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Note

A Matrix for Counting Paths in Acyclic Digraphs

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We define a matrix A associated with an acyclic digraph Γ, such that the coefficient of z^j in $\det(I+zA)$ is the number of j-vertex paths in Γ. This result is actually a special case of a more general weighted version. © 1996 Academic Press, Inc.

We will define a matrix $A=A_\Gamma$ associated with an acyclic digraph Γ, such that the coefficients of the characteristic polynomial of A enumerate the paths in Γ according to their length. Our result is actually an easy consequence of a more general theorem of Goulden and Jackson, but this special case seems never to have been explicitly noted before. We will give two simple proofs, the first of which is essentially a specialization of the proof of Goulden and Jackson.

Let Γ be an acyclic digraph without multiple edges on the vertex set $V=\{x_1, ..., x_n\}$. We say that Γ is *natural* if $i<j$ whenever (x_i, x_j) is an edge. We may assume without loss of generality throughout this paper that Γ is natural. Regard $x_1, ..., x_n$ as indeterminates, and define the diagonal matrix $D=\operatorname{diag}(x_1, ..., x_n)$. Define the $n\times n$ matrix $A=A_\Gamma$ by

$$A_{ij}=\begin{cases}0, & \text{if } (x_i, x_j) \text{ is an edge of } \Gamma\\ 1, & \text{otherwise.}\end{cases} \tag{1}$$

Let I denote the $n\times n$ identity matrix.

THEOREM. *We have*

$$\det(I+z\,DA)=\sum_{j=0}^{n}\left(\sum_P x_{k_1}\cdots x_{k_j}\right)z^j, \tag{2}$$

where P ranges over all paths $x_{k_1}\cdots x_{k_j}$ in Γ with j vertices.

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First Proof. The following result appears in [3, Lemma 3.12]. Let R and S be $n\times n$ matrices such that $R+S=J$, the all 1's matrix. Let X be any $n\times n$ matrix. Then

$$1+\operatorname{trace}(I-XR)^{-1}\, XJ=\frac{\det(I+XS)}{\det(I-XR)}. \tag{3}$$

This result is an immediate consequence of the fact that the matrix $(I-XR)^{-1}\, XJ$ has rank one, and that for any matrix M of rank one we have $1+\operatorname{trace}(M)=\det(I+M)$.

In equation (3) let $X=z\,D$ and $S=A$. Since Γ is natural the matrix XR is strictly upper triangular, so $\det(I-XR)=1$ and the right-hand side becomes $\det(I+z\,DA)$. On the other hand, the left-hand side becomes

$$1+\operatorname{trace}(I-z\,D(J-A))^{-1}\, z\,DJ=1+\operatorname{trace}\sum_{i\geqslant 0} z^{i+1}[D(J-A)]^i\, DJ.$$

Now $J-A$ is just the adjacency matrix of Γ, so

$$\operatorname{trace}[D(J-A)]^i\, DJ=\sum_P x_{k_1}\cdots x_{k_{i+1}},$$

where P ranges over all paths $x_{k_1}\cdots x_{k_{i+1}}$, and the proof follows. ∎

Second Proof. The coefficient of z^j in $\det(I+z\,DA)$ is the sum of the principal $j\times j$ minors of DA. The rows and columns of a principal submatrix $DA[W]$ are indexed by a j-element subset W of the vertex set V. We claim that

$$\det DA[W]=\begin{cases}\prod_{x\in W} x, & \text{if } W \text{ is the set of vertices of a path}\\ 0, & \text{otherwise.}\end{cases}$$

from which the proof of the theorem is immediate. Note that

$$\det DA[W]=\left(\prod_{x\in W} x\right)\det(A[W]).$$

Hence we need to show that

$$\det A[W]=\begin{cases}1, & \text{if } W \text{ is is the set of vertices of a path}\\ 0, & \text{otherwise.}\end{cases} \tag{4}$$

If W is the set of vertices of a path, then since Γ is natural the matrix $A[W]$ is upper triangular with 1's on the diagonal, so (4) is true in this case. Suppose that W is not the set of vertices of a path. Since Γ is natural the matrix $A[W]$ has all its entries on or below the main diagonal equal to 1, and at least one entry on the diagonal just above the main diagonal is equal to 1. The proof of (4) thus follows from the following lemma.

LEMMA. *Let $B=(b_{ij})$ be an $n\times n$ matrix such that $b_{ij}=1$ if $i\geqslant j$. Then*

$$\det B=\prod_{i=1}^{n-1}(1-b_{i,i+1}).$$

Proof of Lemma. The proof is by induction on n, the case $n=1$ being trivial. Expand $\det B$ by the first row. By induction, the first two terms are

$$\prod_{i=2}^{n-1}(1-b_{i,i+1})-b_{12}\cdot\prod_{i=2}^{n-1}(1-b_{i,i+1})=\prod_{i=1}^{n-1}(1-b_{i,i+1}).$$

In the remaining terms, the first two columns of the cofactor are equal, so the term is 0. The proof of the lemma, and with it the theorem, follows. ∎

Define the *path polynomial* $P_\Gamma(z)=\sum c_j z^j$ of a digraph Γ by letting c_j be the number of j-vertex paths in Γ. For instance, if Γ is the graph of proper relations of a poset Q (i.e., (x, y) is an edge of Γ if $x<y$ in Q), then c_j is just the number of j-element chains of Q. We then write $P_Q(z)$ for $P_\Gamma(z)$. The following corollary follows immediately from (2) by setting each $x_i=1$.

COROLLARY. *Let Γ be an acyclic digraph without multiple edges, and let A be as in* (1). *Then*

$$P_\Gamma(z)=\det(I+zA).$$

There is considerable interest in determining when all the zeros of the polynomial $P_\Gamma(z)$ are real, or equivalently, when the matrix A_Γ has real eigenvalues. For instance, it is equivalent to [4, Conj. 1] that if Q is a distributive lattice, then the polynomial $P_Q(z)$ has only real zeros. It is not difficult to see that if Γ is the digraph of strict incidences of a (natural) semiorder, as defined in [2, p. 18], then A_Γ is totally positive (i.e., every minor is nonnegative). Since totally positive matrices have real eigenvalues [1, Thm. 6.2], it follows that the polynomial $P_Q(z)$ has real zeros for any semiorder Q. However, using deeper aspects of the theory of total positivity, an even more general result is proved in [5, Cor. 2.9].

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