A very long scary title

PAUL SEIDEL, MIT

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(Anticanonical) Lefschetz pencils

Let X^n = a smooth complex variety which is Fano (K_X^{-1} is ample); and two linearly independent sections

$$s, t \in H^0(X, K_X^{-1}).$$

This gives rise to a pencil of hypersurfaces

$$Y_z = \left\{ \frac{s}{t} = z \right\} \subseteq X,$$

for $z \in \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$. We assume that $Y = Y_\infty$ is smooth, and that the pencil has only the simplest kind of singularity. The base locus $B = Y_0 \cap Y_\infty$ should also be smooth.

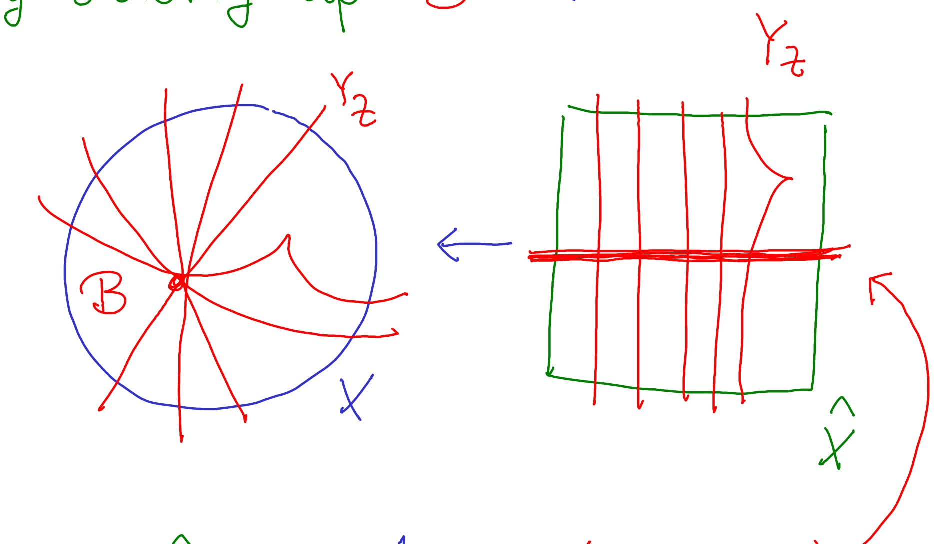
We get two fibrations s/t

$$E = X \setminus Y_\infty \longrightarrow \mathbb{C}$$

and

$$\hat{X} = \{ (x, z) \in X \times \mathbb{CP}^1 : s(x) = z\, t(x) \} \to \mathbb{CP}^1$$

The graph of the pencil, obtained by blowing up $B \subseteq X$

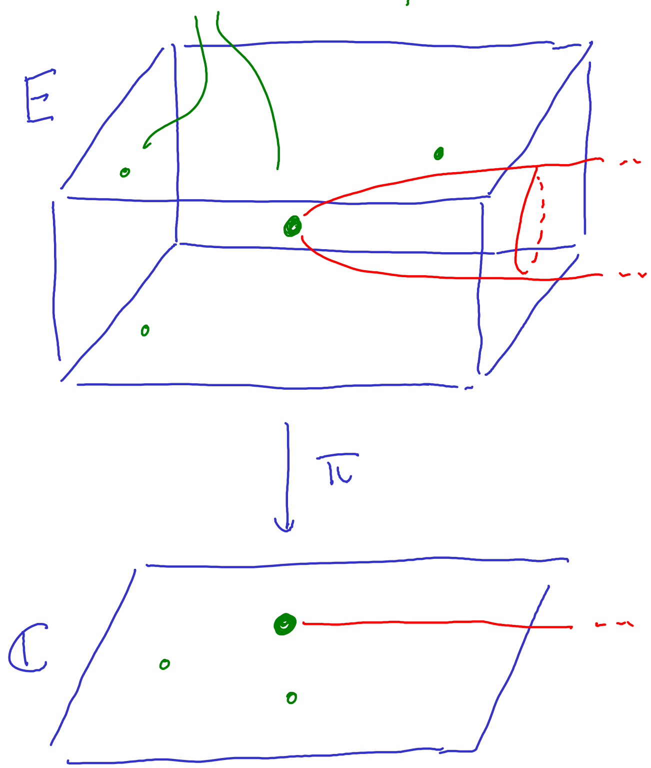


Note $\hat{X} \to \mathbb{CP}^1$ has "trivial" sections, which give a copy of B in each fibre.

Lefschetz thimbles and vanishing cycles

These are the standard tools for analyzing the (symplectic) topology of any Lefschetz fibration. $\pi : E \longrightarrow \mathbb{C}$

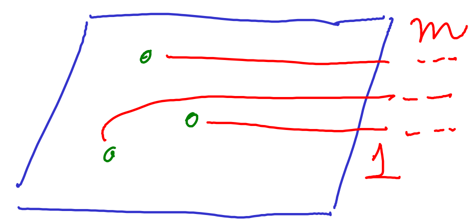
critical point(s)



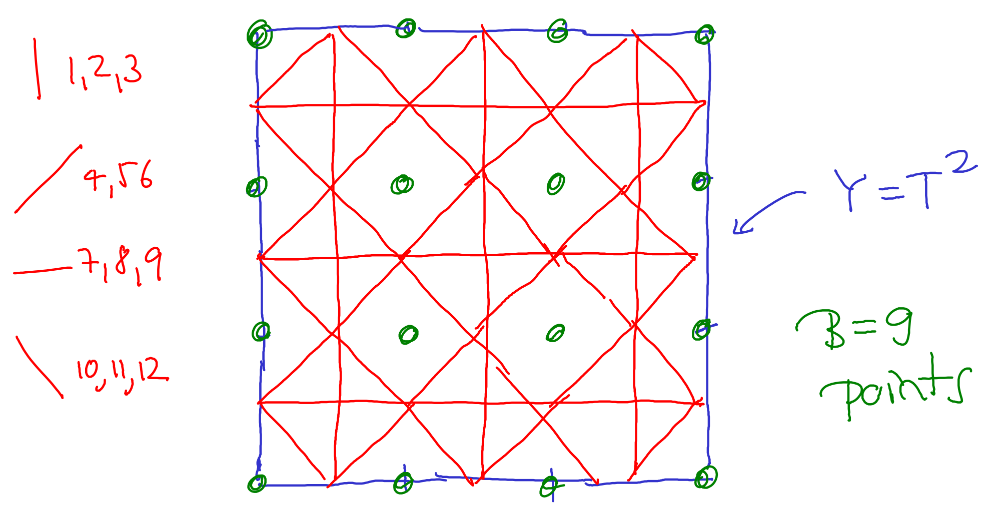
Lefschetz thimbles are properly embedded Lagrangian $\mathbb{R}^n \cong \Delta \subset E$

Vanishing cycles are the corresponding Lagrangian $S^{n-1} \cong V \subset$ smooth fibre.

We choose a basis of such cycles: $(\Delta_1, \dots, \Delta_m)$ or $(V_1, \dots, V_m)$



Example Cubic pencil on $X = \mathbb{CP}^2$.
It has 12 vanishing cycles



The Fukaya category

The Fukaya category $\mathcal{B}_q$ measures intersections between the vanishing cycles (in a fibre).

ordering is irrelevant

$$Ob(\mathcal{B}_q) = \{V_1, \dots, V_m\}$$

$$\hom_{\mathcal{B}_q}(V_i, V_j) = \bigoplus_{x \in V_i \cap V_j} \mathbb{C}((q))\, x$$

if $V_i \pitchfork V_j$

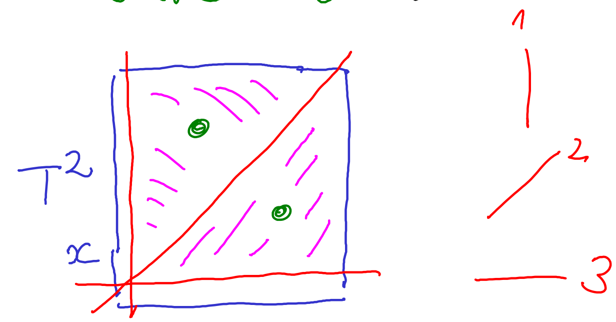
$$\hom_{\mathcal{B}_q}(V_i, V_i) = \mathbb{C}((q))\, x_{min} \oplus \mathbb{C}((q))\, x_{max}$$

Think of the minimum and maximum of a Morse function on $V_i \cong S^{n-1}$

$\mathcal{B}_q$ is an A_∞-category (or A_∞-algebra) over $\mathbb{C}((q))$

The parameter q counts "area", here in a discretized version as intersection number with the base locus.

Example



The composition

$$\begin{array}{ccc}
\hom_{\mathcal{B}_q}(V_2, V_3) & \otimes_{\mathbb{C}((q))} & \hom_{\mathcal{B}_q}(V_1, V_2) \\
\cong & & \\
\mathbb{C}((q))\, x \otimes x & & \\
& \downarrow & \\
& \hom_{\mathcal{B}_q}(V_1, V_3) & \\
& \cong & \\
& \mathbb{C}((q))\, x &
\end{array}$$

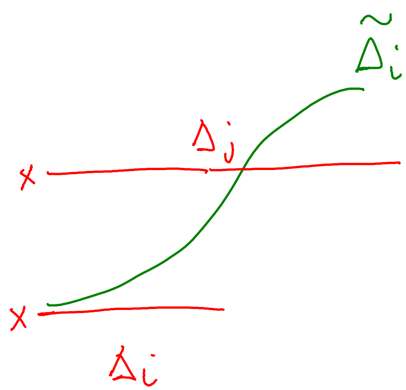
counts holomorphic triangles,

$$x \otimes x \longmapsto \sum_{m=-\infty}^{\infty} q^{m^2} x$$

Kontsevich's version of the Fukaya category

The idea is to consider the intersection of the Lefschetz thimbles – their noncompactness is accounted for by an asymmetric perturbation. If we call the resulting A_∞-category $\mathcal{A}_q$, then

$$\hom_{\mathcal{A}_q}(\Delta_i,\Delta_j) = \bigoplus_{x\in\tilde{\Delta}_i\cap\Delta_j} \mathbb{C}((q))\,x$$

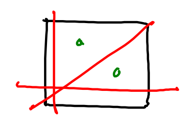


$$\hom_{\mathcal{A}_q}(\Delta_i,\Delta_j) = \begin{cases} \hom_{\mathcal{B}_q}(V_i,V_j) & i<j \\ 0 & i>j \\ \mathbb{C}((q)) & i=j \end{cases}$$

One can think of $\mathcal{A}_q$ as a subalgebra of $\mathcal{B}_q$ (retaining less information).

<u>Observation</u> In many examples (but not always), $\mathcal{A}_q$ depends trivially on q <u>up to isomorphism</u>:

$$\mathcal{A}_q \cong \mathcal{A}\otimes_{\mathbb{C}} \mathbb{C}((q))$$

<u>Ex.</u> For , this is clear (rescale the basis of $\hom(\Delta_1,\Delta_3)$).

<u>Ex.</u> Also true for the cubic pencil on $\mathbb{CP}^2$ (but not on $\mathbb{F}_1$).

<u>Observation</u> The q-dependence of $\mathcal{B}_q$ is nontrivial, but (in examples) highly constrained.

How to analyze the structure more precisely?

Use "mirror geometry" of $\mathcal{A}_q$ and $\mathcal{B}_q$
(in the sense of noncommutative algebraic geometry)

What have we used so far?

(thinking of the original Lefschetz pencil setup)

- [x] Lefschetz fibration $E \to \mathbb{C}$

  $\to$ Fukaya categories $\mathcal{A}_q$ and $\mathcal{B}_q$ (+ more)

? ☐ Fibration extends to $\hat{X} \longrightarrow \mathbb{CP}^1$

? ☐ Existence of trivial sections

Noncommutative geometry of line bundles

Let $\mathcal{A}$ = (dg / A_∞) algebra / $\mathbb{K}$

A "line bundle" $\mathcal{P}$ over $\mathcal{A}$ is characterized by its action on $\mathcal{A}$-modules,

$$(*) \quad - \overset{L}{\underset{\mathcal{A}}{\otimes}} \mathcal{P} : \mathcal{A}^{\mathrm{mod}} \longrightarrow \mathcal{A}^{\mathrm{mod}}$$

A "line bundle" is an $\mathcal{A}$-bimodule which is invertible with respect to tensor product,

$$\mathcal{P} \overset{L}{\otimes}_{\mathcal{A}} \mathcal{P}^{-1} \cong \mathcal{A}$$

"sections of a line bundle" are natural transformations from the identity to $(*)$

"Sections":

$$H^0(\hom_{\mathcal{A}\text{bimod}}(\mathcal{A}, \mathcal{P}))$$

"Dual sections":

$$H^0(\hom_{\mathcal{A}\text{bimod}}(\mathcal{A}, \mathcal{P}^{-1}))$$

$$\cong H^0(\hom_{\mathcal{A}\text{bimod}}(\mathcal{P}, \mathcal{A}))$$

Example The "canonical bundle"

$$\mathcal{A}^\vee = \hom_{\mathbb{K}}(\mathcal{A}, \mathbb{K})$$

if $\dim_{\mathbb{K}} H^*(\mathcal{A}) < \infty$.

Example For smooth $\mathcal{A}$, the "anticanonical bundle"

$$\mathcal{A}^! = \hom_{\mathcal{A}\text{bimod}}(\mathcal{A}, \mathcal{A} \otimes_{\mathbb{K}} \mathcal{A})$$

Noncommutative geometry of divisors

In ordinary algebraic geometry
$s \in H^0(X, \mathcal{L}) \setminus \{0\}$,

$$Y = s^{-1}(0) \iff \mathcal{O}_Y = \mathcal{O}_X/(s)$$

We use a Koszul resolution

$$(Y, \mathcal{O}_Y) \simeq (X, \{\mathcal{L}^{-1} \xrightarrow{s} \mathcal{O}_X\})$$

So, a "divisor" is an extension of $\mathcal{O}_X$ by $\mathcal{L}^{-1}[1]$. This allows the degenerate case $s=0$.

In this (commutative) case, $\mathcal{L}^{-1}[1] \oplus \mathcal{O}_X$ is a sheaf of rank 1 exterior algebras; only the differential depends on s. This is no longer true in noncommutative geometry.

Definition Let $\mathcal{A}$ be a (dg/A_∞) algebra, and $\mathcal{P}$ an invertible bimodule. A noncommutative divisor (associated to $\mathcal{P}^{-1}$) is a (dg/A_∞) algebra structure on

$$\mathcal{B} = \mathcal{P}[1] \oplus \mathcal{A}$$

which contains $\mathcal{A}$ as a subalgebra, and recovers the given $\mathcal{A}$-bimodule structure on $\mathcal{P} = \mathcal{B}/\mathcal{A}$.

A noncommutative divisor has a "leading order term"

$$\sigma \in H^0(hom_{\mathcal{A}^{bimod}}(\mathcal{P}, \mathcal{A})),$$

but (in general) that's not all.

Noncommutative geometry of pencils

We need a notion of family of noncommutative divisors parametrized by $\mathbb{P}^1_{\mathbb{K}}$ (or, a family parametrized by $\mathbb{K}^2$ which is homogeneous, "filling in" with the trivial divisor $\mathbb{B} = \mathbb{A} \oplus \mathbb{P}[1]$ at $(0,0) \in \mathbb{K}^2$). Introduce a new grading ("weight")

$$\mathbb{B} = \underset{\text{weight } 0}{\mathbb{A}} \oplus \underset{\text{weight } -1}{\mathbb{P}[1]}$$

$$V = \mathbb{K}^2 \quad (\text{weight } -1)$$

An nc divisor structure has only pieces that don't decrease weights, e.g. $\mathrm{weight}(\sigma) = 1$.

Definition A noncommutative pencil (associated to $\mathbb{P}^{-1}$) consists of maps, homogeneous with respect to weight,

differential: $\mathbb{B} \to \mathbb{B} \otimes_{\mathbb{K}} \mathrm{Sym}(V)[1]$

product: $\mathbb{B} \otimes_{\mathbb{K}} \mathbb{B} \longrightarrow \mathbb{B} \otimes_{\mathbb{K}} \mathrm{Sym}(V)$

(... for A_∞-structures)

which specialize to a nc divisor at each point $w \in W = V^\vee$

The leading order part consists of

$$\sigma, \theta \in H^0(\hom_{\mathbb{A}\text{bimod}}(\mathbb{P}, \mathbb{A}))$$

Because of homogeneity, we get nc divisors $\mathbb{B}_z$, $z \in \mathbb{P}(W)$. The base locus is the $\mathbb{A}$-bimodule

$$\mathbb{B}_0 \overset{L}{\otimes}_{\mathbb{A}} \mathbb{B}_\infty.$$

(etc. etc.)

Application to Lefschetz pencils

Let $\mathcal{A}_q$ be the Fukaya category of a symplectic Lefschetz fibration $E \longrightarrow \mathbb{C}$, and $\mathcal{B}_q$ the category associated to the fibre. Poincaré duality in Floer theory shows that as vector spaces,

$$\mathcal{B}_q = \mathcal{A}_q \oplus \mathcal{A}_q^\vee [1-n]$$

Fact $\mathcal{B}_q$ is a noncommutative divisor on $\mathcal{A}_q$ associated to the invertible bimodule $\mathcal{A}_q^\vee[-n]$

In the case of an anticanonical Lefschetz pencil, obstruction theory shows that $\mathcal{A}_q$ and the leading order term σ_q determine $\mathcal{B}_q$.

Conjecture For a Lefschetz fibration that extends over $\mathbb{CP}^1$, $\mathcal{A}_q$ carries the structure of a noncommutative pencil (with $\mathcal{B}_q$ the fibre at ∞).

In the case of anticanonical Lefschetz pencils, one can prove this (but the proof is not "nice", it uses obstruction theory – one only has to construct the leading order term)

Conjecture For a Lefschetz fibration that extends to $\mathbb{CP}^1$ with smooth fibre at ∞, the noncommutative pencil has empty base locus.

Geometric origin of the noncommutative pencil structure

We are looking at an nc pencil on $\mathcal{A}_q$ with associated invertible bimodule $\mathcal{A}_q^\vee[-n]$. The leading order part is given by a pair of elements of

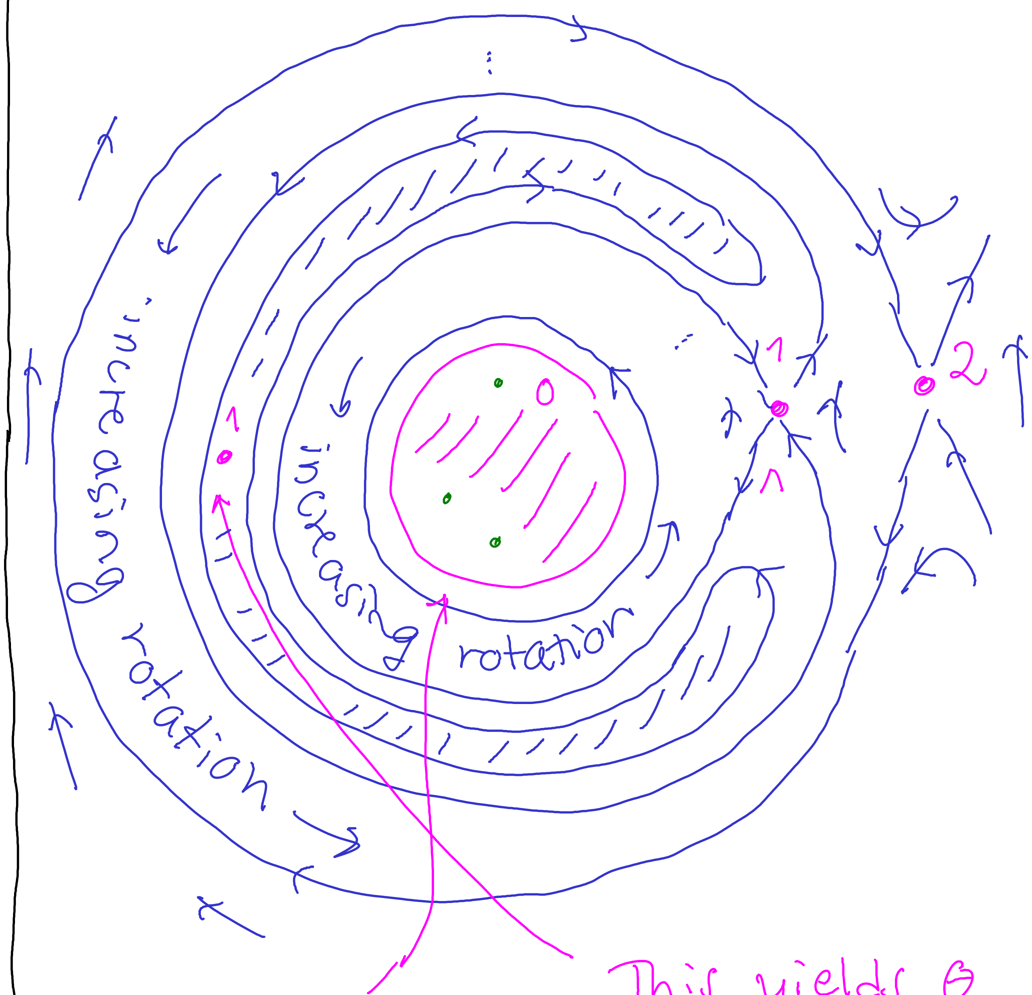
$$H^0(hom_{\mathcal{A}_q}^{bimod}(\mathcal{A}_q^\vee[-n], \mathcal{A}_q))$$

$$\uparrow \text{ open-closed string map}$$

$$HF^*(\mu^2)$$

Here, $HF^*(\mu^2)$ is the fixed point Floer cohomology of a certain symplectic automorphism μ^2 of the total space E, which is fibered over $\mathbb{C}$. The two "generators" of the pencil come from different components of the fixed locus.

Action of μ^2 on the base $\mathbb{C}$:



This yields the first map σ (for any Lefschetz fibration)

This yields θ (because the monodromy is trivial)

What have we used so far?

- ☑ Lefschetz fibration $E \to \mathbb{C}$
  → Fukaya category $\mathcal{A}_q$, and its noncommutative divisor $\mathcal{B}_q$

- ☑ Fibration extends to $\hat{X} \longrightarrow \mathbb{CP}^1$
  → noncommutative pencil on $\mathcal{A}_q$ whose fibre at ∞ is $\mathcal{B}_q$

- ☑ Fibre at ∞ is smooth
  → noncommutative pencil has empty base locus

? ☐ Existence of trivial sections

Differentiating in q-direction

Take $\mathcal{A}_q$ = (dg / A_∞) algebra over $\mathbb{K} = \mathbb{C}((q))$. Think of it as a formal family parametrized by q. There should be a "Kodaira-Spencer class" which measures the q-dependence.

Extend ∂_q to a derivation of $\mathcal{A}_q$ as a $\mathbb{K}$-vector space.

Definition The Kaledin class

$$\kappa = [\partial_q(\text{algebra structure})]$$
$$\in HH^2(\mathcal{A}_q, \mathcal{A}_q)$$
$$\cong H^2(\hom_{\mathcal{A}_q\text{bimod}}(\mathcal{A}_q, \mathcal{A}_q))$$

Vanishing of this class is an infinitesimal rigidity result.

Lemma If we work over $\mathbb{C}[[q]]$, $\kappa = 0 \Leftrightarrow$ the q-dependence is trivial:

$$\mathcal{A}_q \cong \mathcal{A} \otimes_{\mathbb{C}} \mathbb{C}[[q]]$$

Let's apply this to Fukaya categories. There is an open-closed string map

$$H^*(Y; \mathbb{K}) \xrightarrow{\cong} HH^*(\mathcal{B}_q, \mathcal{B}_q)$$
$$q^{-1}[\omega_Y] \longmapsto \kappa$$

Hence, for the Fukaya category of a Calabi-Yau hypersurface, the q-dependence is always nontrivial.

Differentiation in q-direction (continued)

For the Fukaya category of a Lefschetz fibration $E \to \mathbb{C}$, $HH^*(\mathcal{A}_q, \mathcal{A}_q)$ can again be expressed as fixed point Floer cohomology. In the case of fibrations arising from anti-canonical Lefschetz pencil, we get a long exact sequence

holomorphic sections

$$\begin{tikzcd}[column sep=small] & H^*(E) \arrow[dr] & \\ H^{*-2}(Y) \arrow[ur] & & HH^*(\mathcal{A}_q, \mathcal{A}_q) \arrow[ll, "\deg -1"] \end{tikzcd}$$

$$1 \mapsto z^{(1)}|E \qquad q^{-1}[\omega_E] \mapsto K$$

The informal picture is that we use sections to extend cycles from the fibre at ∞ to $\hat{X}$, then restrict to E.

Note that

trivial sections $C \gg 0$ fibre

$$[\omega_{\hat{X}}] = [\mathbb{P}^1 \times B] + C\,[Y]$$

$$\Rightarrow \quad [\omega_E] = [\mathbb{C} \times B]$$

Let $z^{(1)} \in H^2(\hat{X}; \mathbb{C}((q)))$ be the count of holo sections

$$z^{(1)} = \sum_{\substack{A \in H_2(\hat{X}) \\ A \cdot Y = 1}} z_A\, q^{A \cdot [\mathbb{P}^1 \times B]}$$

trivial sections

$$= q^{-1}\,[\mathbb{P}^1 \times B] + O(1)$$

<u>Theorem</u> If $z^{(1)}$ lies in the subspace spanned by $[\mathbb{P}^1 \times B]$ and $[Y]$, then $\mathcal{A}_q$ depends trivially on q.

A second order differential equation

Suppose that the assumption of the previous theorem is satisfied, and write

$$q^{-1}[\mathbb{P}^1 \times B] = \psi z^{(1)} - \eta [Y]$$

for $\psi \in \mathbb{C}[[q]]^\times$, $\eta \in \mathbb{C}[[q]]$.

Consider also the count of holomorphic bisections of $\hat{X} \to \mathbb{CP}^1$,

$$z^{(2)} \in H^0(\hat{X}; \mathbb{C}[[q]]) = \mathbb{C}[[q]]^\times$$

Take the (formal linear 2nd order) ODE

$$\alpha'' + \left(\eta - \frac{\psi'}{\psi}\right)\alpha' - 4z^{(2)}\psi^2\alpha = 0$$

Let $\alpha(q) = q + \cdots$, $\beta(q) = 1 + \cdots$ be solutions, and set

$$\theta = \alpha/\beta \in \mathbb{C}[[q]]^\times$$

The idea is that on

$$\mathcal{A}_q \cong \mathcal{A} \otimes_{\mathbb{C}} \mathbb{C}((q))$$

the q-dependence of the noncommutative divisor is that it moves inside a noncommutative pencil in a way described by $(*)$.

Conjecture The Fukaya category $\mathcal{B}_q$ is defined over $\mathbb{C}[\theta]$, meaning that

$$\mathcal{B}_q \cong \mathcal{B}_\theta \otimes_{\mathbb{C}[\theta]} \mathbb{C}((q))$$

$\mathcal{B}_\theta$ is an invariant of the Lefschetz pencil (it is not, at least not obviously, an invariant of Y by itself)

Schwarzian differential equation

It is a classical observation that quotients $\theta = \alpha/\beta$ are solutions of a nonlinear third order equation: if

$$\mathcal{S}\,\theta = \left(\frac{\theta''}{\theta'}\right)' - \frac{1}{2}\left(\frac{\theta''}{\theta'}\right)^2$$

is the Schwarzian operator, then

$$\mathcal{S}\,\theta + 8\,z^{(2)}\psi^2 + \left(\eta - \frac{\psi'}{\psi}\right)' + \frac{1}{2}\left(\eta - \frac{\psi'}{\psi}\right)^2 = 0.$$

Example For the cubic pencil on $\mathbb{CP}^2$,

$$\mathcal{S}\,\theta + \frac{E_4(q^3) - 1}{2q^2} = 0$$

whose solution is the (inverse) of the "mirror map".

(The end, for now...)