Symplectic topology as sheaf theory ?

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We begin with a form of homological mirror symmetry for hypersurfaces in the torus $(\mathbb{C}^*)^n$. Data:

$P \subset \mathbb{R}^n$ integer polytope (bounded, with nonempty interior)

$\varphi_{\mathbb{Z}} : P_{\mathbb{Z}} = P \cap \mathbb{Z}^n \longrightarrow \mathbb{Z}$

We then define a hypersurface $M = F^{-1}(0) \subset (\mathbb{C}^*)^n$ by

$$F(z_1, \dots, z_n) = \sum_{\nu \in P_{\mathbb{Z}}} \varepsilon^{\varphi(\nu)} z_1^{\nu_1} \cdots z_n^{\nu_n}$$

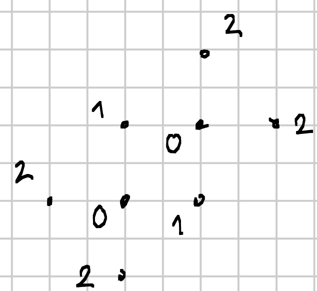
for $\varepsilon > 0$ small. Actually, any two hypersurfaces with a given Newton polytope P and generic coefficients are symplectically isomorphic. However, we find it useful to choose specific coefficients which allow us to analyze M.

Additional assumption:

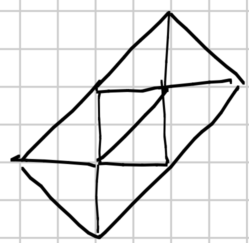
There is a (necessarily unique) decomposition of P into integer simplices P_k of volume $1/n!$, such that the extension of $\varphi_{\mathbb{Z}}$ to a function φ affine on each P_k is convex, and only differentiable on $\mathrm{int}(P_k)$.

In that case, M is smooth for all $\varepsilon << 1$ (Kushnirenko) and has a corresponding decomposition in $(2n-2)$-dimensional pairs-of-pants (Mikhalkin).

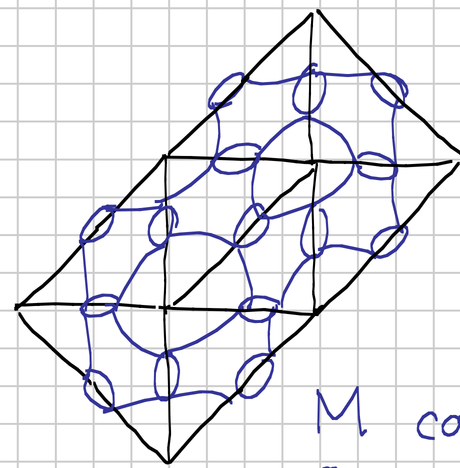
<u>Ex</u>.



$\varphi_{\mathbb{Z}}$



$P = \bigcup_k P_k$



M consists of 8 pairs-of-pants

In the lowest nontrivial dimension $n=2$, M has genus equal to the number of points in $P_{\mathbb{Z}} \setminus \partial P$; the number of punctures is the number of points in $P_{\mathbb{Z}} \cap \partial P$; and the Euler characteristic is $-2\operatorname{vol}(P)$ $\rightarrow$ Pick's formula.

<u>$2n$-dimensional pair-of-pants:</u>

$$N^n = \{ z \in (\mathbb{C}^*)^{n+1} : z_1 + \cdots + z_n + z_{n+1} + 1 = 0 \}$$
$$= \mathbb{CP}^n \setminus (n+2) \text{ hyperplanes}$$

This is an affine variety, $H^*(N^n) \cong H^*(T^{n+1})$ for $* \leq n$.

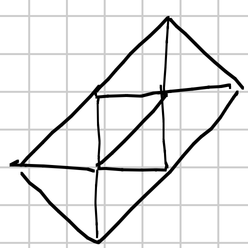
There is a (kind of) simplicial structure to $\{N^n\}$, with embeddings $N^n \times \mathbb{C}^* \hookrightarrow N^{n+1}$

Mirror construction (similar to work of Gross–Siebert, or more classical literature on toric degenerations): take the Legendre transform

$$\psi : \mathbb{R}^n \longrightarrow \mathbb{R},$$
$$\psi(w) = \max_v \; v \cdot w - \varphi(v)$$

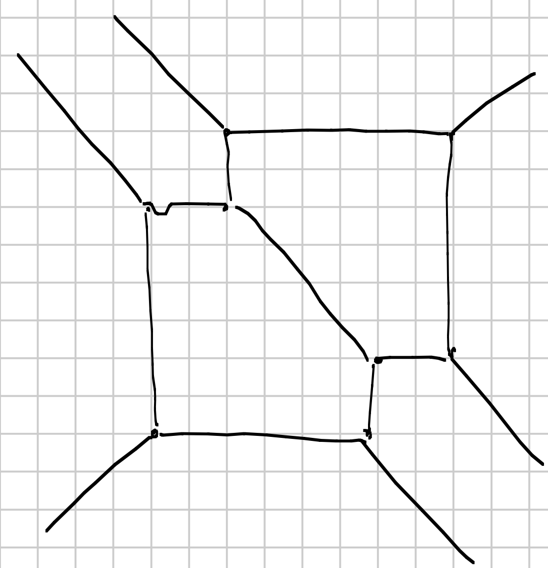
This is again convex, $\psi(\mathbb{Z}^n) \subset \mathbb{Z}$, piecewise affine, and induces a decomposition of $\mathbb{R}^n$ into (partly noncompact) polytopes Q_ℓ

<u>Ex</u>.



$P = \bigcup_k P_k$

Legendre duality



$\mathbb{R}^n = \bigcup_\ell Q_\ell$

We define an associated degeneration of toric varieties $W : \mathcal{X} \longrightarrow \mathbb{A}^1$, by $\mathcal{X} = \mathrm{Proj}(\mathcal{R})$, where $\mathcal{R}$ is a graded $\mathbb{C}[q]$-module, as follows:

Generators of $\mathcal{R}^k$ $\longleftrightarrow$ points $(v_1, \dots, v_{n+1}) \in \frac{1}{k}\mathbb{Z}^{n+1}$ with $v_{n+1} \geq \varphi(v_1, \dots, v_n)$

Multiplication $\mathcal{R}^k \times \mathcal{R}^\ell \longrightarrow \mathcal{R}^{k+\ell}$ $\longleftrightarrow$ weighted interpolation $(v, w) \longmapsto \frac{1}{k+\ell}(kv + \ell w)$

Action of q $\longleftrightarrow$ $v \longmapsto v + (0, \dots, 0, \frac{1}{k})$

The generic fibres are $W^{-1}(\mathbb{G}_m) \cong \mathbb{G}_m^{n+1}$, as one sees by looking at $\mathcal{R} \otimes_{\mathbb{C}[q]} \mathbb{C}[q, q^{-1}]$. The special fibre $W^{-1}(0)$ is the union of toric varieties associated to the Q_ℓ. In our case, there are Zariski charts $\mathbb{A}^{n+1} \hookrightarrow \mathcal{X}$, one for each P_k, in which $W(z) = z_1 \dots z_{n+1}$.

we define the category of Landau-Ginzburg branes (Orlov, earlier work of Eisenbud and others) to be

$$D(W) = D^b Coh(\mathcal{H}_0) / Perf(\mathcal{H}_0)$$

quotient category

There is also an underlying dg category $\mathcal{D}(W)$, and

$\boxed{\text{Conjecture There is a (cohomologically) full and faithful embedding } \mathcal{F}(M) \hookrightarrow \mathcal{D}(W)}$

One can think of this as being glued together from the following local picture: if $U = z_1 \cdots z_{n+1} : \mathbb{C}^{n+1} \longrightarrow \mathbb{C}$, then

$\boxed{\text{Conjecture For the pair-of-pants } N,\ \mathcal{F}(N) \hookrightarrow \mathcal{D}(U)}$

__Explicit description through matrix factorizations__ A matrix factorization of $\mathcal{U}$ is a two-periodic "complex" of free $\mathbb{C}[z_1, \dots, z_{n+1}]$-modules

$$\cdots \longrightarrow E_0 \xrightarrow{d_0} E_1 \xrightarrow{d_1} E_0 \xrightarrow{d_0} E_1 \longrightarrow \cdots$$

with $d_1 d_0 = \mathcal{U} \otimes \mathrm{id}_{E_0}$, $d_0 d_1 = \mathcal{U} \otimes \mathrm{id}_{E_1}$. Matrix factorizations form a differential $\mathbb{Z}/2$-category $\mathcal{MF}(\mathcal{U})$, which turns out to be equivalent to $\mathcal{D}(\mathcal{U})$. In our particular case, there are $(2^{n+1}-2)/2 = 2^n - 1$ obvious "scalar" matrix factorizations

$$\longrightarrow \mathbb{C}[z_1, \dots, z_{n+1}] \xrightarrow{z_{i_1} \dots z_{i_k}} \mathbb{C}[z_1 \dots z_{n+1}] \xrightarrow{\mathcal{U}/z_{i_1} \dots z_{i_k}} \mathbb{C}[z_1 \dots z_{n+1}] \to \cdots$$

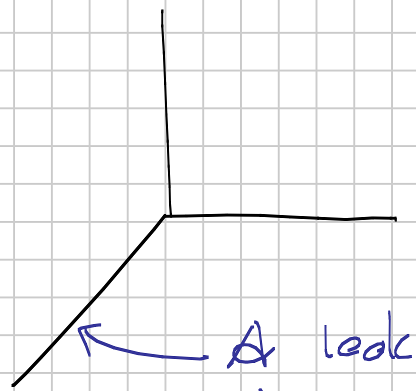
Take the direct sum of all of them, and let $\mathcal{A}$ be its endomorphism dg algebra in $\mathcal{MF}(\mathcal{U})$. Then

$$\mathcal{MF}(\mathcal{U}) \hookrightarrow \mathrm{mod}(\mathcal{A}) \leftarrow \text{dg derived category of modules}$$

In fact, $\mathcal{A}$ is linear over $\mathbb{C}[z_1, \ldots, z_{n+1}]$, hence can be thought of as a sheaf of dga's on affine space.

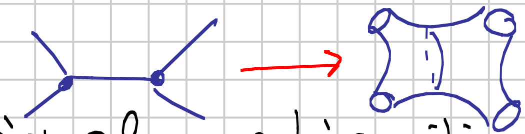
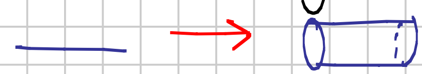
Ex. For $n=2$,

$\mathcal{A}$ acyclic outside the coordinate lines



$\mathcal{A}$ looks like $\mathbb{C}[z] \otimes Cl_2$, where Cl_2 is a (semisimple) Clifford algebra.

Instead of considering symplectic manifolds M given as hypersurfaces, one wants to construct them abstractly from pairs-of-pants, and get a model for $\mathcal{F}(M)$.

__Two-dimensional case__: , special attention must be paid to rotational ambiguities in the gluing. For the smooth part,  follows Kontsevich-Soibelman's model, but the absence of a distinguished zero-section causes problems (in fact, the two issues are clearly related).

__Def__ (Sheridan) signed tropical graph is a trivalent graph G, possibly with some semi-infinite edges, equipped with a sheaf of abelian group $\mathrm{Aff}^{\sigma}_{G}$ and a map $\mathrm{Aff}^{\sigma}_{G} \longrightarrow \mathcal{C}$ into the sheaf of continuous functions, locally as follows:

- over the smooth part, the model is $\mathbb{Z}/2 \oplus \mathrm{Aff}(\mathbb{R})$, where $\mathrm{Aff}(\mathbb{R})$ are integral affine functions. Note the presence of an extra automorphism

$\mathbb{Z}/2 \oplus \mathrm{Aff}(\mathbb{R}) \circlearrowleft$, $\quad (\alpha, f) \longmapsto (\alpha + df \bmod 2, f)$

- near a vertex, the model is $T = \mathbb{R}^+ \times 0 \cup 0 \times \mathbb{R}^+ \cup \{(t,t) : t \leq 0\} \subseteq \mathbb{R}^2$, with

  $\{(\alpha, f) \mid \alpha \in (\mathbb{Z}/2)^2$, f a function on T such that $f = g|T$ for some $\mathbb{Z}$-affine function on $\mathbb{R}^2$, and $\alpha = dg \bmod 2\}$

  This has automorphism group S_3.

One can associate to $(G, \mathrm{Aff}^{\sigma}_{G})$ a symplectic surface M with a <u>real structure</u> (antisymplectic involution) which is essentially unique.

On the other hand, G carries a natural sheaf of $\mathbb{Z}/2$-graded dg algebras $\mathcal{O}_G^\sigma$, which over the smooth part is Kontsevich–Soibelman's $\mathcal{O}$ tensored with a Clifford algebra, and which near the vertices is a non-archimedean analytic version of our previous $\mathcal{A}$.

$$\boxed{\underline{\text{Conjecture}} \quad \mathcal{F}(M) \hookrightarrow \mathcal{D}(\mathcal{O}_G^\sigma)}$$

suitable category of dg module sheaves