How complicated are symplectic manifolds?

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Take $M \subset \mathbb{C}^N$ smooth complex <u>affine algebraic</u> variety. Standard symplectic (Kähler) form

$$\omega_M = \frac{i}{2}\left(dz_1 \wedge d\bar{z}_1 + \cdots\right)$$

Of course $\omega_M = d\theta_M$.

$$\theta_M = \frac{i}{4}\left(z_1 d\bar{z}_1 - \bar{z}_1 dz_1 + \cdots\right)$$

There is a dual <u>Liouville vector field</u> Z_M,

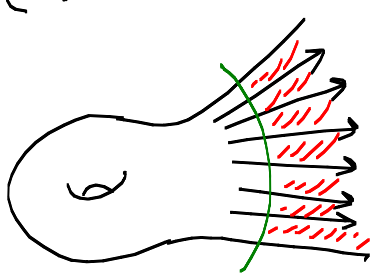
$$i_{Z_M}\omega_M = \theta_M \implies L_{Z_M}\omega_M = \omega_M.$$

Moreover if $h_M = \frac{1}{4}(|z_1|^2 + \cdots)$ then

$$dh_M \circ i = -\theta_M \implies Z_M = \nabla h_M$$

h_M has only <u>finitely many critical values</u> (since it is a polynomial), and the flow of Z_M is <u>complete</u>, since $dh_M(Z_M) = \|Z_M\|^2 \leq 4 h_M$.

Picture:



infinite cone

$h^{-1}([c,\infty)) \approx [c,\infty) \times h^{-1}(c)$.

level set $h^{-1}(c)$, $c \gg 0$

We consider a class of open symplectic manifolds with the same structure, called (complete finite type) <u>Liouville manifolds</u> (M^{2n}, ω).

<u>Remark</u> Any Liouville manifold can be specified up to symplectic isomorphism by a <u>finite</u> amount of data (by a suitable version of the Moser stability argument).

The source of richness of the theory is the dynamics "at infinity".

Fix a relatively compact $\mathcal{U} \subset \cancel{X}\,M$. Let H be a function with support in $\mathcal{U}$, and such that $H(M) = [-C, 0]$ for some $C < 0$.

<u>Idea</u> (Hofer-Zehnder-…; known in many cases)
Fix $\mathcal{U}$; if C is sufficiently large, the Hamiltonian vector field X defined by

$$i_X \omega = -dH$$

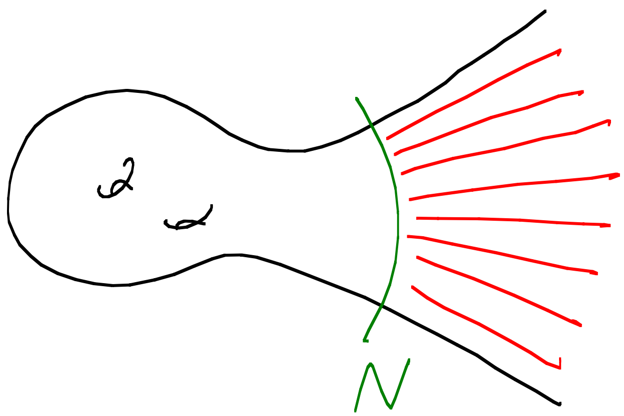
must have non-constant orbits of period 1.

Recall that such orbits have a variational meaning, as critical points of the <u>action</u> functional on the free loop space:

$$x \in C^\infty(\mathbb{R}/\mathbb{Z}, M), \qquad A_H(x) = -\int_0^1 x^*\theta_M + H(x(t))\,dt$$

By considering only those x with $A_H(x) \leq -C$ and then passing to various limits ($C \to \infty$, then U getting larger and larger) one defines a "Morse homology" invariant, called <u>symplectic cohomology</u> $SH^*(M, \omega_M)$ (Cieliebak-Floer-Hofer-Wysocki; Viterbo).

Special choice of H leads to this picture:



structure at ∞ is

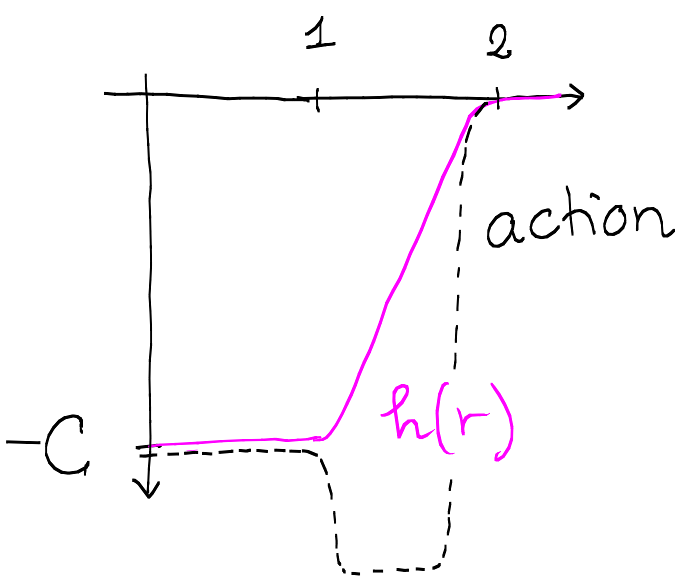
$$[1,\infty) \times N \xhookrightarrow{\iota} M$$

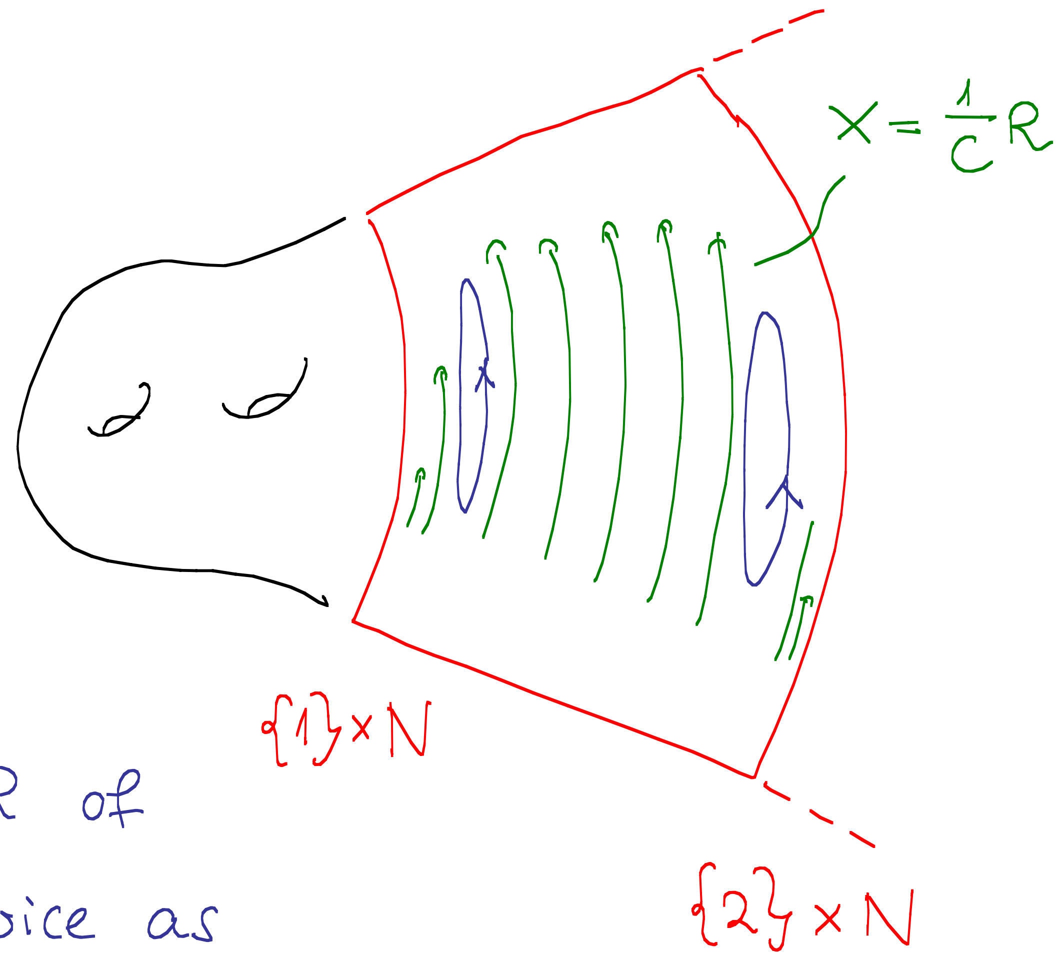
$$\iota^* \omega_M = d(r\, \theta_M | N)$$

r = radial

N carries a canonical one-dimensional foliation. A choice of θ_M singles out a Reeb vector field R tangent to the foliation. Choose $H = \psi(r)$ to be supported in $[1,2] \times N$, and only to depend on the radial variable.

Then $X = h'(r)R$





Periodic orbits of R of period $< 1/C$ occur twice as 1-periodic orbits of X (the action cutoff ensures we use only one of these occurrencess).

$SH^*(M, \omega_M)$ is a $\mathbb{Z}/2\mathbb{Z}$ graded abelian group. It is the cohomology of a periodic complex

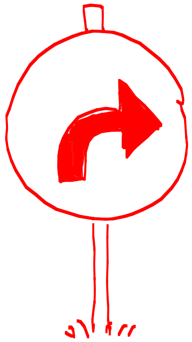
$$\{\cdots \longrightarrow C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} C^0 \longrightarrow \cdots\}$$

where each C^i is a (at most) countably generated free abelian group ($\mathbb{Z} \oplus \mathbb{Z} \oplus \cdots$)

Elementary:

- If R has no periodic orbits, $SH^*(M) \cong H^*(M)$;
- $SH^*(M)$ can be zero (e.g. $M = \mathbb{R}^{2n}$, ω_M standard)

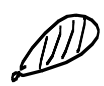
<u>What else</u>?



Digression to the homology of <u>finitely presented groups</u>. Consider

$$G = \langle \underbrace{g_1, \dots, g_k}_{\text{generators}} \mid \underbrace{r_1, \dots, r_\ell}_{\text{relations}} \rangle$$

Let BG be its classifying space. This is a cell complex constructed from the presentation:

- A 1-cell  for each g_i
- A 2-cell  for each r_j
- Cells of dim ≥ 3 to kill $\pi_i(BG)$, $i \geq 2$.

Consider $H_i(G) \stackrel{\text{def}}{=} H_i(BG)$. This is the homology of a complex of (at most) countably generated free abelian groups. E.g. $H_1(G) = G^{ab}$.

$$\boxed{\text{\underline{Theorem} Fix } i \geq 3. \text{ Then for general } G,\ H_i(G) \text{ can be any recursively enumerable abelian group. (Baumslag - Dyer - Miller)}}$$

This means $H_i(G) = \mathbb{Z}^{\infty}/_{\rho_1, \rho_2, \ldots}$ where ρ_i is <u>any</u> sequence of relations that can be produced by a computer program.

Examples $H_i(G)$ could be:

- $\mathbb{Z}^\infty$
- $(\mathbb{Z}/2)^\infty$
- $\mathbb{Z}/2 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/5 \oplus \mathbb{Z}/7 \oplus \mathbb{Z}/11 \oplus \cdots$
- $\mathbb{Q}$
- $\mathbb{Z}/p^\infty = \varinjlim_n \mathbb{Z}/p^n\mathbb{Z}$ (p prime)
- $(\mathbb{Z}/2)^{i_2} \oplus (\mathbb{Z}/3)^{i_3} \oplus \cdots$, where $i_k = 1$ if $k \in \mathbb{N}$ is the ASCII encoding of a grammatically correct English text, and $i_k = 0$ otherwise.

How about $SH^*(M, \omega_M)$?

<u>Theorem</u> (Abouzaid – S.) Let (M, ω_M) be an affine variety of complex dimension $n \geq 6$. Fix a finite set of primes $\mathbb{P}$. Then there is another Liouville type symplectic structure $\tilde{\omega}_M$ (indistinguishable from ω_M by homotopy theoretic means), such that

$$p : SH^*(M, \tilde{\omega}_M) \longrightarrow SH^*(M, \tilde{\omega}_M)$$

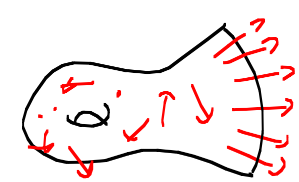
is an isomorphism if and only if $p \in \mathbb{P}$.

Consequence ∞ many different Liouville structures on M (alternative proof of this consequence can be obtained by combining results of McLean and Bourgeois-Ekholm-Eliashberg, as pointed out by Smith; this method is a little more general, but relies on multiplicative structures).

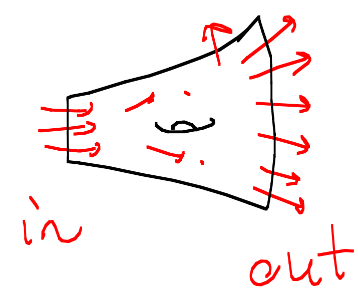
Problem Understand the dynamics of the Reeb flow in these examples better.

All this discussion was fueled by the "dynamics at infinity". Do our symplectic manifolds also have inner complexity? For convenience, switch point of view slightly:

Liouville domain M compact with boundary, with a symplectic form ω_M and a Liouville vector field Z_M which points strictly outwards along the boundary.



__Liouville cobordism__ (C, ω_C) with $\partial C = \partial_{in} C \sqcup \partial_{out} C$, and Z_C points inwards/outwards along the components $\partial_{in}/\partial_{out}$.



__Trivial cobordism__ Every point of $\partial_{in} C$ gets carried to $\partial_{out} C$ by the flow of Z_C.

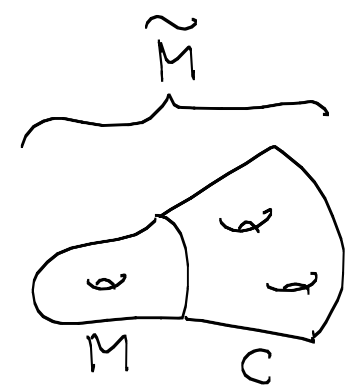
"Liouville manifolds = Liouville domains up to attaching trivial cobordisms to the boundary"

<u>Hypothetical invariant of Liouville domains</u>:

the "complexity" $\xi(M, \omega_M)$ should satisfy

- $0 \leq \xi(M) < \infty$
- If $\tilde{M} = M \cup_{\text{boundary}} C$, then

$$\xi(\tilde{M}) \geq \xi(M)$$



  with <u>equality</u> if the cobordism is <u>trivial</u>.

- Takes on unbounded values for <u>some</u> Liouville domains with fixed topology.

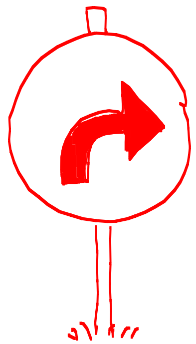
__Theorem__ (Auroux) Let M_1 and M_2 be two four-dimensional Liouville domains, with

$$\chi(M_1) = \chi(M_2), \qquad \sigma(M_1) = \sigma(M_2)$$

and $\partial M_1 \cong \partial M_2$ (respecting all the structure). Then there is a cobordism C such that

$$M_1 \cup_{\text{boundary}} C \cong M_2 \cup_{\text{boundary}} C$$

This puts serious restraints on what $\xi(M)$ can be (but does not rule it out).



<u>Digression</u> to using additional structures on SH^* (Symplectic Field Theory). First, there is an equivariant version $SH^*_{S^1}(M)$. This carries a Lie bracket of degree -1, so one can form the Chevalley-Eilenberg complex

$$\cdots\ \mathrm{Sym}^3 SH^*_{S^1}(M) \longrightarrow \mathrm{Sym}^2 SH^*_{S^1}(M) \xrightarrow[{[-,-]}]{} SH^*_{S^1}(M) \to 0$$

This is in fact an approximation to a richer invariant, $\widehat{SH}^*_{S^1}(M)$, which comes with spectral sequence

$$E_1 = \bigoplus_k \mathrm{Sym}^k(SH^*_{S^1}(M)) \Rightarrow \widehat{SH}^*_{S^1}(M)$$

Moreover, there is a canonical element $e \in SH^0_{S^1}(M)$

<u>Definition</u> Let $\rho(M)$ be $= r$ if e survives precisely to the E_r page

<u>Ex</u> $\rho(\mathbb{R}^{2n}) = 0$, $\rho(T^*S^2) = 0$, $\rho(\mathbb{C}^*) = 1$.

<u>Conjecture</u> (McLean) For contractible affine algebraic surfaces, this is related to log Kodaira dimension

How can dynamical ideas apply to <u>closed</u> symplectic manifolds (M,ω)? Classical

<u>Definition</u> The Hofer-Zehnder capacity

$$c(M) = \sup\{ c : \text{there is an } H : M \to \mathbb{R}, \max(H) - \min(H) = c, \text{ whose Hamiltonian vector field has no nonconstant 1-periodic orbits} \}$$

We know $c(M) > 0$ (easy), but it can be ∞ (Zehnder, Usher)

Upper bounds provided by Gromov-Witten invariants

$$\begin{tikzcd} GW_{g,n,A} : H_*(M)^{\otimes n} \arrow[r] \arrow[dr] & \mathbb{Q} \\ & H_*(\overline{\mathcal{M}}_{g,n}) \arrow[u, "\text{collapse}"'] \end{tikzcd}$$

$A \in H_2(M;\mathbb{Z})$

$g \geq 0$

$n \geq 0$

except for $\begin{cases} g=0, \; n<3 \\ g=1, \; n=0 \end{cases}$

$$\boxed{\begin{array}{l} \underline{\text{Theorem}} \text{ (Liu-Tian, Lu) Suppose that} \\ \qquad GW_{g,n+2,A}(\text{point}, \text{point}, x_1, \dots, x_n) \neq 0. \\ \text{Then} \\ \qquad c(M) \leq \int_A \omega_M \end{array}}$$

<u>Example</u> $M = \mathbb{CP}^n$, $[\omega_M] = PD(\text{hyperplane}) \rightsquigarrow c(M) = 1$.

However, for very large classes of manifolds all the GW invariants vanish, so the bound is trivial (and $c(M)$ mostly unknown).

Different dynamical aspect: the <u>flux group</u>.
Let $\{\phi_r\}_{0\le r\le 1}$ be an isotopy of symplectic diffeomorphisms of (M,ω). Then

$$\omega\left(D\phi_r(\cdot), \frac{\partial \phi_r}{\partial r}\right) = \alpha_r$$

is a closed one-form.

The flux of the isotopy is

$$F(\{\phi_r\}) \overset{\text{def}}{=} \int_0^1 [\alpha_r]\, dr \in H^1(M;\mathbb{R})$$

<u>Definition</u> The flux group $\Gamma \subset H^1(M;\mathbb{R})$ is

$$\Gamma = \{ F(\{\phi_r\}) : \phi_0 = \phi_1 = id \}$$

<u>Theorem</u> (Ono) Γ is a discrete subgroup.

For each $\phi : (M, \omega_M) \longrightarrow (M, \omega_M)$ we have a <u>fixed point Floer cohomology</u> $HF^*(\phi)$, which is a $\mathbb{Z}/2$-graded vector space.

<u>Properties</u>

- Euler characteristic $\chi(HF^*(\phi))$ is the Lefschetz fixed point number

- Invariant under isotopies, $HF^*(\phi_0) \cong HF^*(\phi_1)$, <u>if</u> the flux vanishes.

- $HF^*(\mathrm{id}_M) \cong H^*(M)$

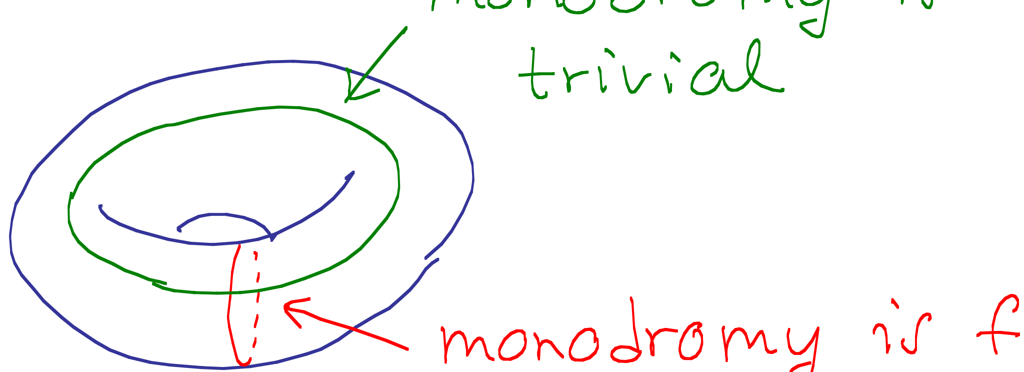
So, given any $\gamma \in H^1(M;\mathbb{R})$, take $\{\phi_t\}$ with $\phi_0 = \mathrm{id}_M$ and $F(\{\phi_r\}) = \gamma$. $HF^*(\phi_1)$ depends only on γ, and $\boxed{\text{if } \gamma \in \Gamma,\ HF^*(\phi_1) \cong H^*(M).}$

Application to <u>symplectic mapping tori</u>. Take (F, ω_F), for simplicity $\pi_1(F) = \{1\}$, and an automorphism $f : (F, \omega_F) \longrightarrow (F, \omega_F)$. Form

$$M = F \times \mathbb{R}^2 \Big/ \begin{matrix} (x,s,t) \sim (x, s-1, t) \\ (x,s,t) \sim (f(x), s, t-1) \end{matrix}$$

$$\omega_M = \omega_F + ds \wedge dt$$

$$\begin{tikzcd} F \arrow[r] & M \arrow[d] \\ & T^2 \end{tikzcd}$$



If we take $\gamma = [ds]$, there is an isotopy $\{\phi_r\}$ with flux γ and $\phi_0 = id_M$,

$$\phi_1(x, s, t) = (f(x), s, t).$$

<u>Lemma</u> $HF^*(\phi_1) \cong HF^*(f) \otimes H^*(T^2)$.

This can be used to prove, in suitable examples, that $[ds] \notin \Gamma$, which distinguishes M from the trivial mapping torus $F \times T^2$ (it seems that this goes beyond what GW invariants can do).