $(a_1, \ldots, a_n) \in \mathbb{N}^n$ such that at least one $a_i = 0$. Hence, letting $\mathbb{P} = \{1, 2, \ldots\}$, we get

$$\begin{aligned} D_{\tilde{\mathcal{B}}_n}(q) &= \sum_{\substack{(a_1,\ldots,a_n)\in\mathbb{N}^n \\ \text{not all } a_i>0}} q^{a_1+\cdots+a_n} \\ &= \sum_{(a_1,\ldots,a_n)\in\mathbb{N}^n} q^{a_1+\cdots+a_n} - \sum_{(a_1,\ldots,a_n)\in\mathbb{P}^n} q^{a_1+\cdots+a_n} \\ &= \frac{1}{(1-q)^n} - \frac{q^n}{(1-q)^n} \\ &= \frac{1+q+\cdots+q^{n-1}}{(1-q)^{n-1}}, \end{aligned}$$

agreeing with (2). Björner and Brenti [1, §4] describe a labeling of elements of $\tilde{S}_n$ by sequences in $\mathbb{N}^n \backslash \mathbb{P}^n$; presumably this labeling is equivalent to ours.

3. Enumeration of k-parking functions. Corollary 2.2 is not an entirely satisfactory "determination" of $D_{\mathcal{S}_n^k}(q)$ since it does not lead immediately to any explicit formulas, generating functions, recurrences, etc. We need a better understanding of k-parking functions. First let us recall the well-known situation for the case $k = 1$. A *rooted forest on* $[n]$ is a graph on the vertex set $[n] = \{1, 2, \ldots, n\}$ for which every connected component is a rooted tree. An *inversion* of a rooted forest F is a pair (i, j) for which $i < j$, and j lies on the unique path connecting k to i, where k is the root of the tree to which i belongs. Let $\operatorname{inv}(F)$ denote the number of inversions of F. The *inversion enumerator* $I_n(q)$ for labelled forests on $[n]$ is defined to be the polynomial

$$I_n(q) = \sum_F q^{\operatorname{inv}(F)},$$

where F ranges over all labelled forests on $[n]$. (Often $I_n(q)$ is called the inversion enumerator of trees on $n+1$ (labelled) vertices. A tree T can be obtained from the rooted forest F by adjoining a new vertex 0 and connecting it to the roots of F.) Since it is well-known that there are $(n+1)^{n-1}$ rooted forests on $[n]$, we have $I_n(1) = (n+1)^{n-1}$. Some values of $I_n(q)$ for small n are

$$\begin{aligned}
I_1(q) &= 1 \\
I_2(q) &= q+2 \\
I_3(q) &= q^3+3q^2+6q+6 \\
I_4(q) &= q^6+4q^5+10q^4+20q^3+30q^2+36q+24 \\
I_5(q) &= q^{10}+5q^9+15q^8+35q^7+70q^6+120q^5+180q^4 \\
&\quad +240q^3+270q^2+240q+120.
\end{aligned}$$

The next result summarizes the fundamental properties of $I_n(q)$. Property (a) is implicit in Mallows and Riordan [13], and appears more explicitly in [12]. An elegant bijective proof was given by Gessel and Wang [7]. Property (b) is equivalent to [13, equation (2)], and appears more explicitly in [6, equation (14.6)]. Finally, property (c) is due to Kreweras [12].

3.1 Theorem. (a) *We have*

$$I_n(1+q) = \sum_G q^{e(G)-n},$$

where G ranges over all connected graphs (without loops or multiple edges) on $n+1$ labelled vertices, and where $e(G)$ denotes the number of edges of G.
(b) *We have the generating function identity*

$$\sum_{n\geq 0} I_n(q)(q-1)^n \frac{x^n}{n!} = \frac{\sum_{n\geq 0} q^{\binom{n+1}{2}} \frac{x^n}{n!}}{\sum_{n\geq 0} q^{\binom{n}{2}} \frac{x^n}{n!}}.$$

(c) *We have*

$$q^{\binom{n}{2}} I_n(1/q) = \sum_{(a_1,\ldots,a_n)} q^{a_1+\cdots+a_n},$$

where $(a_1,\ldots,a_n)$ ranges over all parking functions of length n. Hence from equation (3) and Corollary 2.2 there follows

$$D_{\mathcal{S}_n^1}(q) = q^{\binom{n}{2}} I_n(1/q).$$

We want to extend Theorem 3.1 to $\mathcal{S}_n^k$. First we need to generalize the notion of an inversion of a forest. Define a *rooted k-forest* to be a rooted forest on vertices $1, 2, \ldots, n$ with edges colored with the colors $0, 1, \ldots, k-1$. There is no additional restriction on the possible colors of the edges. Denote the color of an edge e by $\kappa(e)$. Define the *length* $\ell(F)$ of a rooted k-forest F by

$$\ell(F) = \mathrm{inv}(F) + \sum_{(v,e)} \kappa(e), \tag{6}$$

where $\mathrm{inv}(F)$ denotes the number of inversions of F (ignoring the edge colors), and where (v,e) ranges over all vertices v and edges e such that e lies on the unique path from v to the root of the component of F to which v belongs. Define the inversion enumerator $I_n^k(q)$ by

$$I_n^k(q) = \sum_F q^{\ell(F)},$$

where F ranges over all rooted k-forests on $[n]$.

It is easy to see by standard enumerative arguments that there are $(kn+1)^{n-1}$ rooted k-forests on $[n]$, so

$$I_n^k(1) = (kn+1)^{n-1}. \tag{7}$$

Some values of $I_n^k(q)$ for small n and $k > 1$ are as follows:

$$\begin{aligned}
I_1^k(q) &= 1 \\
I_2^2(q) &= q^2 + 2q + 2 \\
I_3^2(q) &= q^6 + 3q^5 + 6q^4 + 9q^3 + 12q^2 + 12q + 6 \\
I_4^2(q) &= q^{12} + 4q^{11} + 10q^{10} + 20q^9 + 34q^8 + 52q^7 + 74q^6 + 96q^5 \\
&\quad +114q^4 + 120q^3 + 108q^2 + 72q + 24 \\
I_2^3(q) &= q^3 + 2q^2 + 2q + 2 \\
I_3^3(q) &= q^9 + 3q^8 + 6q^7 + 9q^6 + 12q^5 + 15q^4 + 18q^3 + 18q^2 + 12q + 6.
\end{aligned}$$

There is a formula for $I_n^k(q)$ in terms of *unlabelled* rooted forests (though in Theorem 3.3(b) we will give a more explicit formula in terms of generating functions). Let φ be an unlabelled rooted forest, with vertex set $V(\varphi)$. Regard φ as a poset whose maximal elements are the roots. Given a vertex $v \in V(\varphi)$, let $h_v = \#\{u \in V(\varphi) : u \le v\}$. Let $e(\varphi)$ denote the number of linear extensions of φ (as defined e.g in [19, p. 110]), and let $[j] = 1 + q + \cdots + q^{j-1}$, the q-analogue of the nonnegative integer j. If $\#V(\varphi) = n$, then it is well-known [18, §22] that

$$e(\varphi) = \frac{n!}{\prod_{v \in V(\varphi)} h_v}.$$

It was observed by Björner and Wachs [2, Thm. 1.3] that

$$\sum_F q^{\mathrm{inv}(F)} = e(\varphi) \prod_{v \in V(\varphi)} [h_v],$$

where F ranges over all $n!$ labelings of φ. If $a(\varphi)$ denotes the order of the automorphism group of φ, then the $n!$ labelings of φ include $a(\varphi)$ copies of each nonisomorphic labelled rooted forest whose underlying unlabelled rooted forest is φ. Hence

$$\sum_F q^{\mathrm{inv}(F)} = \frac{e(\varphi)}{a(\varphi)} \prod_{v \in V(\varphi)} [h_v], \tag{8}$$

where now F ranges over all nonisomorphic labelled rooted forests whose underlying unlabelled rooted forest is φ.

3.2 Theorem. *We have*

$$I_n^k(q) = \sum_\varphi \frac{e(\varphi)}{a(\varphi)} \prod_{\substack{v \in V(\varphi) \\ v \text{ not a root of } \varphi}} [kh_v] \cdot \prod_{\substack{v \in V(\varphi) \\ v \text{ a root of } \varphi}} [h_v],$$

where φ ranges over all nonisomorphic (unlabelled) rooted forests with n vertices.

Proof. By the definition (6) of $\ell(F)$ for a labelled rooted forest F, there is a contribution to $\ell(F)$ from the vertex labeling, and a completely independent contribution from the edge coloring. Given F, denote by $\Upsilon(F)$ the underlying unlabelled rooted forest, i.e., erase the vertex labels and edge colors. It follows that for fixed φ we have

$$\sum_{F : \Upsilon(F) = \varphi} q^{\ell(F)} = \left(\sum_{F'} q^{\mathrm{inv}(F')} \right) \left(\sum_\kappa q^{\sum_{(v,e)} \kappa(e)} \right), \tag{9}$$

where (a) F' ranges over all nonisomorphic vertex labelings of φ, (b) κ ranges over all edge k-colorings of φ, and (c) (v, e) is as in equation (6). By (8), the sum over F' is equal to $\frac{e(\varphi)}{a(\varphi)} \prod_{v\in V(\varphi)}[h_v]$. Let e be an edge of φ, and let t be the vertex of e farthest from the root of its component. If e is colored $\kappa(e)$ in the labeling F of φ, then $\kappa(e)$ is counted h_t times in the sum on the right-hand side of (6). Since all edges are colored independently, we get that the sum over κ in (9) is equal to

$$\prod_{\substack{v\in V(\varphi)\\ v \text{ not a root of } \varphi}} \left(1 + q^{h_v} + q^{2h_v} + \cdots + q^{(k-1)h_v}\right).$$

Since $[h_v]\cdot\left(1 + q^{h_v} + q^{2h_v} + \cdots + q^{(k-1)h_v}\right) = [kh_v]$, the proof follows by summing (9) over all φ. $\square$

Next we define an extension of the notion of a connected graph (in order to generalize Theorem 3.1(a)). A *multirooted k-graph* is a graph G on the vertex set $\{1, 2, \ldots, n\}$ such that (a) a subset S of the vertices is chosen as a set of "roots," with the restriction that every connected component of G contains at least one root, and (b) the edges are colored from a set of k colors. We do not allow loops (edges from a vertex to itself) and multiple edges of the same color. However, it is permissible to have several edges between two distinct vertices as long as they all have different colors. Denote by $e(G)$ the number of edges of G and by $r(G)$ the number of roots. The concept of a multirooted 2-graph is due to Huafei Yan [21], who proved Theorem 3.2(a) below in the case $k = 2$. It was then routine to extend this result to arbitrary k.

There is a simple bijection between multirooted 1-graphs G on $[n]$ and connected graphs G' on $[n+1]$, as follows. Since $k = 1$ the color of the edges of G are all the same and can be ignored. Adjoin a new vertex $n + 1$ to G, and connect it to all the roots of G, yielding a connected graph G' on $[n + 1]$. This gives the desired bijection. Note that $e(G') = e(G) + r(G)$. Thus multirooted k-graphs are indeed a generalization of connected graphs.

We can now give our extension of Theorem 3.1.

3.3 Theorem. (a) *We have*

$$I_n^k(1 + q) = \sum_G q^{e(G)+r(G)-n}, \tag{10}$$

where G ranges over all multirooted k-graphs on $[n]$.

(b) *We have the generating function identity*

$$\sum_{n\geq 0} I_n^k(q)(q-1)^n \frac{x^n}{n!} = \frac{\sum_{n\geq 0} q^{k\binom{n}{2}+n} \frac{x^n}{n!}}{\sum_{n\geq 0} q^{k\binom{n}{2}} \frac{x^n}{n!}}.$$

(c) *We have*

$$q^{k\binom{n}{2}} I_n(1/q) = \sum_{(a_1,\ldots,a_n)} q^{a_1+\cdots+a_n},$$

where $(a_1, \ldots, a_n)$ ranges over all k-parking functions of length n. Hence from equation (3) and Theorem 3.1 there follows

$$D_{\mathcal{S}_n^k}(q) = q^{k\binom{n}{2}} I_n^k(1/q).$$

Proof. (a) We follow the proof of the $k=1$ case in [12]. For $k=2$ the argument is due to H. Yan [21]. We first claim that $I_n^k(q)$ satisfies the recurrence

$$I_{n+1}^k(q) = \sum_{a_1+a_2+\cdots+a_{k+1}=n} \binom{n}{a_1,a_2,\ldots,a_{k+1}} q^{a_2+2a_3+3a_4+\cdots+(k-1)a_k}$$

$$\cdot\left(1+q+\cdots+q^{a_1+a_2+\cdots+a_k}\right) I_{a_1}^k(q) I_{a_2}^k(q)\cdots I_{a_{k+1}}^k(q), \qquad (11)$$

where $a_1,a_2,\ldots,a_{k+1}$ are nonnegative integers, and where $\binom{n}{a_1,a_2,\ldots,a_{k+1}}$ denotes a multinomial coefficient. To prove (11), let $F_1,\ldots,F_{k+1}$ be rooted k-forests on disjoint vertex sets whose union is $[n]$. Let a_i be the number of vertices of F_i. We will "merge" these rooted k-forests into a single rooted k-forest F on $[n+1]$ as follows. The components of F_{k+1} remain components of F. Let the vertices of $F_1,\ldots,F_k$ be $u_1<u_2<\cdots<u_r$, where $r=a_1+\ldots+a_k$. Define $u_{r+1}=n+1$. Choose an integer $1\le j\le r+1$, and for all $m\ge j$ replace vertex u_m in whatever forest it appears by u_{m+1}. (If $j=r+1$ then there is nothing to replace.) We have replaced $F_1,\ldots,F_k$ with isomorphic rooted k-forests $F_1',\ldots,F_k'$ whose vertices are $u_1,\ldots,u_{r+1}$ with u_j omitted. Now let $F_1',\ldots,F_k'$ be the subtrees of a root u_j, and put color $i-1$ on the edge connecting u_j with the root of F_i'. Putting together this tree T with the forest F_{k+1} gives a rooted k-forest F on $[n+1]$.

For each solution to $a_1+\cdots+a_{k+1}=n$ in nonnegative integers, there are $\binom{n}{a_1,\ldots,a_{k+1}}$ choices for the vertex sets of $F_1,\ldots,F_{k+1}$. There are also $r+1$ choices for the integer j. We then get an additional $j-1$ ordinary inversions of F (each involving the root vertex u_j of T), in addition to the inversions already appearing in $F_1,\ldots,F_{k+1}$. Moreover, for each $1\le i\le k$, we get an additional a_i pairs (v,e), where v is a vertex of T and e is an edge colored $i-1$ on the path from v to the root u_j. Namely, v is any vertex of F_i', and e is the last edge on the path from v to u_j. Thus the length enumerator for those rooted k-forests F obtained by fixing $a_1,\ldots,a_{k+1}$ and j is given by

$$q^{a_2+2a_3+3a_4+\cdots+(k-1)a_k}\cdot q^{j-1} I_{a_1}^k(q) I_{a_2}^k(q)\cdots I_{a_{k+1}}^k(q).$$

Summing over all $a_1+\cdots+a_{k+1}=n$ and all $1\le j\le r$ yields equation (11).

Let $J_n^k(q)$ denote the right-hand side of equation (10). It is clear that $J_0^k(q) = I_0^k(q+1)=1$. Hence it suffices to show that $J_n^k(q)$ satisfies the same recurrence as $I_n^k(1+q)$, viz.,

$$J_{n+1}^k(q) = \sum_{a_1+a_2+\cdots+a_{k+1}=n} \binom{n}{a_1,a_2,\ldots,a_{k+1}} (1+q)^{a_2+2a_3+3a_4+\cdots+(k-1)a_k}$$

$$\cdot\frac{(1+q)^{a_1+a_2+\cdots+a_k+1}-1}{q} J_{a_1}^k(q) J_{a_2}^k(q)\cdots J_{a_{k+1}}^k(q), \qquad (12)$$

To prove (12), let $G_1,\ldots,G_{k+1}$ be multirooted k-graphs on disjoint vertex sets whose union is $[n]$. Let the colors of the edges of G be $1,2,\ldots,k$. Let a_i be the number of vertices of G_i. We will "merge" these multirooted k-graphs into a single multirooted k-graph G on $[n+1]$, as follows. Adjoin a new vertex $n+1$, and for each $1\le i\le k$, draw an edge colored i from $n+1$ to the roots of G_i. Also draw any number of edges with colors less than i from $n+1$ to the vertices of G_i (as long as there are no multiple edges of the same color). We now have a connected graph H with colored edges. "Erase" the roots of H, and choose any nonempty subset of the vertices of H

to be a new set of roots. Taking the disjoint union of H with G_{k+1} gives a multirooted k-graph G on $[n+1]$.

The above procedure yields a bijection between multirooted k-graphs G on $[n+1]$ and sequences $\Gamma = (G_1, \ldots, G_{k+1}, E_1, \ldots, E_k, S)$, where the G_i's are multirooted k-graphs on disjoint vertex sets whose union is $[n]$, where E_i is a set of edges colored $1, 2, \ldots, i-1$ connecting $n+1$ with vertices in G_i, and where S is a nonempty subset of the union of the vertices of $G_1, \ldots, G_k$, together with the vertex $n+1$. Write $\nu(K)$ for the number of vertices of the graph K. We then have

$$\begin{aligned}
\sum_G q^{e(G)+r(G)-n} &= \sum_{a_1+\cdots+a_{k+1}=n} \sum_{\substack{(G_1,\ldots,G_{k+1},E_1,\ldots,E_k,S)\\ \nu(G_i)=a_i}} q^{\#E_1} \cdots q^{\#E_k} q^{\#S} \\
&\quad \cdot \quad q^{e(G_1)+r(G_1)-a_1} \cdots q^{e(G_{k+1})+r(G_{k+1})-a_{k+1}} \\
&= \sum_{a_1+\ldots+a_{k+1}=n} (1+q)^{a_2}(1+q)^{2a_3} \cdots (1+q)^{(k-1)a_k} \\
&\quad \cdot \left((1+q)^{a_1+\ldots+a_k} - 1\right) J^k_{a_1}(q) \cdots J^k_{a_{k+1}}(q).
\end{aligned}$$

Now divide both sides by q. The left-hand side becomes $J^k_{n+1}(q+1)$, while the right-hand side agrees with the right-hand side of equation (12). This completes the proof of (a).

(b) Let

$$C^k_n(q) = \sum_G q^{e(G)},$$

where G ranges over all connected graphs on $[n]$ with k-colored edges, with no loops and with no multiple edges of the same color. (We do not choose a set of roots of G.) Without the condition that G is connected, the corresponding generating function is clearly $(1+q)^{k\binom{n}{2}}$. Hence by the exponential formula (e.g., [17, Cor. 6.2]), we have

$$\begin{aligned}
F^k(q) &:= \sum_{n\geq 1} C^k_n(q) \frac{x^n}{n!} \\
&= \log \sum_{n\geq 0} (1+q)^{k\binom{n}{2}} \frac{x^n}{n!}.
\end{aligned}$$

We get a multirooted k-graph on $[n]$ by choosing a partition $\pi = \{B_1, \ldots, B_j\}$ of the set $[n]$, placing a graph enumerated by $C^k_n(q)$ on each block B_i, and choosing a nonempty subset of B_i. Hence

$$q^n J^k_n(q) = \sum_{\pi=\{B_1,\ldots,B_j\}} C^k_{b_1}(q) \cdots C^k_{b_j}(q)[(1+q)^{b_1} - 1] \cdots [(1+q)^{b_j} - 1],$$

where π ranges over all partitions of $[n]$ and $b_i = \#B_i$. Again by the exponential formula we get

$$\begin{aligned}
\sum_{n\geq 0} q^n J^k_n(q) \frac{x^n}{n!} &= \exp\left(F^k((1+q)x) - F^k(x)\right) \\
&= \exp\left(\log \sum_{n\geq 0} (1+q)^{k\binom{n}{2}} \frac{(1+q)^n x^n}{n!} - \log \sum_{n\geq 0} (1+q)^{k\binom{n}{2}} \frac{x^n}{n!}\right) \\
&= \frac{\sum_{n\geq 0} (1+q)^{k\binom{n}{2}+n} \frac{x^n}{n!}}{\sum_{n\geq 0} (1+q)^{k\binom{n}{2}} \frac{x^n}{n!}}.
\end{aligned}$$

Now substitute $q-1$ for q and use (a) to get the desired formula.

(c) A proof was given by Huafei Yan and appears in [21]. (Another proof was later found by Igor Pak.) Yan's proof is based on the following. Let

$$P_n^k(q) = \sum_{(a_1,\dots,a_n)} q^{k\binom{n}{2}-a_1-\cdots-a_n},$$

where $(a_1,\dots,a_n)$ ranges over all k-parking functions of length n. Yan then gives a combinatorial proof of the recurrence

$$P_{n+1}^k(q) = \sum_{i=0}^n \binom{n}{i} \left(1+q+\cdots+q^{k-1}\right)^{n-i} \left(1+q+\cdots+q^{ki}\right) P_i^k(q) P_{n-i}^1(q^k).$$

She then shows combinatorially that

$$J_{n+1}^k(q) = \sum_{i=0}^n \binom{n}{i} \left(1+(1+q)+\cdots+(1+q)^{k-1}\right)^{n-i} \left(1+(1+q)+\cdots+(1+q)^{ki}\right) J_i^k(q) J_{n-i}^1((1+q)^k-1),$$

and the proof follows from (a). $\square$

NOTE. It follows from Theorem 3.3(c) and equation (7) that there are $(kn+1)^{n-1}$ k-parking functions of length n. A direct way to see this, generalizing an argument due essentially to Pollack [4, §2] for the case $k=1$, is as follows. Let H be the subgroup of $(\mathbb{Z}/(kn+1)\mathbb{Z})^n$ generated by $(1,1,\dots,1)$. Then it is not difficult to show that every coset of H contains exactly one k-parking function, and the result follows. This argument shows that the set of k-parking functions of length n has the natural structure of an abelian group isomorphic to $(\mathbb{Z}/(kn+1)\mathbb{Z})^{n-1}$. It might be interesting to see if this group structure can be exploited in some way in the study of the extended Shi arrangements and rooted k-trees.

NOTE. There is a natural two variable polynomial $D_n^k(q,t)$ that refines the distance enumerator $D_{\mathcal{S}_n^k}(q)$. Namely, define

$$D_n^k(q,t) = \sum_R q^{a(R)} t^{b(R)},$$

where (a) R ranges over all regions of $\mathcal{S}_n^k$, (b) $a(R)$ is the number of hyperplanes $x_i - x_j = m$, where $1 \le i < j \le n$ and $m > 0$, which separate R from R_0, and (c) $b(R)$ is the number of hyperplanes $x_i - x_j = m$, where $1 \le i < j \le n$ and $m \le 0$, which separate R from R_0. Thus $D_n^k(q,q) = D_{\mathcal{S}_n^k}(q)$. The coefficients of $D_n^1(q,t)$ for $2 \le n \le 4$ are given by the following tables:

$t \backslash q$	0	1
0	1	1
1	1	

$t \backslash q$	0	1	2	3
0	1	1	2	1
1	2	2	2	
2	2	2		
3	1			

$t \backslash q$	0	1	2	3	4	5	6
0	1	1	2	3	3	3	1
1	3	3	6	7	6	3	
2	5	5	8	9	5		
3	6	7	9	6			
4	5	6	5				
5	3	3					
6	1						

We do not know a direct interpretation of $D_n^k(q,t)$ in terms of rooted k-forests or k-parking functions, nor do we know of any simple recurrences or generating functions for $D_n^k(q,t)$ (though it is easy to describe $D_n^k(q,0)$, $D_n^k(0,t)$, and the coefficients of the terms of total degree $k\binom{n}{2}$). We also don't know a generalization of Theorem 3.3(a) involving $D_n^k(q,t)$. Let us note that Haiman [8] has a (conjectured) two variable refinement of $(n+1)^{n-1}$ which is completely different from our $D_n^1(q,t)$. We don't know of any direct connection between our work and Haiman's, and such a connection remains an intriguing area of investigation.

Acknowledgment. Research partially supported by NSF grant DMS-9500714. I am grateful to Donald Knuth for pointing out several inaccuracies in the original version of this paper.

References

[1] A. Björner and F. Brenti, Affine permutations of type A, *Elec. J. Combinatorics* **3(2)** (1996). Available at the URL http://ejc.math.gatech.edu:8080/Journal/Volume_3/foatatoc.html.

[2] A. Björner and M. L. Wachs, q-hook length formulas for forests, *J. Combinatorial Theory (A)* **52** (1989), 165–187.

[3] N. Bourbaki, *Groupes et algèbres de Lie*, Ch. 4, 5, et 6, Éléments de Mathématique, fasc. XXXIV, Hermann, Paris, 1968.

[4] D. Foata and J. Riordan, Mappings of acyclic and parking functions, *aequationes math.* **10** (1974), 10–22.

[5] J. Françon, Acyclic and parking functions, *J. Combinatorial Theory (A)* **18** (1975), 27–35.

[6] I. Gessel, A noncommutative generalization and q-analog of the Lagrange inversion formula, *Trans. Amer. Math. Soc.* **257** (1980), 455–482.

[7] I. Gessel and D.-L. Wang, Depth-first search as a combinatorial correspondence, *J. Combinatorial Theory (A)* **26** (1979), 308–313.

[8] M. Haiman, Conjectures on the quotient ring by diagonal invariants, *J. Algebraic Combinatorics* **3** (1994), 17–76.

[9] P. Headley, Reduced expressions in infinite Coxeter groups, Ph.D. thesis, University of Michigan, Ann Arbor, 1994.

[10] P. Headley, On reduced expressions in affine Weyl groups, in *Formal Power Series and Algebraic Combinatorics, FPSAC '94, May 23–27, 1994*, DIMACS preprint, pp. 225–232.

[11] A. G. Konheim and B. Weiss, An occupancy discipline and applications, *SIAM J. Applied Math.* **14** (1966), 1266–1274.

[12] G. Kreweras, Une famille de polynômes ayant plusieurs propriétés énumeratives, *Periodica Math. Hung.* **11** (1980), 309–320.

[13] C. L. Mallows and J. Riordan, The inversion enumerator for labeled trees, *Bull Amer. Math. Soc.* **74** (1968), 92–94.

[14] A. Postnikov and R. Stanley, Deformations of Coxeter hyperplane arrangements, in preparation.

[15] J.-Y. Shi, The Kazhdan-Lusztig cells in certain affine Weyl groups, *Lecture Note in Mathematics*, no. 1179, Springer, Berlin/Heidelberg/New York, 1986,

[16] J.-Y. Shi, Sign types corresponding to an affine Weyl group, *J. London Math. Soc.* **35** (1987), 56–74.

[17] R. Stanley, Generating functions, in *Studies in Combinatorics* (G.-C. Rota, ed.), Mathematical Association of America, Washington, DC, 1978, pp. 100–141.

[18] R. Stanley, Ordered structures and partitions, *Mem. Amer. Math. Soc.*, no. 119, 1972.

[19] R. Stanley, *Enumerative Combinatorics*, vol. 1, Wadsworth and Brooks/Cole, Pacific Grove, CA, 1986; second printing, Cambridge University Press, Cambridge/New York, 1996.

[20] R. Stanley, Hyperplane arrangements, interval orders, and trees, *Proc. Nat. Acad. Sci.* **93** (1996), 2620–2625

[21] C. H. Yan, Generalized tree inversions and k-parking functions, in preparation.

Department of Mathematics 2-375
Massachusetts Institute of Technology
Cambridge, MA 02139