LECTURE I The symplectic topologist as a dynamicist

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Motivated by recent work of Çineli - Ginzburg - Gürel, Shelukhin (all errors and omissions due to my ignorance: I am not a specialist)

Symplectic linear algebra

$Sp(2n) \subset SL(2n,\mathbb{R})$ linear maps which preserve

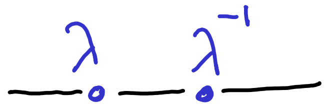
$$\omega_{\mathbb{R}^{2n}} = dp_1 \wedge dq_1 + \dots + dp_n \wedge dq_n$$

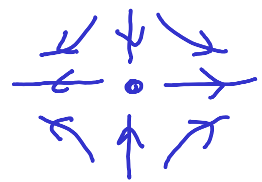
i.e. $A \in Sp(2n)$ satisfies

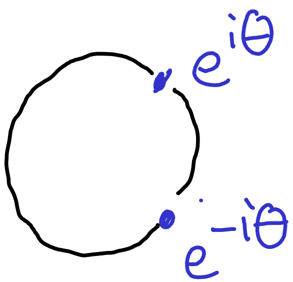
$$\langle v, Jw \rangle = \langle Av, AJw \rangle$$

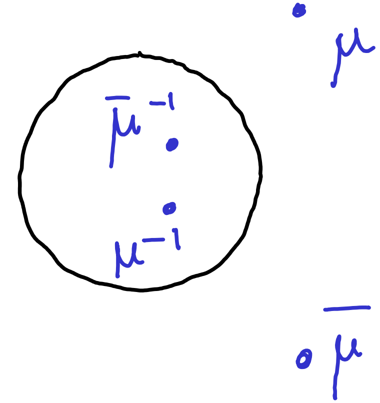
where $J = i$ on $\mathbb{R}^{2n} \cong \mathbb{C}^n$.

Eigenvalues come in blocks

1) 



2) 

3) 

Intuitively, only 2) shows a nonzero "net amount of rotation". To measure that, we pass to the universal cover:

$$U(n) = Sp(2n) \cap O(2n) \subseteq Sp(2n)$$

$$\pi_1(Sp(2n)) \cong \pi_1(U(n)) = \mathbb{Z}$$

so

$$\begin{array}{ccc} U(n) & \hookrightarrow & Sp(2n) \\ \uparrow & & \uparrow \\ \widetilde{U}(n) & \hookrightarrow & \widetilde{Sp}(2n) \end{array}$$

Rotation number

There is a unique homogeneous quasi-morphism ρ

$$\begin{tikzcd}
U(n) \arrow[r, "\det"] & S^1 \\
\widetilde{U}(n) \arrow[u] \arrow[r, "\widetilde{\det}"] \arrow[d, hook] & \mathbb{R} \arrow[u, "e^{-\pi i\theta}"'] \\
\widetilde{Sp}(2n) \arrow[ur, bend right, "\rho"'] &
\end{tikzcd}$$

homogeneous : $\rho(\tilde{A}^k) = k\rho(\tilde{A})$

quasimorphism :

$$|\rho(\tilde{A})\rho(\tilde{B}) - \rho(\tilde{A}\tilde{B})| < \text{Const.}$$

This is a key notion in studying iterations $k \longmapsto \tilde{A}^k$

Conley-Zehnder index

Take

$$Sp(2n)^* = \{A \in Sp(2n) : 1 \text{ not an eigenvalue}\}$$

Then there is a quasimorphism

$$\begin{tikzcd}
Sp(2n)^* \arrow[r, "\operatorname{sign}(\det(\mathbb{1}-A))"'] & \{\pm 1\} \\
\widetilde{Sp}(2n)^* \arrow[u] \arrow[r, "\mu"'] & \mathbb{Z} \arrow[u, "(-1)^m"']
\end{tikzcd}$$

what the index $\mu(\tilde{A})$ counts: when going from $\mathbb{1}$ to $\tilde{A}$ in $\widetilde{Sp}(2n)$, how often do the eigenvalues cross 1 (with signs).

Examples:

- $\tilde{A} = \begin{pmatrix} e^t & \\ & e^{-t} \end{pmatrix}$ $\quad \mu(\tilde{A}) = 1$

- $\tilde{A} = \exp(tJH)$, H nondegenerate symmetric and $t>0$ small

  $\mu(\tilde{A})$ = Morse index of H
  = number of negative eigenvalues

- $\tilde{A} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

  $\mu(\tilde{A}) = -2\left\lfloor \frac{\theta}{2\pi} \right\rfloor$

- $\mu(\tilde{A}^{-1}) = 2n - \mu(\tilde{A})$

μ is not homogeneous, but we can instead use the homogeneity of ρ and the fact

$$\mu(\tilde{A}) = \rho(\tilde{A}) + \text{correction in } (0,2n)$$

For instance, as $k \to \infty$,

$$\begin{cases} \mu(\tilde{A}^k) \longrightarrow +\infty \text{ ; or} \\ \mu(\tilde{A}^k) \in (0,2n) \text{ for all } k \text{ ; or} \\ \mu(\tilde{A}^k) \longrightarrow -\infty \end{cases}$$

This reduces to $\begin{cases} \rho(\tilde{A}) > 0 \\ \rho(\tilde{A}) = 0 \\ \rho(\tilde{A}) < 0 \end{cases}$

Hamiltonian diffeomorphisms

Imagine a time-dependent system in classical mechanics,

$$H = H_t(p,q) \quad t \in \mathbb{R}, \quad (p,q) \in \mathbb{R}^{2n}$$

Equations of motion:

$$\dot{p} = -\frac{\partial H}{\partial q}, \quad \dot{q} = \frac{\partial H}{\partial p}$$

Example:

$$H_t(p,q) = \frac{1}{2}\|p\|^2 + V_t(q)$$

$$\dot{p} = -\nabla V_t, \quad \dot{q} = p \Rightarrow \ddot{q} = -\nabla V_t$$

Suppose $H_t = H_{t+1}$. Then it makes sense to look for k-periodic trajectories $(k = 1, 2, 3, \ldots)$

Global analogue: (M^{2n}, ω_M) is a compact symplectic manifold

$\omega_M \in \Omega^2(M)$ closed,

locally $\omega_M = dp_1 \wedge dq_1 + \cdots + dp_n \wedge dq_n$

Every $H \in C^\infty(M, \mathbb{R})$ gives rise to a Hamiltonian vector field X.

Time-dependent $H_t \Rightarrow X_t$.

Solving

$$\dot{x} = X_t(x)$$

gives a family (φ_t) of diffeomorphisms of M. Assume $H_t = H_{t+1}$

$$\Rightarrow \varphi_{t+k} = \varphi_t \circ \varphi_1^k \qquad (k \in \mathbb{Z})$$

We set $\varphi = \varphi_1$, and look for k-periodic points of φ.

Nondegeneracy The aim is to make the local structure of fixed / periodic points simple

$\varphi(x) = x$ nondegenerate fixed point: 1 not an eigenvalue of $D\varphi_x$
$\Leftrightarrow D\varphi_x \in Sp(2n)^*$

$\varphi^k(x) = x$ nondegenerate periodic point: no root of unity is an eigenvalue of $D\varphi_x^k$
$\Leftrightarrow x$ is a nondegenerate fixed point of $\varphi^k, \varphi^{2k}, \varphi^{3k}, \dots$

A "generic" φ has only nondegenerate periodic points.

Example: $H \in C^\infty(M, \mathbb{R})$ a Morse function (time-independent) and small. Remember the local formula

$$\dot{p} = -\frac{\partial H}{\partial q}, \quad \dot{q} = \frac{\partial H}{\partial p}$$

If $x = (p, q)$ is a critical point of H, it is a fixed point of φ (nondegenerate, and there are no other fixed points)

Thm ("Arnold conjecture") If φ has nondegenerate fixed points,

$$\# \mathrm{Fix}(\varphi) \geq \sum_i \dim H^i(M; \mathbb{F})$$

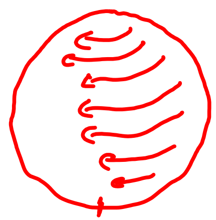
for any field $\mathbb{F}$ (long story; recent breakthrough by Abouzaid-Blumberg for $\mathrm{char}(\mathbb{F}) > 0$)

From now on, always assume φ has nondegenerate periodic points.

<u>Question</u> ("Nondegenerate Conley conjecture") Are there necessarily infinitely many periodic points?

<u>Yes</u> For $M = T^{2n} = \mathbb{R}^n / \mathbb{Z}^{2n}$ with $\omega_M = dp_1 \wedge dq_1 + \cdots + dp_n \wedge dq_n$ (Conley - Zehnder 1984)

<u>No</u> For $M = S^2$, φ an irrational rotation



(or more generally, any M which has a Hamiltonian torus action with isolated fixed points)

Topological data:

$$[\omega_M] \in H^2(M;\mathbb{R})$$
$$c_1(M) = c_1(TM) \in H^2(M;\mathbb{Z})$$

Answer is <u>yes</u>:

- $c_1(M) = 0$ (Salamon - Zehnder 1992)
- $c_1(M) = -[\omega_M]$ (Ginzburg - Gürel 2010)
- ...

Take $M = \mathbb{CP}^2$ blown up at $m \leq 8$ points, $c_1(M) = [\omega_M]$ (del Pezzo)

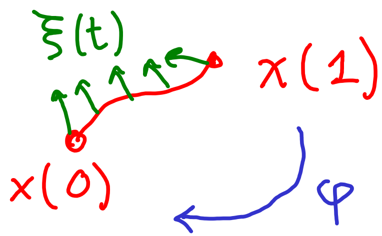
$m \leq 3$ <u>No</u>, they admit torus action

$m > 3$ <u>Unknown</u>, the symplectic topology changes dramatically at $m = 5$ (algebro-geometric moduli)

Hamiltonian Floer cohomology

Take the twisted free loop space

$$\mathcal{L}_\varphi = \{\, x : \mathbb{R} \to M,\ x(t) = \varphi(x(t+1)) \}$$



It carries the (multivalued) action functional à la Hamilton-Jacobi

$$dA(x)\,\xi = -\int_0^1 \omega_M(x', \xi)\, dt$$

Critical points have $x' = 0 \iff$ x is constant at a fixed point of φ. Nondegenerate fixed points $\iff A$ is Morse.

Floer cohomology is (formally) a kind of Morse theory for A. One defines a chain complex

$$CF^*(\varphi) = \bigoplus_x \mathbb{K}\cdot x$$

($\mathbb{K}$ = coefficient field, as in ordinary cohomology, let's say $\mathrm{char}(\mathbb{K}) = 0$; x are the fixed points), which with a suitable differential defines Floer cohomology $HF^*(\varphi)$.

<u>Theorem</u> $HF^*(\varphi) \cong H^*(M; \mathbb{K})$ (canonical given $\varphi = \varphi_1 \xleftrightarrow{\varphi_t} \varphi_0 = id_M$)

The Arnol'd conjecture follows directly from that.

Conley conjecture for $c_1(M) = 0$

In this case, if $\varphi(x) = x$, $\widetilde{D\varphi_x}$ has a preferred lift to $\widetilde{Sp}(2n)$. The Conley-Zehnder index

$$\mu(\varphi, x) = \mu(\widetilde{D\varphi_x}) \in \mathbb{Z}$$

defines the degree of $x \in CF^*(\varphi)$ (if $c_1(M) \neq 0$, $HF^*(\varphi)$ is only graded mod 2).

Suppose that φ only has finitely many periodic points. After passing to an iterate, these all become fixed points. We look at the Floer complex for φ^k,

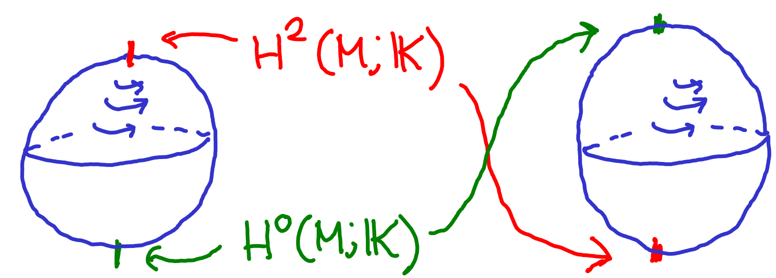
$$CF^*(\varphi^k) = \bigoplus_x \mathbb{K}x$$

fixed point of φ — in degree $\mu(\varphi^k, x)$

For $k \gg 0$, the iteration rule says that $\mu(\varphi^k, x) \notin \{0, 2n\}$. But that is a contradiction to

$$HF^*(\varphi^k) \cong H^*(M; \mathbb{K}) \neq 0, \quad * = 0, 2n$$

Situation for $M = S^2$, φ rotation



angle $\theta \in (0, 2\pi)$ — angle $\theta \in (2\pi, 4)$

Floer cohomology as a TQFT

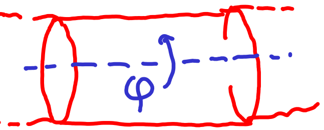
(topological quantum field theory)

Remember $CF^*(\varphi) = \bigoplus \mathbb{K}x$ for fixed points $\varphi(x) = x$, thought of as constant loops in $\mathcal{L}_\varphi$. The differential is defined using "gradient flow lines" $\mathbb{R} \longrightarrow \mathcal{L}_\varphi$, or maps

$$\begin{cases} u: \mathbb{R}\times\mathbb{R} \longrightarrow M \\ u(s,t) = \varphi(u(s,t+1)) \\ \bar\partial u = \dfrac{\partial u}{\partial s} + J_t(u)\dfrac{\partial u}{\partial t} = 0 \end{cases}$$

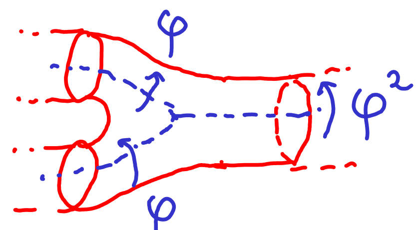
where J_t is an almost complex structure on M (necessarily t-dependent, $J_t = \varphi_* J_{t+1}$)

Formal structure: think of this as maps from a cylinder, with a "seam" marking φ-periodicity

$\mathbb{R}\times S^1$ 

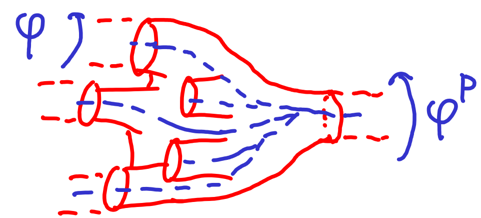
differential for $HF^*(\varphi)$

One can use other Riemann surfaces as well:



"pair-of-pants" product

$HF^*(\varphi)\otimes HF^*(\varphi) \longrightarrow HF^*(\varphi^2)$



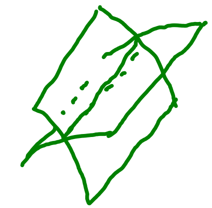
p-fold product

$HF^*(\varphi)^{\otimes p} \longrightarrow HF^*(\varphi^p)$

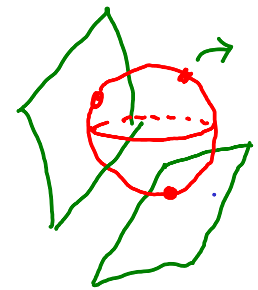
The proof of the Conley conjecture in the case $[\omega_M] = -c_1(M)$ uses the product structure. Slightly more precisely,

- Assume only finitely many periodic orbits;
- Show that then, the p-fold pair-of-pants power of any class in $HF^*(\varphi)$ becomes zero, $p \gg 0$;
- Contradiction! Because we know what the pair-of-pants product corresponds to in $H^*(M;\mathbb{K}) \cong HF^*(\varphi)$: the (small) quantum product.

Small quantum product is a deformation of the cup product by contributions from J-holomorphic maps $\mathbb{C}P^1 \longrightarrow M$



ordinary cup product is the intersection of cycles



non-local contribution of a holomorphic map

Example: $H^*(S^2) = \mathbb{K}\cdot 1 \oplus \mathbb{K}\cdot a$

classical product $a \cdot a = 0$

quantum product $a * a = 1$

Unfortunately, <u>no known relation between $*$ and Conley conjecture</u>!

(Quantum) Steenrod operations

From now on, we work with a coefficient field $\mathbb{K}$, $\mathrm{char}(\mathbb{K}) = p > 0$

Recall group cohomology of the cyclic group,

$$H^*_{\mathbb{Z}/p}(\text{point}; \mathbb{K}) = H^*\left(\frac{\text{contractible}}{\mathbb{Z}/p}; \mathbb{K}\right)$$

$$= \mathbb{K}[[t]] \oplus \mathbb{K}[[t]]\theta$$

$|t| = 2, \quad |\theta| = 1$

is one-dimensional in each degree (for $p=2$, this is $H^*(\mathbb{RP}_\infty; \mathbb{Z}/2)$).

The classical Steenrod operation is

$$St : H^i(M;\mathbb{K}) \longrightarrow H^{pi}_{\mathbb{Z}/p}(M;\mathbb{K})$$

$$= \left(H^*(M;\mathbb{K}) \otimes H^*_{\mathbb{Z}/p}(\text{point};\mathbb{K})\right)^{pi}$$

It exploits the symmetry of the ordinary cup-product

$$St(x) = \underbrace{x \smile \cdots \smile x}_{p} + \text{terms with } t \text{ or } \theta$$

On the other hand,

$$St(x) = (\text{nonzero constant}) \, x \, t^{\overbrace{\frac{p-1}{2}|x|}^{\text{degree}}}$$

$$+ \text{ cohomology classes of higher degree}$$

Therefore, St is injective. There is (under certain assumptions on M) a quantum counterpart QSt

Example: $M = S^2$, $a \in H^2(S^2;\mathbb{K})$

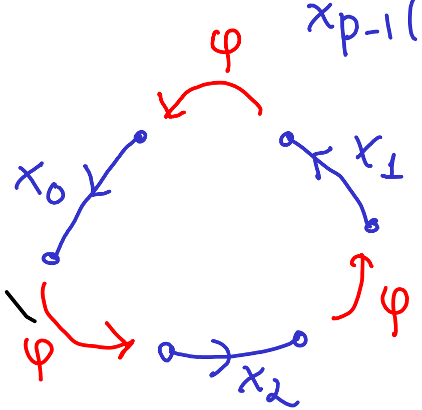
classical $St(a) = (\text{nonzero const.})\, at$

quantum $p=2$, $QSt(a) = at + 1$

Quantum Steenrod and iteration

Remember that Floer cohomology is defined using the loop space $\mathcal{L}_\varphi$. For an iterate φ^p, we have

$$\mathcal{L}_{\varphi^p} = \{ x : [0,1] \to M,\ x(0) = \varphi^p(x(1))$$
$$\cong \{ x_0 : [0, 1/p] \to M,\ x_0(0) = \varphi(x_1(\tfrac{1}{p}))$$
$$x_1 : [1/p, 2/p] \to M,\ x_1(0) = \varphi(x_2(\tfrac{1}{p}))$$
$$\dots$$
$$x_{p-1} : [p-1/p, 1] \to M,$$
$$x_{p-1}(0) = \varphi(x_0(\tfrac{1}{p})) \}$$



$p = 3$

It carries a natural $\mathbb{Z}/p$-action, which is compatible with the p-fold pair-of-pants product,

$$HF^*(\varphi)^{\otimes p} \longrightarrow HF^*(\varphi^p)$$

By exploiting that symmetry, we get an equivariant p-th power map on Floer cohomology

$$\begin{array}{ccc} HF^*(\varphi) & \xrightarrow[p\text{-th power}]{} & HF^{p*}_{\mathbb{Z}/p}(\varphi^p) \\ \cong & & \cong \\ H^*(M;\mathbb{K}) & \xrightarrow{QSt} & \left(H(M;\mathbb{K}) \otimes H_{\mathbb{Z}/p}(\text{point})\right)^{p*} \end{array}$$

unlike the ordinary pair-of-pants product, the equivariant version "knows that $\varphi(x) = x \Rightarrow \varphi^p(x) = x$"

Quasi-rotations This is an extreme case of the Conley conjecture. Suppose $H^*(M;\mathbb{Z})$ is torsion-free. $\varphi : M \to M$ is called a pseudo-rotation if

$$\# \mathrm{Fix}(\varphi) = \operatorname{rank} H^*(M)$$

(the least number of fixed points allowed by the Arnol'd conjecture). As usual, we also impose a non-degeneracy assumption. The consequence is that

$$CF^*(\varphi) = CF^*(\varphi^k) \quad \text{for all } k$$

(ignoring grading issues), with zero differentials.

Theorem (Shelukhin; May 2019) Let M satisfy

(1) $c_1(M) = [\omega_M]$

(2) For $p=2$, $QSq([\text{point}])$ is equal to $Sq([\text{point}])$

(3) $c_1(M) \in H^2(M;\mathbb{Z})$ is divisible by some integer $> n = \frac{1}{2}\dim(M)$

true for $M = \mathbb{C}P^n$, but otherwise rare

Then M has no pseudorotations

Theorem (Çineli-Ginzburg-Gürel, Shelukhin; September 2019)

(3) can be dropped ← important

Recall that

$$H^*_{\mathbb{Z}/p}(\text{point};\mathbb{K}) = \mathbb{K}[[t]] \oplus \theta\,\mathbb{K}[[t]]$$

For a given prime p, let $\ell_p(M)$ be the largest integer such that

$$H^*(M;\mathbb{K}) \xrightarrow{QSt} H^*(M;\mathbb{K}) \otimes \frac{H^*_{\mathbb{Z}/p}(\text{point})}{t^{\ell_p(M)}}$$

is not injective. For classical Steenrod operations,

$$St([\text{point}]) = t^{n(p-1)}[\text{point}]$$

so the counterpart of our quantum notion would be $n(p-1)$.

Unpublished (S–Shelukhin)

If M admits a pseudorotation,

$$\limsup_{p} \frac{\ell_p(M)}{p} < n.$$

The end (for now...)