Cotangent bundles and their relatives

(after Fukaya-Oh, Kontsevich-Soibelman, Nadler-Zaslow, Kontsevich...)

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Morse Lecture #1

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<u>This lecture</u>: three situations in which the symplectic topology of M^{2n} can be described in terms of (different kinds of) sheaves on a space B^n.

① Cotangent bundles $\mathbb{R}^n \longrightarrow M = T^*B \longrightarrow B$
(ordinary sheaves of vector spaces) (exact)

② Lagrangian torus fibrations $T^n \longrightarrow M \longrightarrow B$
(sheaves of modules over a ring of nonarchimedean analytic functions)

mirror symmetry (lecture 3)

③ "Singular cotangent bundles" (exact)

1 Cotangent bundles

B a connected n-dimensional C^∞ manifold, $M = T^*B$ its cotangent bundle, with the standard symplectic form $\omega = d\theta$, θ the universal one-form. In coordinates p_j (fibre) and q_j (base),

$$\theta = \sum_j p_j \, dq_j, \quad \omega = \sum_j dp_j \wedge dq_j.$$

The conjectural picture is

$$\boxed{\begin{array}{l}\text{differential} \\ \text{topology of } B\end{array}} \longleftrightarrow \boxed{\begin{array}{l}\text{symplectic} \\ \text{topology of } M = T^*B\end{array}}$$

A Lagrangian submanifold $L^n \subset M^{2n}$ is <u>exact</u> if $\theta|L \in H^1(L;\mathbb{R})$ vanishes.

__Conjecture__ (Arnol'd) Suppose that B is compact. Then, every closed exact $L \subset M = T^*B$ is isotopic to the zero-section (through such submanifolds).

__Conjecture__ (folk) Cotangent bundles of compact manifolds are symplectomorphic $\Leftrightarrow$ their bases are diffeomorphic.

Many partial results on Arnol'd's conjecture, restricting the topology of L (Laudenbach-Sikorav, Viterbo, Buhovski, Ritter, ...), in particular:

__Theorem__ (Nadler-Zaslow and Fukaya-S-Smith)
If $\pi_1(B) = \{1\}$ and $L \subset M = T^*B$ is closed, exact, and has vanishing Maslov class $\mu_L \in H^1(L)$, then the projection $H^*(B) \longrightarrow H^*(L)$ is an isomorphism.

Categorical setup underlying this result:
There is a natural differential graded category associated to $(M, \omega = d\theta)$, the Fukaya category $\mathcal{F}(M)$. For simplicity take $\mathbb{Q}$-coefficients. <u>Objects</u> are Lagrangian submanifolds which are closed, <u>exact</u>, have <u>vanishing Maslov class</u>, and are <u>relatively spin</u>:

$$w_2(L) = w|L \in H^2(L; \mathbb{Z}/2)$$

for some fixed $w \in H^2(M; \mathbb{Z}/2)$. <u>Morphisms</u> come from Floer cohomology theory.

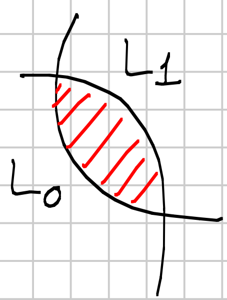
<u>Example</u> $H^*(\hom_{\mathcal{F}(M)}(L, L)) \cong H^*(L; \mathbb{Q})$. In fact, we have an underlying quasi-isomorphism of dga's (Fukaya-Oh) $\hom_{\mathcal{F}(M)}(L, L) \simeq C^*(L; \mathbb{Q})$.

Generally,

as usual, there are different cochain models

$$\hom^*_{\mathcal{F}(M)}(L_0, L_1) \simeq CF^*(L_0, L_1)$$

Floer cochains, in the transverse intersection case there is one generator per intersection point,



$$H(\hom^*_{\mathcal{F}(M)}(L_0, L_1)) = HF^*(L_0, L_1)$$

Floer cohomology, the differential uses holomorphic discs to – intuitively – cancel excess intersection points.

$\chi(HF^*(L_0, L_1)) = \pm L_0 \cdot L_1$. More crucially, isotopies give rise to (quasi)isomorphic objects.

Arnol'd's conjecture predicts that $\mathcal{F}(M)$ for $M = T^*B$ is not very interesting: up to (quasi) isomorphism there is only one object, the zero-section.

Let $\mathcal{D}(\underline{\mathbb{Q}}_B)$ be the differential graded derived category of <u>dg local systems</u>: complexes of sheaves of vector spaces whose cohomology sheaves are bounded and locally free of finite rank.

<u>Theorem</u> (Nadler; alternatively Abouzaid) There is a (cohomologically) full and faithful embedding

$$\mathcal{F}(M = T^*B) \hookrightarrow \mathcal{D}(\underline{\mathbb{Q}}_B).$$

Here, $\mathcal{F}(M)$ is set up using $w = w_2(B)$ pulled back to M

Suppose now that $\pi_1(B) = \{1\}$, and let E be a dg local system.

<u>Lemma</u> If $H^*(\hom_{\mathcal{D}(\underline{\mathbb{Q}}_B)}(E,E))$ "looks like" the cohomology of an n-manifold, then $E \cong \underline{\mathbb{Q}}_B$ up to shift.

This leads to the previously stated theorem:
L an object of $\mathcal{F}(M = T^*B)$, E its associated dg local system on B

$\Rightarrow$ $H^*(\hom_{\mathcal{D}(\underline{\mathbb{Q}}_B)}(E,E)) \cong H(\hom_{\mathcal{F}(M)}(L,L)) = H^*(L;\mathbb{Q})$

$\Rightarrow$ $E \cong \underline{\mathbb{Q}}_B$ and hence $L \cong B$ (zero-section) in the Fukaya category. This is a Floer-cohomology-level version of the statement in Arnol'd's conjecture.

② Lagrangian torus fibrations (with sections)

Let B be a $\mathbb{Z}$-affine manifold. This means that it has an atlas with transition functions in

$$\mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n) = \mathbb{R}^n \rtimes GL(n,\mathbb{Z}).$$

In particular, TB contains a canonical integer lattice $TB_{\mathbb{Z}}$. Take the dual $T^*B_{\mathbb{Z}} \subset T^*B$, and form the symplectic manifold

$$M = T^*B / T^*B_{\mathbb{Z}}, \quad \omega = dp \wedge dq$$

which is a locally trivial Lagrangian torus fibration over B (together with a global Lagrangian section; not all Lagrangian torus fibrations admit one).

The conjectural picture is

$$\boxed{\begin{array}{c}\mathbb{Z}\text{-affine}\\ \text{geometry of } B\end{array}} \longleftrightarrow \boxed{\begin{array}{c}\text{symplectic}\\ \text{topology of } M\end{array}}$$

<u>Example</u> Consider $B \subset \mathbb{R}^n$ open subset. Then $M = T^n \times B \subseteq T^n \times \mathbb{R}^n \cong T^*(T^n)$. In this context, we have the theorem of Benci–Sikorav:

> $B, \tilde{B} \subseteq \mathbb{R}^n$ open connected, zero first Betti number. If $T^n \times B \cong T^n \times \tilde{B}$ symplectically $\Rightarrow B = \phi(\tilde{B})$ for some $\mathbb{Z}$-affine map ϕ.

This is an interesting question for more general $\mathbb{Z}$-affine manifolds (answer not as obvious).

Note that $\omega \in \Omega^2(M)$ is no longer always exact. In this situation, infinite sums indexed by symplectic areas $r = \int \omega$ appear naturally in Floer cohomology. To organize these, we need:

<u>Definition</u> The <u>Novikov field</u> Λ_q is the field of formal series $a = \sum a_r q^r$ where $r \in \mathbb{R}$, $a_r \in \mathbb{Q}$, and:

for every C there are only finitely many $r \leq C$ such that $a_r \neq 0$

In other words, infinitely many a_{r_k} can be $\neq 0$ but only if the $r_k \longrightarrow +\infty$. We write $val_q : \Lambda_q \to \mathbb{R} \cup \{+\infty\}$ for the <u>q-valuation</u>.

Kontsevich - Soibelman associate to every $\mathbb{Z}$-affine manifold B a sheaf $\mathcal{O}_B$ of commutative Λ_q-algebras (the sheaf of <u>non-archimedean holomorphic functions</u>).

Suppose $B \subset \mathbb{R}^n$ is open and connected. Sections of $\mathcal{O}_B$ are of the form

$$f(z_1, \ldots, z_n) = \sum_{k_1, \ldots, k_n} a_{k_1 \ldots k_n} z_1^{k_1} \ldots z_n^{k_n}$$

where $(k_1, \ldots, k_n) \in \mathbb{Z}^n$, $a_{k_1 \ldots k_n} \in \Lambda$, and satisfying the following condition for all $b \in B$:

$$\boxed{\begin{array}{l}\text{Given } C, \text{ there are only finitely many } (k_1, \ldots, k_n) \\ \text{such that} \\ \qquad \mathrm{val}_q(a_{k_1, \ldots, k_n}) - \sum_{i=1}^{n} b_i k_i \leq C.\end{array}}$$

__Ex__ For $B = (-\infty, 0) \subseteq \mathbb{R}$, we get functions on the __non-archimedean punctured disc__: $f(z) = \sum_k a_k z^k$, and

As $k \to -\infty$, rapid decay $\mathrm{val}_q(a_k)/k \longrightarrow -\infty$

As $k \to +\infty$, at most slow growth $\underline{\lim}\ \mathrm{val}_q(a_k)/k \geq 0$.

__Remark__ Given $B \subset \mathbb{R}^n$ open, sections of $\mathcal{O}_B$ depend only on the convex hull of B (non-archimedean analogue of the far deeper Hartogs phenomenon in complex function theory).

From now on B is compact. The Fukaya category $\mathcal{F}(M)$ is linear over Λ_q. Note its definition is far more involved. For instance, $HF^*(L,L) \cong H^*(L;\mathbb{Q})$ is no longer true (or even makes sense) in general.

As before, the definition involves $w \in H^2(M; \mathbb{Z}/2)$, which we take to be the pullback of $w_2(B)$.

<u>Conjecture</u> There is a (cohomologically) full and faithful embedding

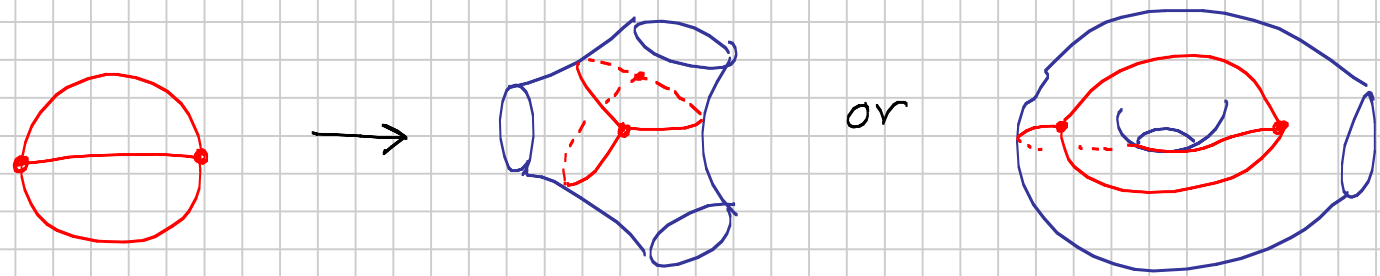
$$\mathcal{F}(M) \longrightarrow \mathcal{D}(\mathcal{O}_B)$$

The right hand side is the dg derived category of $\mathcal{O}_B$-module sheaves. Kontsevich-Soibelman, to whom the conjecture is due, proved it for the subcategory of $\mathcal{F}(M)$ consisting of Lagrangian sections. We have full proofs for $B = S^1$ (Polishchuk-Zaslow) and $B = S^1 \times S^1$ (square torus, Abouzaid-Smith).

3 <u>Ribbon graphs</u>

Take a finite ribbon graph B (an unoriented graph, whose vertices are all at least trivalent, together with a cyclic ordering of the edges at any vertex). One can "fatten up" B to an open oriented surface M, and equip that surface with a symplectic form $\omega = d\theta$, where $\theta|B$ is exact.

<u>Ex.</u>



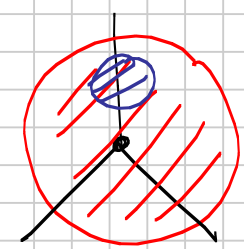
(depending on the ribbon structure)

The Fukaya category $\mathcal{F}(M)$ is defined over $\mathbb{Q}$ but (in the absence of additional choices) $\mathbb{Z}/2$-graded. Kontsevich's idea: introduce a sheaf of $\mathbb{Z}/2$-graded differential categories $\mathcal{K}_B$ on B, such that

$$\mathcal{F}(M) \longrightarrow \Gamma(\mathcal{K}_B) \leftarrow \text{global section dg category}$$

On the smooth part, $\mathcal{K}_B$ is the same as in the cotangent bundle case. What can we say about the dg category associated to a d-valent vertex?

- It needs to come with d dg functors to chain complexes, $\longrightarrow$
- It carries an action of $\mathbb{Z}/d$



__Ex__ $d=3$, $A_2 = \{\bullet \longrightarrow \bullet\}$ which means that representations of the algebra are triples

$$(V_1, V_2 \text{ vector spaces}, \ F: V_1 \to V_2 \text{ map})$$

In $\mathcal{D}(A_2)$ these are replaced by $\mathbb{Z}/2$-graded chain complexes. There are 3 natural indecomposable objects

$$\mathbb{Q} \longrightarrow 0, \qquad 0 \longrightarrow \mathbb{Q}, \qquad \mathbb{Q} \xrightarrow{1} \mathbb{Q}$$

These give three functors from $\mathcal{D}(A_2)$ to vector spaces. There is a <u>hidden symmetry</u> which permutes these 3 objects cyclically.

For a general d-valent vertex, use A_{d-1} in the same way.

Conclusion: The origin of these sheaves is in Floer cohomology for noncompact Lagrangian "submanifolds".

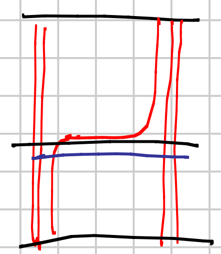
(1)

B

$$HF^*(T^*_x B, T^*_x B) \cong \mathbb{Q}$$

"conormal" ↘ ↙ zero-section

(2)

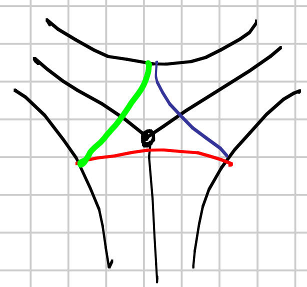


U

$$HF^*(\nu^* U, O_U) \cong \mathcal{O}(U)$$

at least for convex $U \subseteq B = \mathbb{R}^n$

(3)



$$\begin{tikzcd} & \bullet \arrow[dl, "x_3"'] & \\ \bullet \arrow[rr, "x_1"'] & & \bullet \arrow[ul, "x_2"'] \end{tikzcd}$$

higher order product $\mu^3(x_3, x_2, x_1) = id$ etc. makes this derived equivalent to A_2