Geometric setup

Start with a Fano variety carrying a Lefschetz pencil of anticanonical hypersurfaces. We blow up the base locus of the pencil to obtain a fibration with Calabi-Yau fibres

$$\begin{tikzcd}
\mathbb{C}\times\delta M \arrow[d, phantom, "\cong" rotate=90] & & \mathbb{CP}^1\times\delta M \arrow[d, phantom, "\cong" rotate=90] & & \\
\delta E \arrow[r, phantom, "\subseteq"] \arrow[d, phantom, "\cap"] & & \delta\overline{E} \arrow[r, phantom, "\supseteq"] \arrow[d, phantom, "\cap"] & \delta M \arrow[d, phantom, "\cap"] \\
E \arrow[r, phantom, "\subseteq"] \arrow[d] & & \overline{E} \arrow[r, phantom, "\supseteq"] \arrow[d] & \overline{M} \arrow[d] \\
\mathbb{C} \arrow[r, phantom, "\subseteq"] & & \mathbb{CP}^1 \arrow[r, phantom, "\ni"] & \infty
\end{tikzcd}$$

Ex Pencil of cubics on $\mathbb{CP}^2$, then $\overline{E} \longrightarrow \mathbb{CP}^1$ is a rational elliptic surface, $\overline{M}$ = torus, δM = 9 points.

Assumptions $\tilde{H}^0(\delta M) = 0$
$H^1(\overline{M}) = 0$

If necessary, can replace this with $\tilde{H}^0(\delta M)^\Gamma = 0$, $H^1(\overline{M})^\Gamma = 0$ for a finite group Γ

Fukaya categories ($\mathbb{Z}$-graded)

$\mathcal{F}_K(\bar{M})$ Fukaya category, over the $\mathbb{C}$-coefficient 1-variable Novikov field K

$\mathcal{F}_q(\bar{M})$ Relative Fukaya category of $(\bar{M}, \delta M)$, over $\mathbb{Q}[[q]]$
$\mathcal{F}_q(\bar{M}) \otimes_{\mathbb{Q}[[q]]} K \subseteq \mathcal{F}_K(\bar{M})$

$\mathcal{B}_q$ Full subcategory of $\mathcal{F}_q(\bar{M})$ consisting of a basis of vanishing cycles

For idempotent-triangulated closures $(\ )^{\pi}$,

$$\mathcal{B}_q^{\pi} \simeq \mathcal{F}_q(\bar{M})^{\pi}$$

$$(\mathcal{F}_q(\bar{M}) \otimes_{\mathbb{Q}[[q]]} K)^{\pi} \simeq \mathcal{F}_K(\bar{M})^{\pi}$$

Definition We say that $\mathcal{B}_q$ is "defined over" a ring $R \subset \mathbb{Q}[[q]]$ if there is a $\tilde{\mathcal{B}}_q$ over R and a quasi-iso. $\tilde{\mathcal{B}}_q \otimes_R \mathbb{Q}[[q]] \simeq \mathcal{B}_q$

Note If $\mathcal{B}_q$ is defined over R, the part of $\mathcal{F}_K(\bar{M})$ consisting of Lagrangians with $H^1(L)=0$ is defined over $\overline{R} \subseteq K$.

The statement Let $r+2 = \dim H^2(\overline{E})$

There are $f, g_1, \dots g_r \in \mathbb{Q}[[q]]$, explicitly given by Gromov-Witten invariants of $\overline{E}$, so that $\mathcal{B}_q$ is defined over the sub-ring of $\mathbb{Q}[[q]]$ generated by $(f, g_1^{\pm 1}, \dots, g_r^{\pm 1})$.

Example Pencil of cubics. By using a suitable symmetry group Γ, this can be treated as if $r=0$.

The function is

$$f(q) = \left(\left(\frac{\eta(q)}{\eta(q^9)}\right)^3 + 3\right)^{-1} = q - 5q^4 + \dots$$

(a version of the mirror map)

Example Pencil of quintics in $\mathbb{C}P^4$

This has $r=0$, and

$$f = q - 154q^6 - 13127q^{11} + \dots$$

again is the mirror map (up to q^{30})

Relation with classical mirror symmetry depends on comparing quantum Lefschetz for

$$\text{quintic } \overline{M} \subseteq \mathbb{C}P^4$$
$$\overline{E} \subseteq \mathbb{C}P^1 \times \mathbb{C}P^4$$

and that will also be the general situation

Other approaches A general theorem of Toën ensures that $B_q \otimes_{\mathbb{Q}[[q]]} K$ is defined over a finitely generated subfield of K (but provides no specific information)

Ganatra-Perutz-Sheridan $(r=0)$ characterize the "canonical coordinate" q intrinsically in terms of the nc Hodge theory of $\mathcal{F}_q(\bar{M})$. If one knows HMS and that the mirror family extends over $\mathbb{A}^1$, a result equivalent to ours follows.

However, Hodge theory plays no part in our construction. Rather, what we do is to "show that the mirror extends over $\mathbb{A}^1$" intrinsically in terms of B_q.

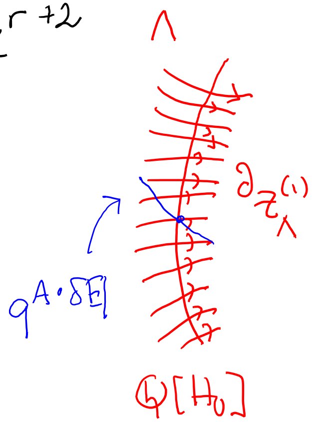
The functions $g_1,\dots,g_r$. Take

$$H = H_2(\overline{E}|;\mathbb{Z})/\mathrm{tors} \cong \mathbb{Z}^{r+2}$$

Λ = associated graded Novikov ring $\ni q^A$

$|q^A| = 2(A\cdot\overline{M})$ (intersection with fibre)

$\mathrm{val}(q^A) = A\cdot\delta E|$ (intersection with exceptional)



The valuation gives a filtration $\Lambda_{\geq k}\subseteq\Lambda$, and $\Lambda_{\geq 0}/\Lambda_{\geq 1}\cong\mathbb{Q}[H_0]$

$$H_0 = \{A\in H : A\cdot\delta E| = 0\}$$

Every class $z\in H^2(\overline{E}|;\Lambda^j)$ gives a derivation $\partial_z : \Lambda^*\longrightarrow\Lambda^{*+j}$. Take

$$z^{(1)}_\Lambda = \sum_{A\cdot\overline{M}=1} z_A\, q^A$$

$z_A \in H^2(\overline{E}|)$, one-point genus zero GW invariant

Then $\partial_{z^{(1)}_\Lambda}(\Lambda_{\geq 0})\subseteq\Lambda_{\geq 0}$.

Lemma Any $\bar{h}\in\mathbb{Q}[H_0]$ has a unique extension $h\in\Lambda_{\geq 0}$, $\partial_{z^{(1)}_\Lambda}h = 0$.

Take a basis $\bar{h}_1,\dots,\bar{h}_r$ of $\mathbb{Q}[H_0]$, extend to $h_1,\dots,h_r$, and then apply $q^A\mapsto q^{(A\cdot\delta E|)}$ to get $g_1,\dots,g_r$ from the theorem.

The function f Recall the classical Schwarzian

$$S_q f = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2$$

An inhomogeneous Schwarzian equation

$$S_q f + h = 0 \qquad h \in \mathbb{Q}[[q]]$$

has a solution $f(0)=0$, $f'(0)\neq 0$, unique up to

$$f \longmapsto \frac{f}{af+b} \qquad a \in \mathbb{Q}^\times,\ b \in \mathbb{Q}.$$

f is the quotient of solutions of $s'' - (g/2)\, s = 0$.

In our situation, there is a Schwarzian with differentiation replaced by $\partial_{Z^{(1)}_\Lambda}$. Call it

$$S_{Z^{(1)}_\Lambda} : \{ k \in \Lambda^i_{\geq 1} \text{ such that } q^{A_*} k \in \Lambda_{\geq 0} \text{ is invertible} \} \longrightarrow \Lambda^4_{\geq 0}$$

$A_* \in H_2(\overline{E})$ the class $[\mathbb{CP}^1] \times pt$ in $\mathbb{CP}^1 \times \delta M = \overline{\delta E}$

We take $Z^{(2)}_\Lambda = \sum_{A \cdot \overline{M} = 2} q^A z_A \in \Lambda^4_{\geq 0}$, solve

$$k \in \Lambda^0_{\geq 1}, \quad \boxed{S_{Z^{(1)}_\Lambda} k + 8 Z^{(2)}_\Lambda = 0}$$

and get f by $q^A \longmapsto q^{A \cdot \delta E|}$.

Mirror symmetry intuition

The mirror of

$$(\bar{E}, \bar{M}) \longrightarrow (\mathbb{CP}^1, \infty)$$

(when it exists, and in a suitably generalized sense) is a variety $X|$ over Λ carrying anticanonical sections σ, τ with no common zero,

base point free

$$\frac{\tau}{\sigma} : X| \longrightarrow \mathbb{P}^1$$

$Y = \sigma^{-1}(0)$ is the mirror of $\bar{M}$

$(X|) \setminus Y$ is the mirror of $\bar{E}$, with $\frac{\tau}{\sigma}$ = superpotential

One can formalize this statement without mirror symmetry: it says that the Fukaya category of $\bar{E} \longrightarrow \mathbb{C}$ should carry a pair of natural transformations

$$\sigma, \tau : \text{Serre functor}[-n] \longrightarrow \text{identity}$$

which are the leading terms of a noncommutative linear system.

In this context, $\partial_{z^{(1)}_{\Lambda}}$ describes a direction in which $X|$ does not change, but (σ, τ) do, by a linear differential equation ($\to$ Schwarzian).

The $r=0$ special case Then,

$$z^{(1)} = \sum_{A\cdot\overline{\delta M}=1} z_A\, q^{A\cdot\delta E|}$$

from A_*

$$\in [\delta E|]\, q^{-1} + H^2(\overline{E}|)[[q]]$$

$$z^{(2)} = \sum_{A\cdot\overline{\delta M}=2} z_A\, q^{A\cdot\delta E|}$$

$$\in \mathbb{Q}[[q]].$$

Write

$$q^{-1}[\delta E|] = \psi\, z^{(1)} - \eta\,[\tilde{M}]$$

$$\psi \in 1 + q\mathbb{Q}[[q]],\quad \eta \in \mathbb{Q}[[q]]$$

The category $\mathcal{B}_q$ is then defined over $\mathbb{Q}[f]$, where f is a solution of

$$S_q f + \left[8 z^{(2)}\psi^2 + \left(\eta - \frac{\psi'}{\psi}\right)' + \frac{1}{2}\left(\eta - \frac{\psi'}{\psi}\right)^2 \right] = 0$$

Cor If $z^{(1)}, z^{(2)}$ are locally convergent, so is f, and we get a locally convergent version of the Fukaya category of $\overline{M}$ (for Lagrangians with $H^1(L)=0$).

<u>Filtering $\mathcal{B}_q$</u> As $\mathbb{Q}[[q]]$-module

$\mathcal{B}_q \cong \mathcal{B}[[q]]$,

$$\mathcal{B} = \bigoplus_{i,j=1}^{m} CF^*(V_i, V_j)$$

the V_i are spheres, so

unit

$CF^*(V_i, V_i) = \mathbb{Q} e_i \oplus \mathbb{Q} \cdot t_i$

$|e_i| = 0$, $|t_i| = n-1$. write

$\mathcal{B} = \mathcal{A} \oplus \mathcal{P}[1]$,

$$\mathcal{A} = \bigoplus_i \mathbb{Q} e_i \oplus \bigoplus_{i<j} CF^*(V_i, V_j)$$

$$\mathcal{P}[1] = \bigoplus_i \mathbb{Q} t_i \oplus \bigoplus_{i>j} CF^*(V_i, V_j)$$

actually, need σ only up to multiplication with $\mathbb{Q}[[q]]^\times$

Take the A_∞-structure of $\mathcal{B}$ (for $q=0$, so working in M)

- $\mathcal{A}$ is an A_∞-subalgebra
- $\mathcal{P}[1] = \mathcal{B}/\mathcal{A}$ is an $\mathcal{A}$-bimodule quasi-isomorphic to $\mathcal{A}^\vee[-n]$
- The next piece of $\mu_{\mathcal{B}}$,

$$\mathcal{A} \otimes \cdots \otimes \mathcal{A} \otimes \mathcal{P}[1] \otimes \cdots \otimes \mathcal{A} \longrightarrow \mathcal{A}$$

gives an $\mathcal{A}$-bimodule map

$\sigma : \mathcal{A}^\vee[-n] \longrightarrow \mathcal{A}$.

<u>Lemma</u> Those three pieces determine $\mathcal{B}$ up to quasi-isomorphism.

Turning on the parameter q, one has $\mathcal{B}_q \cong \mathcal{A}_q \oplus \mathcal{P}_q[1]$ with corresponding properties, in particular

of bimodules

$$\sigma_q \in H^n(hom(\mathcal{A}_q^\vee, \mathcal{A}_q))$$

Importantly, these have equivalent interpretations in terms of $\overline{E} \longrightarrow \mathbb{C}$: $\mathcal{A}_q$ is part of the Fukaya category of that Lefschetz fibration, and σ_q is the "diagonal class" of that category, a natural structure in the noncompact context.

<u>Remark</u> $H^*(hom(\mathcal{A}_q^\vee, \mathcal{A}_q))$ has a natural $\mathbb{Z}/2$-action, and σ_q lies in

$$H^*(hom(\mathcal{A}_q^\vee, \mathcal{A}_q))^{\mathbb{Z}/2} \cong HH^{*,(n)}(\mathcal{A}_q, 2)$$

Higher Hochschild cohomology (Kontsevich-Vlassopoulos, (n) = sign).

<u>Theorem</u> The A_∞-deformation $\mathcal{A}_q$ is trivial.

(First observed in examples by Auroux-Katzarkov-Orlov). There is a geometric reason behind vanishing of

$$[\partial_q \mu^*_{\mathcal{A}_q}] \in HH^2(\mathcal{A}_q).$$

From triviality of the deformation, we get a q-differentiation operator

$$\nabla_q : H^*(\hom(A_q^\vee, A_q)) \circlearrowleft$$

$$\text{\rotatebox{90}{$\cong$}}$$

$$\partial_q : H^*(\hom(A^\vee, A))[[q]] \circlearrowleft$$

Theorem σ_q satisfies

$$\boxed{\nabla_q^2 \sigma_q + \left(\eta - \frac{\psi'}{\psi}\right)\nabla \sigma_q - 4 z^{(2)} \psi^2 \sigma_q = 0}$$

So,

$$\sigma_q = s_0 \sigma_0 + s_1 \sigma_1$$

$$\Rightarrow \quad \frac{\sigma_q}{s_0} = \sigma_0 + \left(\frac{s_1}{s_0}\right)\sigma_1$$

where $\sigma_0, \sigma_1 \in H^*(\hom(A^\vee, A))$ and s_0, s_1 are solutions of

$$s'' + \left(\eta - \frac{\psi'}{\psi}\right) s' - 4 z^{(2)} \psi s = 0.$$

with $s_0|_{q=0} = 1$, $s_1|_{q=0} = 0$, $s_1'|_{q=0} \neq 0$.

It follows that for $f = s_1/s_0$,

which satisfies a Schwarzian equation

$$\frac{\sigma_q}{s_0} \in H^*(\hom(A^\vee, A)) \otimes_{\mathbb{Q}} (\mathbb{Q} \oplus \mathbb{Q} f)$$

$$\subseteq H^*(\hom(A^\vee, A))[[q]]$$

Correspondingly, the part of $\mu^*_{\mathcal{B}_q}$ which "eats up r P's" is polynomial in f of degree $\leq r$.

<u>Homological mirror symmetry</u>

Take a pencil of Calabi-Yau hypersurfaces in $\mathbb{CP}^n$. We use only a subcategory $\mathcal{B}_q^*$ (vanishing cycles for the pencil with toric fibre at ∞) and correspondingly $\mathcal{A}_q^*$.

First, for $q=0$:

<u>Theorem</u> (Futaki-Ueda)

$$\mathcal{A}^* \simeq D^b \mathrm{Coh}_\Gamma(\mathbb{P}^n)$$

$$\Gamma = (\mathbb{Z}/_{n+1})^{n-1}$$

derived

Now, $\mathcal{B}$ is determined by $\mathcal{A}$ and an element of

$$H^0(\mathbb{P}^n, K_{\mathbb{P}^n}^{-1})^\Gamma$$

In fact, one can show this must be torus-invariant, so a multiple of $z_0 \cdots z_n$. Assume we know the multiple is nonzero. Similarly, $\mathcal{B}_q$ is determined by $\mathcal{A}_q = \mathcal{A}[[q]]$ and an element of

$$H^0(\mathbb{P}^n, K_{\mathbb{P}^n}^{-1})^\Gamma \oplus H^0(\mathbb{P}^n, K_{\mathbb{P}^n}^{-1})^\Gamma f$$

we only need to figure out this element, that it's $z_0^{n+1} + \cdots + z_n^{n+1}$.