Affine Hecke Algebras via DAHA

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Abstract. This is the lecture delivered at the conference "Algebraic Analysis and Representation Theory" in honor of Professor Masaki Kashiwara's 70th birthday. Its main topic is the project aimed at obtaining the Plancherel formula for the regular representation of Affine Hecke Algebras (AHA) as the limit $q \to 0$ of the integral-type formulas for DAHA inner products in the polynomial and related modules. The integrals for the latter as $\Re(k) > 0$ (in the DAHA parameters) must be analytically continued to negative $\Re(k)$, which is a q-generalization of "picking up the residues" due to Arthur, Heckman, Opdam and others, which can be traced back to Hermann Weyl. We arrive at finite sums of integrals over double affine residual subtori. This is not related to the DAHA irreducibility of the polynomial and similar modules, though such formulas can be used for their DAHA stratification when these modules become reducible for singular (negative) k.

The decomposition of the regular AHA representation in terms of (unitary) irreducible modules is an important part of algebraic harmonic analysis, involving highly non-trivial geometric methods (Kazhdan-Lusztig and others). As a possible application, our

approach would allow to interpret formal degrees of AHA discrete series via DAHA. We mainly discuss the spherical case and provide the analytic continuations for A_1. The key is that the uniqueness of the DAHA inner product fixes the corresponding AHA decomposition (and q-counterparts of formal degrees) uniquely.

Even in the spherical case, the procedure of analytic continuation to $\Re(k) < 0$ is technically involved. There are no significant theoretical challenges here, but finding double affine residual subtori and their contributions is performed (partially) only for A_n at the moment. The passage to the whole regular representation will presumably require the technique of hyperspinors, which we outline a bit in the lecture. Importantly, there is no canonical AHA-type trace in the DAHA theory; instead, we analyze coinvariants serving DAHA anti-involutions. There are of course other aspects of DAHA harmonic analysis (the unitary dual, calculating Fourier transforms of DAHA modules and so on); we touch them a bit but mostly focus on the AHA aspects in the case of A_1. Only few (basic) references are provided; see there for further information.

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Affine Hecke Algebras via DAHA
(Integral formulas for DAHA inner products)

A. On Fourier Analysis
B. AHA-decomposition
C. Shapovalov pairs
D. Rational DAHA (A_1)
E. General DAHA (A_1)
F. Inner products
G. Analytic continuation
H. P-adic limit
I. Jantzen filtrations etc.

Warmest congratulations to Kashiwara sensei (70!)
A GREAT MASTER OF HARMONIC ANALYSIS **!!**

HARMONIC ANALYSIS, AHA vs. DAHA:

HA on AHA	HA on DAHA
Unitary(sph) dual	Polynomial/induced modules
Fourier transform	$\mathcal{HH}$- automorphism $X\to Y^{-1}$
Trace formulas, $L^2(\mathcal{H})$	Inner products as integrals

One aim: formal degrees of discrete series via DAHA?!
Kazhdan,Lusztig,Reeder,Shoji,Opdam,Ciubotaru,S.Kato

A. ON FOURIER ANALYSIS

For A_1, FT $= \int e^{2\lambda x}\{\cdot\}\,dx$ is associated with the automorphism $x \leftrightarrow -y$ of "Heisenberg". Its spherical generalization can be related to $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $\in SL_2$, though this interpretation seems a special feature of A_1. Similarly, Weyl algebras at $q = e^{\frac{2\pi\imath}{N}}$ can be used to study $F_N = \sum_{j=0}^{N-1} q^{\lambda j}\{\cdot\}$.

FAMOUS CHALLENGES HERE:

P1. Extending Lie theory from spherical to (any) hypergeometric functions (Gelfand's Program).

P2. Fourier Theory. Can FT be interpreted as a reflection in the Weyl group? Unlikely so. Say, there are 3 candidates (reflections) for FT in SL_3, but FT *must* be unique: polynomials $\mapsto$ δ-functions.

P3. A counterpart of $F(e^{-x^2}) = \sqrt{\pi}e^{+\lambda^2}$ at roots of unity is $F_N(q^{j^2}) = \zeta\,\sqrt{N}q^{-\lambda^2}$ for $\zeta \in \{0, 1, \imath, 1+\imath\}$. The Weyl algebra gives $\sqrt{N}$ but doesn't catch ζ, i.e. it "fails" to calculate the Gauss sums.

DAHA APPROACH: P1,2: *Global* hypergeometric fncts are reproducing kernels of DAHA-FT (its square is essentially id); any root systems were managed. P3 can be settled too (DAHA-Gauss-Selberg sums). What does this give for AHA?

GLOBAL FUNCTIONS $\Phi_{q,t}(X,\Lambda),\ q<1.$[2]

$\widetilde{\Phi}_{q,t}(X,\Lambda) \stackrel{\mathbf{def}}{=\!=} \theta_R(X)\theta_R(\Lambda)\Phi_{q,t}(X,\Lambda)$ **is a "globally" convergent series in terms of $P_\mu(X)P_\mu(\Lambda)$, W-invariant (X and Y), and $X \leftrightarrow \Lambda$-symmetric. Here θ_R is theta associated with a root system R, $\mu \in P$. Letting $X = q^x, \Lambda = q^\lambda$ and assuming that $\lambda = w(\lambda_+)$ for λ_+ such that $\Re(\lambda,\alpha_i)>0$ (i.e. λ is generic) and that $\Re(x,\alpha_i)\to+\infty$, asymptotically:**

$C_{q,t}\Phi_{q,t}(X,\Lambda) \asymp \Phi^{as}_{q,t}(X,\Lambda) \stackrel{\mathbf{def}}{=\!=} q^{-(x,\lambda_+)}(1+\ldots),$ **where $C_{q,t}$ is some explicit product via Λ, q, t.**

Harish-Chandra theory (I.Ch[4, 6], J.Stokman[13]):

$$\Phi_{q,t}(X,\Lambda) = \sum_{w\in W}\sigma_{q,t}(w(\Lambda))\,\Phi^{as}_{q,t}(X,w(\Lambda))$$

for the q,t-extension $\sigma_{q,t}(\Lambda)$ of the Harish-Chandra c-function. Also, $\mathcal{L}_p\Phi_{q,t}(X,\Lambda) = p(\Lambda)\Phi_{q,t}(X,\Lambda)$ for $p\in\mathbb{C}[X]^W$ and Macdonald-Ruijsenaars operators $\mathcal{L}_p$ (Ch. for non-A root systems); but this is NOT used in the formula/theory of $\Phi_{q,t}$. However this eigenvalue problem is important for $\Phi^{as}_{q,t}$ and the justification of the Harish-Chandra formula.

As a matter of fact Φ is an entirely algebraic object, uniquely determined by its asymptotic behavior, including the walls$\sim$resonances (when $\Re(\alpha,\lambda)=0$ for some roots α). Φ^{as} generalizes the *basic* hypergeometric function (Heine, 1846), … , Givental-Lee's q-Whittaker one.

B. AHA-DECOMPOSITION

$R \in \mathbb{R}^n$ **- a root system,** $Q \subset P$, $W = < s_\alpha >$,
$\widetilde{W} = W \ltimes Q \subset \widehat{W} = W \ltimes P = \widetilde{W} \ltimes \Pi$, $\Pi = P/Q$,
$\mathcal{H} \stackrel{\text{def}}{=\!=} < \Pi, T_i (0 \le i \le n) > / \{$**homogeneous Coxeter relations,** $(T_i - t^{\frac{1}{2}})(T_i + t^{-\frac{1}{2}}) = 0 \}$,

$T_{\widehat{w}} = \pi T_{i_l} \cdots T_{i_1}$, $\widehat{w} = \pi s_{i_l} \cdots s_{i_1} \in \widehat{W}$, $l = l(\widehat{w})$.
$\boldsymbol{T^*_{\widehat{w}} \stackrel{\text{def}}{=\!=} T_{\widehat{w}^{-1}}, \langle T_{\widehat{w}} \rangle = \delta_{id,\widehat{w}}}$,
$\langle f, g \rangle \stackrel{\text{def}}{=\!=} \langle f^* g \rangle = \sum_{\widehat{w} \in \widehat{W}} c_{\widehat{w}} d_{\widehat{w}}$, **where**
$f = \sum c_{\widehat{w}} T_{\widehat{w}}$, $g = \sum d_{\widehat{w}} T_{\widehat{w}} \in L^2(\mathcal{H}) = L^2(\mathbb{R}\widehat{W})$.

Dixmier: $\langle f, g \rangle = \int_{\pi \in \mathcal{H}^\vee} \mathbf{Tr}(\pi(f^* g)) d\nu(\pi)$.

SPH-case: $f, g \in P_+ \mathcal{H} P_+$, $P_+ = \sum_{w \in W} t^{\frac{l(w)}{2}} T_w$.

Macdonald: $\nu_{sph}(\pi)$, $t > 1$. **Its extension to** $0 < t < 1$ **(due to ... Arthur, Heckman-Opdam ...):**
$\int \{\cdot\}\, d\nu^{an}_{sph}(\pi) = \sum C_{s,S} \cdot \int_{s+iS} \{\cdot\}\, d\nu_{s,S}$, [8, 15, 16]
summed over $s + S$ =**residual subtori. Residual points** $\sim$ **square integrable irreps (as** χ_π **extends to** $L^2(\mathcal{H})$**). Kazhdan-Lusztig**[11, 12]**: Deep Alg-Geom!**

The q, t-generalization of AKLHO becomes DAHA-invariant (the whole sum is necessary for this!) and the C-coefficients are uniquely determined by this.

C. SHAPOVALOV PAIRS [5]

DAHA INTEGRATIONS (plus $\Gamma_q \rightsquigarrow p$-adic Γ):

imaginary $(\lvert q\rvert \neq 1)$	real $(\lvert q\rvert \neq 1)$
$\Downarrow$	$\Downarrow$
constant term $(\forall q)$	Jackson sums
$\Uparrow$	$\Downarrow$
the case $\lvert q\rvert = 1 \ \Rightarrow$	$\Rightarrow$ roots of unity

$R \in \mathbb{R}^n$ a root system (reduced),
$W =< s_i, 1 \leq i \leq n >$, $P =$ weight lattice.

$\mathcal{HH} =< X_b, T_w, Y_b >_{q,t}$, $b \in P, w \in W$.
Over $\mathbb{R} \ni q, t$, $q = exp(-1/a)$, $a > 0$.

Shapovalov anti-involution $\varkappa$ of $\mathcal{HH}$ (for Y):
such that $T_w^{\varkappa} = T_{w^{-1}}$ and "PBW" holds:
$\mathcal{HH} \ni H = \sum c_{awb} Y_a^{\varkappa} T_w Y_b$ ($\exists$ and unique!).

Example. $\varkappa : X_b \leftrightarrow Y_b^{-1}, T_w \to T_{w^{-1}} (w \in W)$.
Coinvariant: $\{H\}_{\varkappa}^{\varrho} = \sum c_{awb}\, \varrho(Y_a)\varrho(T_w)\varrho(Y_b)$,
where $\varrho : \mathbb{R}[Y^{\pm 1}] \to \mathbb{R}$(or $\mathbb{C}[\cdot] \to \mathbb{C}$) is a character,
and $\varrho(T_w) = \varrho(T_{w^{-1}})$. Then $\{\varkappa(H)\}_{\varkappa}^{\varrho} = \{H\}_{\varkappa}^{\varrho}$ by
construction and $\{A, B\} \stackrel{\text{def}}{=\!=} \{A^{\varkappa} B\}_{\varkappa}^{\varrho} = \{B, A\}$.

OUR PROBLEM: Integral formulas for $\{H\}_{\varkappa}^{\varrho}$?

Let ϱ be the $1d$ character of affine $\mathcal{H}_Y$ sending $T_i \mapsto t^{1/2}$. Then $\{A, B\}$ acts via $\mathcal{X} \times \mathcal{X}$ for the *polynomial representation* $\mathcal{X} = \mathbb{R}[X^{\pm 1}] = Ind_{\mathcal{H}_Y}^{\mathcal{HH}}(\varrho)$**.**

Level-one anti-involution $\varkappa$: when dim$\mathcal{HH}/(\mathcal{J} + \mathcal{J}^{\varkappa}) = 1$ for $\mathcal{X} = \mathcal{HH}/\mathcal{J}$ (Shapovalov $\varkappa \Rightarrow$ level-one). Let $* : g \mapsto g^{-1}$ for $g = X, Y, T_w, q, t$ (it serves the Macdonald-type inner product in $\mathcal{X}$). It is level-one for *generic* q, t **but not Y-Shapovalov.**

D. RATIONAL DAHA

For rational DAHA, $*$ is **not** *level-one.*

$\mathcal{HH}'' \stackrel{\textbf{def}}{=\!=} \langle x, y, s\rangle/$**relations :**

$[y, x] = \frac{1}{2} + ks,\ s^2 = 1,\ sxs = -x,\ sys = -y.$

Polynomial representation $\mathbb{R}[x]$:

$s(x) = -x,\ x =$ **mult by** $x,\ y \mapsto D/2,$

$D = \frac{d}{dx} + \frac{k}{x}(1 - s)$ **(Dunkl).**

The anti-involution $x^* = x, y^* = -y, s^* = s$ formally serves $\int f(x)g(x)|x|^{2k}$, but it diverges at ∞. Algebraically, $\mathbb{R}[x]$ has NO $*$-form for $k \notin -1/2 - \mathbb{Z}_+$: $\{1, y(x^{p+1})\} = 0 = \{1, c_{p+1}x^p\}$, where $c_{2p} = p$, $c_{2p+1} = p + 1/2 + k$ (direct from the Dunkl operator). Hence, $\{1, x^p\} = 0$ $(\forall p)$ and $\{\ ,\ \} = 0$.

Replacing y **by** $y+x$**,** $*$ **becomes Shapovalov:**
PBW THM. $h = \sum c_{a\delta b}((y+x)^*)^a s^\delta (y+x)^b$.
Letting now $\{f\} \stackrel{\text{def}}{=} \sum_{\delta=0,1} c_{o\delta o}$, $\{f,g\} \stackrel{\text{def}}{=} \{f^* g\}$ **acts through** $\mathbb{R}[x]e^{-x^2} \times \mathbb{R}[x]e^{-x^2}$**:** $(y+x)e^{-x^2} = 0$; **therefore** $\mathbb{R}[x]e^{-x^2} = \mathcal{H}''/(\mathcal{H}''(y+x), \mathcal{H}''(s-1))$.

Explicitly, let $p = \frac{a+b}{2}$ **for** $a,b \in Z_+$. **Then** $\{x^a, x^b\} = (\frac{1}{2})^p (\frac{1}{2}+k) \cdots (\frac{1}{2}+k+p-1)$; **use PBW.**

Integral formula for this form: $\{f\,,\,g\} =$
$\frac{1}{i}\int_{-\epsilon+i\mathbb{R}} (fge^{-2x^2}(x^2)^k)dx/(\cos(\pi k)C)$,
$C = \Gamma(k+1/2)\,2^{k+1/2}$, $\forall k \in \mathbb{C}$, $\epsilon > 0$.

For real $k > -1/2$, **one can simplify this:**
$\{f\,,\,g\} = \frac{1}{iC}\int_{i\mathbb{R}} fge^{-2x^2}|x|^{2k}dx$.

Let $k = -\frac{1}{2} - m$ **and** $\int_{-\epsilon+i\mathbb{R}} \rightsquigarrow \frac{1}{2}(\int_{-\epsilon+i\mathbb{R}} + \int_{\epsilon+i\mathbb{R}})$;
$\{f\,,\,g\}$ **becomes** $=$ **const Res**$_0\,(fge^{-2x^2}x^{-2m-1}dx)$.
Its Radical is $(x^{2m+1}e^{-x^2})$**: a unitary** $\mathcal{H}''$**-module w.r.t.** $\frac{1}{i}\int_{i\mathbb{R}} fge^{-2x^2}|x|^{-2m-1}dx$**; the** $*$**-form of the quotient** $\mathbb{R}[x]/(x^{2m+1})$ **is non-positive! See** [5]**.**

E. GENERAL DAHA (A_1)[1]

ELLIPTIC BRAID GROUP AND DAHA:

$\mathcal{B}_q \stackrel{\textbf{def}}{=\!=} \langle T, X, Y, q^{1/4}\rangle / \{\ \ \boldsymbol{TXT = X^{-1}},$
$\boldsymbol{TY^{-1}T = Y,\ Y^{-1}X^{-1}YXT^2 = q^{-\frac{1}{2}}}\ \}.$
$\mathcal{B}_1 = \pi_1^{\mathrm{orb}}(\{E \setminus 0\}/\mathbf{S}_2) \cong$
$\pi_1(\{E \times E \setminus \textbf{diag}\}/\mathbf{S}_2),\ E =$**elliptic curve.**

$\boldsymbol{\mathcal{H}} \stackrel{\textbf{def}}{=\!=} \mathbb{R}[\boldsymbol{\mathcal{B}_q}]/((\boldsymbol{T} - \boldsymbol{t^{1/2}})(\boldsymbol{T} + \boldsymbol{t^{-1/2}}));$
we use $q = \exp(-1/a), a > 0,\ t = q^k, k \in \mathbb{R}.$

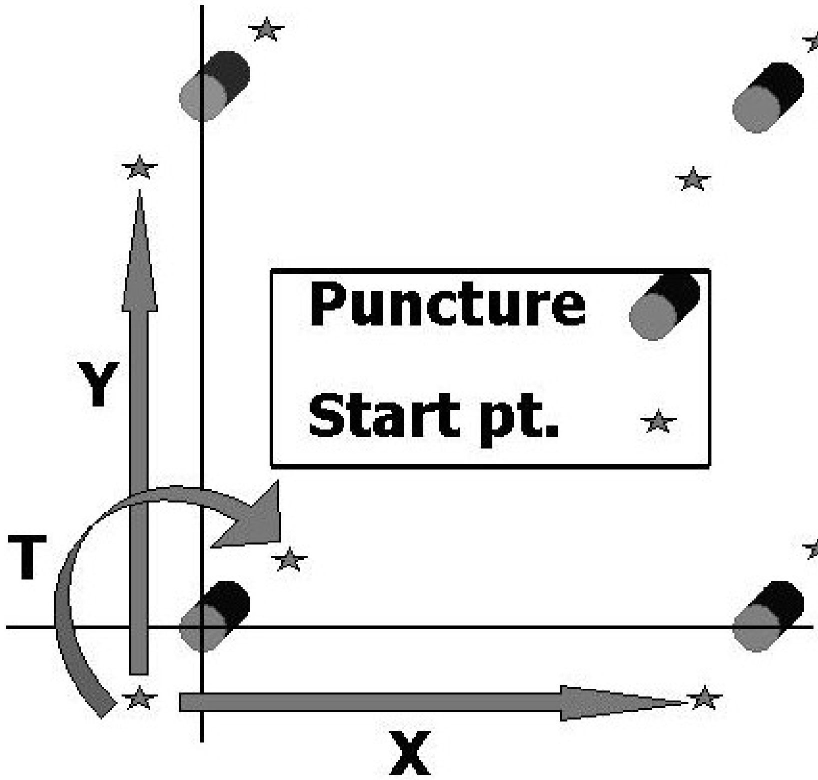
For $t^{\frac{1}{2}} = 1,\ \mathcal{H} =$**Weyl algebra** $\rtimes \mathbf{S}_2, T \to s:$

$$sXs = X^{-1}, sYs = Y^{-1}, Y^{-1}X^{-1}YX = q^{-1/2}.$$

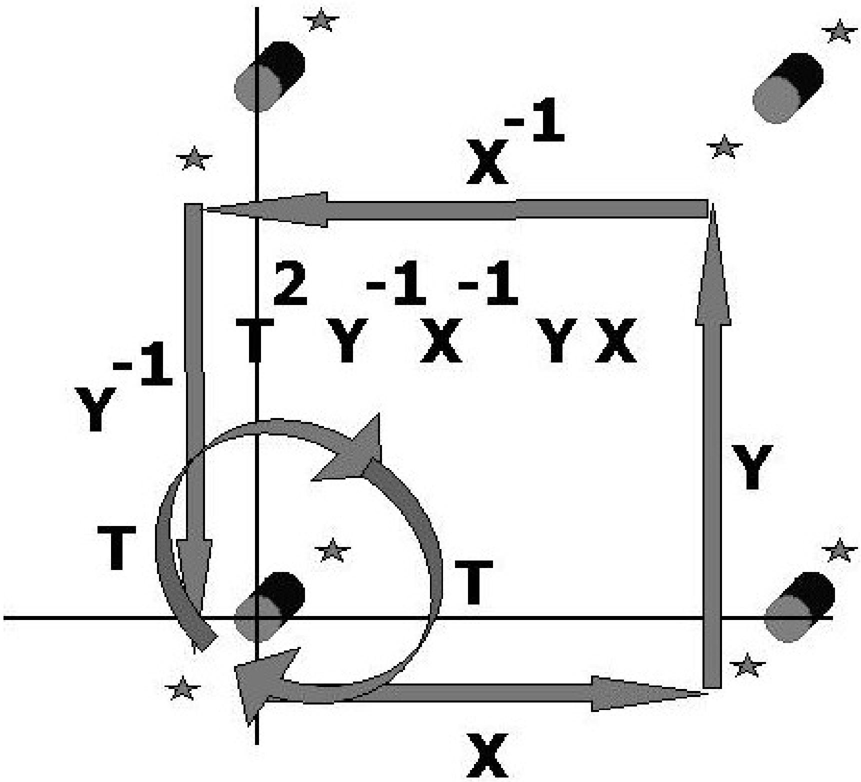
We "coupled" Weyl algebra with Hecke one. Heisenberg and Weyl algebras (non-commutative tori) are the main tools in quantization of symplectic varieties. So DAHA is "next", a general tool for their "refined quantization".

Rational DAHA $\mathcal{H}''$ $([y, x] = 1/2 + ks, \ldots)$ **:**
$X = e^{\sqrt{\hbar}x}, Y = e^{-\sqrt{\hbar}y}, q = e^{\hbar}, t = q^k, \hbar \to 0, T \to s.$

Operator Fourier transform is the DAHA automorphism sending: $q^{1/2} \mapsto q^{1/2}, t^{1/2} \mapsto t^{1/2},$
$Y \mapsto X^{-1},\ X \mapsto TY^{-1}T^{-1},\ T \mapsto T;$
topologically, the transposition of the periods of E**!**



Generators and relation $Y^{-1}X^{-1}YXT^2=1$



The whole $PSL_2(\mathbf{Z})$ acts projectively :

$$\begin{pmatrix}1 1\\ 0 1\end{pmatrix} \sim \tau_+ : Y \mapsto q^{-1/4} XY,\ X \mapsto X,\ T \mapsto T,$$

$$\begin{pmatrix}1 0\\ 1 1\end{pmatrix} \sim \tau_- : X \mapsto q^{1/4} YX,\ \ Y \mapsto Y,\ T \mapsto T.$$

They are directly from topology. The key for us is a pure algebraic fact that τ_+ is the conjugation by q^{x^2}, where $X = q^x$; use $\mathcal{X}$ below to see this. DAHA FT is for $\tau_+^{-1}\tau_-\tau_+^{-1} = \sigma^{-1} = \tau_-\tau_+^{-1}\tau_-$.

Polynomial representation $= Ind_{\mathcal{H}_Y}^{\mathcal{H\!H}}(\varrho)$ **is in** $\boldsymbol{\mathcal{X}}$ **= Laurent polynomials of $\boldsymbol{X = q^x}$,**

$$\boldsymbol{T \mapsto t^{1/2}s + \frac{t^{1/2} - t^{-1/2}}{q^{2x} - 1}(s-1),}$$

$$\boldsymbol{Y \mapsto \pi T, \pi = sp, sf(x) = f(-x),}$$

$$\boldsymbol{pf(x) = f(x + 1/2),\ t = q^k.}$$

Y **is the difference Dunkl Operator.**

$Y + Y^{-1}$ **preserves** $\mathcal{X}_{sym} \overset{\text{def}}{=\!=}$
sym (even) Laurent polynomials.
$Y + Y^{-1}\,|_{sym}$ **is the** q, t**-radial part.**

F. INNER PRODUCTS[1]

Macdonald Truncated θ-function: $\mu(x) =$
$= \prod_{i=0}^{\infty} \frac{(1-q^{i+2x})(1-q^{i+1-2x})}{(1-q^{i+k+2x})(1-q^{i+k+1-2x})}$, $X = q^x$,

$\langle f, g\rangle_{1/4} \stackrel{\textbf{def}}{=\!=} \frac{1}{2\pi a i} \int_{1/4+P} f(x)\, T(g)(x)\mu(x)dx,$
$P = [-\pi i a, \pi i a],\ q = \exp(-1/a).$

THM. For $k > -\frac{1}{2}$ **(generally,** $\Re k > -\frac{1}{2}$**),** $\langle f, g\rangle_{1/4} = (fT(g)\mu)_{\mathrm{CT}}$**; the later serves the anti-involution** $\diamond$**:** $T^\diamond = T$, $Y^\diamond = Y$, $X^\diamond = T^{-1}XT$**; the former is symmetric for any** k **and** *positive* **on** $\mathcal{X} = \mathbb{R}[X^{\pm 1}]$**.**

Proof. The coincidence and the relation $\diamond$ **to** $(fT(g)\mu)_{\mathrm{CT}}$ **are standard. The positivity is straightforward via the norm-formulas for** E**-polynomials; let us see this directly using** $\pi(\mu) = \mu(1/2-x) = \mu$**.**

a) $Y(E_n) = q^{-n_\sharp} E_n,\ E_n = X^n + (l.t.), X = q^x,$

$$n_\sharp = \frac{n-k}{2} \ \textbf{as}\ n \le 0, \ = \frac{n+k}{2} \ \textbf{as}\ n > 0.$$

b) $\langle E_n, E_m\rangle_{1/4} = C_n \delta_{nm}$ **due to** $Y^\diamond = Y$**.**

c) $C_n = q^{-n_\sharp} \frac{1}{i} \int_{1/4+P} E_n \overline{E_n} \mu(x)dx > 0$**, since** $\pi(x) = \overline{x}$ **(bar=c.c.) and** $\mu(x) > 0$ **at** $1/4 + P$**; use** $T(E_n) = \pi Y(E_n) = q^{-n_\sharp}\pi(E_n) = q^{-n_\sharp}\overline{E_n}$. $\square$

Imaginary Integration. $k>-\frac{1}{2}$, $f,g\in\mathcal{X}$:
$\langle f,g\rangle_{1/4}^{\gamma,\infty} \overset{\mathbf{def}}{=\!=} \frac{1}{i}\int_{\frac{1}{4}+i\mathbb{R}} fT(g)q^{-x^2}\mu(x)dx =$
$\frac{1}{2i\sqrt{\pi a}}\int_{\frac{1}{4}+P} fT(g)\sum_{j=-\infty}^{\infty} q^{j^2/4+jx}\mu(x)dx$
is a symmetric and positive form, serving
$T^{\varkappa}=T$, $X^{\varkappa}=T^{-1}XT$, $Y^{\varkappa}=q^{-1/4}XY$.

G. ANALYTIC CONTINUATION

Ingredients: The Shapovalov $\varkappa$ **above and**
$\varrho(\sum_{a,b\in\mathbb{Z}}^{\epsilon=0,1} c_{a\epsilon b}(Y^{\varkappa})^a T^{\epsilon} Y^b) \overset{\mathbf{def}}{=\!=} \sum c_{a\epsilon b} t^{\frac{a+\epsilon+b}{2}}$,
$\{A,B\}_{\varkappa}^{\varrho} \overset{\mathbf{def}}{=\!=} \varrho(A^{\varkappa}B) = \{B,A\}_{\varkappa}^{\varrho}$ **on** $\mathcal{H}$**.**
It "acts" via $\mathcal{X}\times\mathcal{X}$, $\mathcal{X}=\mathbb{R}[X^{\pm 1}]$, **and** $\{1,1\}=1$.
This form is (obviously) analytic for all $k\in\mathbb{C}$**.**

THM. $G(k)\{f,g\}_{\varkappa}^{\varrho} = \langle f,g\rangle_{1/4}^{\gamma,\infty}$, **where**
$G(k)=\sqrt{\pi a}\prod_{j=1}^{\infty}\frac{1-q^{k+j}}{1-q^{2k+j}}$, $\Re k>-1/2$.

Proof. **For** $\mathcal{C} \overset{\mathbf{def}}{=\!=} \{\epsilon+i\mathbb{R}\}$, **we set** $\boldsymbol{\Phi_{\epsilon}^{k}(f,g)} \overset{\mathbf{def}}{=\!=}$ $\frac{1}{i}\int_{\epsilon+i\mathbb{R}} fT(g)q^{-x^2}\mu(x)dx$**. For this path, bad** k **are** $\{\,2\mathcal{C}-1-\mathbb{Z}_+,\ -2\mathcal{C}-\mathbb{Z}_+\,\}$ **(when poles of** μ **belong to** $\mathcal{C}$**); so** $\{\Re k>-1/2\}$ **are all good as** $\epsilon=1/4$ **(and** $\Re k\gg 0$ **provide the required).** $\square$

Case $\epsilon = 0$. $\Phi_0^k(f,g)$ **is** $G(k)\{f,g\}_{\varkappa}^{\varrho}$ **only for** $\Re k > 0$**! For any** k**, it is symmetric and "sends"** $T \mapsto T$, $X \mapsto X^{\varkappa} = T^{-1}XT$ **(not for** Y **if** $\Re k < 0$**).**

Comparing $\epsilon = 0, \frac{1}{4}$ **for** $0 > \Re k > -\frac{1}{2}$ **(the key step):**

$$\Phi^k_{\frac{1}{4}} = \Phi^k_0 + A(-\tfrac{k}{2})\mu^{\bullet}(-\tfrac{k}{2})F(-\tfrac{k}{2}) \stackrel{\text{def}}{=\!=} \widehat{\Phi}^k, \text{ where}$$

$$A(\widetilde{k}) = \sqrt{\pi a}\,\sum_{m=-\infty}^{\infty} q^{m^2+2m\widetilde{k}} \text{ (from } q^{-x^2}\text{)},$$

$$F(-\tfrac{k}{2}) = fT(g)(x \mapsto -\tfrac{k}{2}), \text{ here } F \in \mathbb{R}[X^{\pm 2}],$$

$$\mu^{\bullet}(-\tfrac{k}{2}) = \prod_{j=0}^{\infty} \frac{(1-q^{k+j})(1-q^{-k+j+1})}{(1-q^{1+j})(1-q^{-2k+j+1})}.$$

Since $\widehat{\Phi}^k$ **is meromophic for** $0 > \Re k > -1$ **(i.e. beyond** $-1/2$**!), it coincides with** $G(k)\{f,g\}_{\varkappa}^{\varrho}$ **there. And it is also symmetric for ANY** k**; justification:**

$$fT(g)(-\tfrac{k}{2}) = t^{1/2} fg(-\tfrac{k}{2}) = T(f)g(-\tfrac{k}{2})$$

due to $T = \frac{q^{2x+k/2} - q^{-k/2}}{q^{2x}-1}\, s - \frac{q^{k/2} - q^{-k/2}}{q^{2x}-1}$,
where $(q^{2x+k/2} - q^{-k/2})(x \mapsto -k/2) = 0$.

MAIN THM. For $\Re k < 0$**,** $G(k)\{f,g\}_{\varkappa}^{\varrho} = \Phi_0^k +$ $\mu^{\bullet}(-\frac{k}{2}) \sum_{\widetilde{k} \in \widetilde{K}} A(\widetilde{k})\, t^{-j_{\pm}} \prod_{i=1}^{j_{\pm}} \frac{1-t^2q^i}{1-q^i} F(\widetilde{k})$, $j_+ = j-1$, $j_- = j$**,** $\widetilde{K} = \{-k/2\} \cup \{\pm\frac{k+j}{2}, 1 \le j \le m\}$, $m \stackrel{\text{def}}{=\!=} [\Re(-k)]$**,** $\widetilde{K} = \{n_{\sharp}, |n| \le m\}$, $[\cdot]$ **=integer part. [If** $F \in \mathbb{R}[X^{\pm 1}]$**, the poles of** μ **are** $q^{-\frac{1}{2}}X \in \pm q^{\mathbb{Z}_+/2} t^{\frac{1}{2}} \ni X^{-1}$**].**

Only the total sum is an $\mathcal{H}$-form (cf. "AKLHO")!

 COR. $\{f,g\}^{\varrho}_{\varkappa}$ **is degenerate exactly at the poles of** $G(k)\ :\ k=-\frac{1}{2}-m, m\in\mathbb{Z}_+$. **For such** k, $\mathcal{X}/$**Radical**$\{\ ,\ \}$ **is a sum of** 2 **irreps of dim**$=2m+1$ **("perfect** $\mathcal{H\!\!H}$**-modules",** $X\mapsto -X$ **transposes them).**

The radical is unitary(!) with respect to Φ_0^k**.**

Rational Limit: $q=e^{\hbar}, t=q^k,\ \hbar\to 0,$
$Y=e^{-\sqrt{\hbar}y},\ X=e^{\sqrt{\hbar}x},\ q^{x^2}=e^{x^2}, \mu\rightsquigarrow x^{2k},$
Funct$(\widetilde{K})\rightsquigarrow \mathbb{R}[x]/(x^{2m+1})$, $\mathcal{H\!\!H}''=\lim \mathcal{H\!\!H}$.

GENERAL THEOREM. For Shapovalov or level-one $\varkappa$**, the corresponding DAHA form is a finite sum of integrals over** q,t**-residual subtori.**

H. P-ADIC LIMIT[1]

In AHA $\mathcal{H}$ **of type** A_1 $(s=s_1,\omega=\omega_1,\pi=s\omega)$, **let** $\psi_n \stackrel{\mathbf{def}}{=\!=} t^{-\frac{|n|}{2}}T_{n\omega}\mathcal{P}_+$, $\mathcal{P}_+=(1+t^{1/2}T)/(1+t)$ **for** $n\in\mathbb{Z}$**, considered as polynomials in** $Y \stackrel{\mathbf{def}}{=\!=} T_\omega=\pi T$**; they are actually the** *Matsumoto spherical functions*. **Accordingly, the** *Satake-Macdonald p-adic spherical functions* **become** $\mathcal{P}_+\psi_n (n\geq 0)$.

THM. For $n\in\mathbb{Z}$, $E_n(X)/E_n(t^{-\frac{1}{2}})$ **become** ψ_n **as** $q\to 0$ **upon** $f(X)\mapsto f'(Y)\stackrel{\mathbf{def}}{=\!=} \boldsymbol{f(X\mapsto Y, t\mapsto \frac{1}{t})}$.

Let $\mu_0 = \mu(q \to 0) = \frac{1-X}{1-tX}$, $\{f,g\}_0 = (fT(g)\mu_0)_{ct}$. **Then for** $\langle T_{\widehat{w}}\rangle = \delta_{id,\widehat{w}}$ **and** $* : T_{\widehat{w}} \mapsto T_{\widehat{w}^{-1}}$ **in** $\mathcal{H}$, $\{f,g\}_0' = (t^{1/2}+t^{-1/2})\langle (f'\mathcal{P}_+)(g'\mathcal{P}_+)^*\rangle$ **for** $f,g \in \mathcal{X}$ **(nonsym AHA Plancherel formula);** $\boldsymbol{t' = \frac{1}{t}}$, $\boldsymbol{X' = Y}$.

THM. For $q = e^{-\frac{1}{a}}$, $M \in \mathbb{N}$, $F = fT(g) \in \mathbb{R}[X^{\pm 2}]$, $(F\mu)_{ct} = \frac{1}{2\pi M a\imath}\int_{-\pi M a\imath}^{+\pi M a\imath} F\mu\, dx + \mu^\bullet(-\frac{k}{2})\times$ $\Big(F(-\frac{k}{2}) + \sum_{j=1,\,\pm}^{[\Re(-k)]} F(\pm\frac{k+j}{2})\, t^{-j_\pm} \prod_{i=1}^{j_\pm} \frac{1-t^2q^i}{1-q^i}\Big)$, *the MAIN THM without Gaussians.* **The corresponding Jantzen filtration is in terms of AHA submodules of** $\mathcal{X}$ **($\mathcal{H}\!\!\mathcal{H}$-modules for** $k = -\frac{1}{2} - m$**).**

Making $a = 1/M$ **and sending** $M \to \infty$ **(then** $q = e^{-1/a} \to 0$**), we set** $k = -ca$ **for** $c > 0$**. Then** $t = e^{-\frac{k}{a}} \to e^c$ **and the formula above for** $\Re k \to 0_-$ **becomes the AKLHO one; recall that DAHA** $t > 1$ **is replaced by AHA** $1/t < 1$**. Here and generally only AHA residual subtori contribute for such** k**.**

A generalization to the whole (non-spherical) regular representation of $\mathcal{H}$ **requires the induced DAHA module corresponding to** $\mathcal{X}$ **and the technique of** W**-*spinors*; "supermathematics" is for** A_1**.**

Spinors[5, 7, 14]**. The W-spinors (or *hyperspinors*) are simply collections $\{f_w, w \in W\}$ of elements $f_w \in A$ with a natural action of W on the indices. If A (a space or an algebra) has its own (inner) action of W and $f_w = w^{-1}(f_{id})$, they are called principle spinors (or simply "functions"). The technigue of spinors is a direct generalization of *supermathematics*, which is the case of the root system A_1. It allows to deal with spinors as with usual functions (including any algebraic operations, differentiation, integration and so on and so forth); do we have "hypersymmetric theories" in physics?**

They proved to be very useful. One of the first instances was the Cherednik-Matsuo theorem (1991, a connection of AKZ and QMBP; see [1] and also [14],[5]). The theory of *non-symmetric* q-Whittaker functions is a convincing example [5],[7]. By the way, x^{2k} for complex k in the rational theory above is a typical *complex spinor*, i.e. a collection of two its (independent) branches in the upper and lower half-planes. To give another example, the Dunkl eigenvalue problem always has $|W|$ independent spinor solutions; generally only one of them is a *function*. In the case of $\mathcal{HH}''$ for A_1 (above), both fundamental spinor solutions for singular $k = -1/2 - m,\ m \in \mathbb{Z}_+$ are *functions*. See [5] for these and further examples.

I. JANTZEN FILTRATIONS etc.

To summarize, the ingredients are (a) *Shapovalov anti-involution* $\varkappa$ of $\mathcal{H\!\!H}$ (w.r.t. $\mathcal{Y} = \mathbb{R}[Y_b]$), i.e. s.t. $\{\varkappa(Y_a)T_wY_b\}$ form a (PBW) basis of $\mathcal{H\!\!H}$;

(b) the corresponding *Coinvariant*: $\varrho : \mathcal{H\!\!H} \to \mathbb{R}$ satisfying $\varrho(\varkappa(H)) = \varrho(H)$ (for any character of $\mathcal{Y}$ and any ϱ on H s.t. $\varrho(T_w - T_{w^{-1}}) = 0$); and

(c) *Shapovalov form*, a combination of (a) and (b), normalized by $\{1,1\} = 1$ and analytic for any k, which is $\{A,B\} \stackrel{\mathbf{def}}{=\!=} \varrho(A^{\varkappa}B)$ for $A,B \in \mathcal{H\!\!H}$.

PROBLEMS: Express $\{f,g\}^{\varrho}_{\varkappa}$ as a sum of integrals over the DAHA residual subtori for any (negative) $\Re k$. Generalize to arbitrary DAHA anti-involutions (any "levels") and any induced modules.

Analytic Jantzen filtration of $\mathcal{X}$ (AHA (!) but not DAHA modules) for $\Re k < 0$. The top(first) term is the sum over the smallest subtori, the bottom(last) term is the pure integration over $i\mathbb{R}^n$.

Algebraic Jantzen filtration of $\mathcal{X}$ is in terms of DAHA (!) modules at singular k_o w.r.t. $\widetilde{k} = k - k_o$. The top form is then for $\widetilde{k} = 0$, the bottom one is (in known examples) the pure integration over $i\mathbb{R}^n$.

For the second, restrict the m-th form to the radical of the $(m-1)$-th form and consider its radical, continue. The construction gives the so-called Kasatani decomposition of $\mathcal{X}$ for A_n [9, 10]. For arbitrary root systems, this is not done; generally the constituents can be DAHA reducible.

Rational case. Represent $\{f,g\}^{\varrho}_{\varkappa}$ as an integral over the boundary of the tube neighborhood of the resolution of the cross $\prod x_\alpha = 0$, e.g., $\pm i\epsilon + \mathbb{R}$ for A_1. This resolution (presentation as a divisor with normal crossings) is due to Ch (Publ. of RIMS, 1991), de Concini - Procesi, Beilinson - Ginzburg.

Conjecturally: singular k_o are the k when this integral can be reduced to integrations "over smaller subtori" (a sum over certain points for A_1).

Conjecturally: for singular $k_o = -\frac{s}{d_i}$, the bottom module is "semi-simple" (à la Suzuki for A_n); see [3] (a q,t-theory of the *bottom module* related to the "wheel conditions"). It can be unitary for $s=1$ (not for any s) w.r.t. the pure $\int_{\mathbb{R}^n}\{\cdot\}e^{-x^2}\,dx$ (as for A_1); see Etingof *et al.* in the case of A_n [10].

Note a relation to singularities à la Shokurov.

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