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# STEENROD SQUARES AND SYMPLECTIC FIXED POINTS

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## Symplectic diffeomorphisms

Let $M^{2n}$ be a closed manifold with a symplectic form $\omega \in \Omega^2(M)$, $d\omega = 0$ (locally $M = \mathbb{R}^{2n}$, $\omega = dp_1 \wedge dq_1 + \cdots$ )

Let $\varphi : M \to M$ be a diffeomorphism which preserves the symplectic form, $\varphi^* \omega = \omega$.

Question Is it possible for $\varphi$ to have only finitely many periodic points? ("Conley conjecture")

Answer In general, yes, e.g. irrational rotation of $M = S^2$ or $M = \mathbb{C}P^n$.

For simplicity, assume all periodic points are nondegenerate:

$(*)$ $\varphi^k(x) = x \implies \det(\mathbb{1} - D\varphi^k_x) \neq 0$

Chern class

Thm (Salamon-Zehnder) Suppose that $c_1(M) = 0$, $H^1(M;\mathbb{R}) = 0$, and $\varphi$ is isotopic to $id$ (symplectically). If $(*)$ holds, there must be $\infty$ many periodic points.

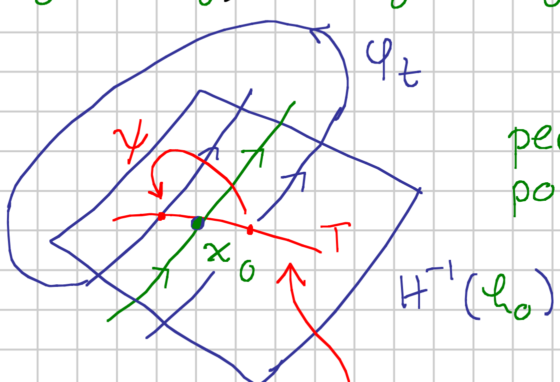
unnecessary (Ginzburg-Gürel-Hein)

Thm (Ginzburg-Gürel) Same, if $c_1(M) = -[\omega]$.

Thm (Franks) For $M = S^2$, either 2 or $\infty$ many periodic points.

Historical origin "Poincaré return maps" of Hamiltonian systems: given $M$ and $H \in C^\infty(M,\mathbb{R})$, consider the Hamiltonian vector field $X = X_H$ and its flow $(\varphi_t)$. Suppose $X_{x_0} \neq 0$, $\varphi_{t_0}(x_0) = x_0$, $H(x_0) = h_0$

periodic point




local transverse slice $T$ comes with a symplectic diffeomorphism, the Poincaré return map $\psi$.

Now suppose $M$ carries a rotational $(S^1)$ symmetry with momentum map $\mu$, $\{H, \mu\} = 0$, and that $x_0$ is an $S^1$-fixed point. One can choose $T$ and the Poincaré return map $\psi$ to be $S^1$-invariant, and then consider the dynamics on the reduced spaces

$$\bar{T} = \mu^{-1}(\theta) \cap T / S^1 \circlearrowleft \bar{\psi}$$

Example $M^{S^1}$ a surface, weights on the normal bundle are all $+1$

$\Rightarrow$ $\bar{T} \cong \mathbb{C}P^{n-2}$ for $\theta > \mu(x_0)$.

$\bar{\psi}$ is isotopic to the identity.

# Algebra background

$V$ a vector space over a field $\mathbb{K}$, with an action of the group $\mathbb{Z}/2$, given by an involution

$$\iota : V \longrightarrow V, \qquad \iota^2 = \mathrm{id}.$$

$\mathrm{char}(\mathbb{K}) \neq 2$ is boring: vector spaces with $\mathbb{Z}/2$-actions form a semisimple abelian category. More concretely, projectors $\pi_\pm = \frac{1}{2}(\mathrm{id} \pm \iota)$ split $V = V_+ \oplus V_-$, so the theory is just that of pairs of vector spaces.

$\mathrm{char}(\mathbb{K}) = 2$ is nontrivial. We still have the fixed part $V^\iota \subseteq V$, and for any $v \in V$, $v - \iota(v) \in V^\iota$. Hence, $V/V^\iota$ carries the trivial induced action, and we have

$$\begin{tikzcd} 0 \arrow[r] & V^\iota \arrow[r, hook] & V \arrow[r, two heads] & V/V^\iota \arrow[r] \arrow[ll, bend left, "\sigma = \mathrm{id} - \iota"] & 0 \end{tikzcd}$$

non-canonical

$$\Rightarrow \quad V \cong V^\iota \oplus V/V^\iota, \qquad \iota \cong \begin{pmatrix} \mathrm{id} & \sigma \\ 0 & \mathrm{id} \end{pmatrix}$$

Roughly, this is the theory of pairs of vector spaces with an injective map between them.

Group cohomology $\mathrm{char}(\mathbb{K})=2$ from now on. Let $u$ be a formal variable of degree 1; the group cochain complex is

$$C^*(\mathbb{Z}/2;V) = V[[u]] = \prod_{i=0}^{\infty} u^i V$$

$$d_{\mathbb{Z}/2} = u(\mathrm{id}-\iota)$$

with cohomology $H^*(\mathbb{Z}/2;V)$; explicitly

$$C^*(\mathbb{Z}/2;V) \cong \left(V^\iota \oplus V/V^\iota\right) \times u\left(V^\iota \oplus V/V^\iota\right) \times \cdots$$

$$d_{\mathbb{Z}/2} = u\sigma$$

$$H^*(\mathbb{Z}/2;V) \cong \mathrm{coker}(u\sigma) = V^\iota \oplus u\,\mathrm{coker}(\sigma)[[u]]$$

$u$ may have kernel

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Tate version

negative powers of $u$

$$\hat{C}^*(\mathbb{Z}/2;V) = V((u)) = C^*(\mathbb{Z}/2;V) \otimes_{\mathbb{K}[[u]]} \mathbb{K}((u))$$

$$\hat{H}^*(\mathbb{Z}/2;V) = H^*(\mathbb{Z}/2;V) \otimes_{\mathbb{K}[[u]]} \mathbb{K}((u)) \cong \mathrm{coker}(\sigma)((u))$$

Example Trivial action $\Rightarrow$ Tate cohomology is $V((u))$

Example $V$ has a basis on which $\mathbb{Z}/2$ acts freely
$\Leftrightarrow$ $\sigma$ is an isomorphism
$\Leftrightarrow$ Tate cohomology vanishes.

__Example__ $V$ any vector space, take $V\otimes V$ with the $\mathbb{Z}/2$-action which exchanges the two factors. Then

$$H^*(\mathbb{Z}/2; V\otimes V) \cong \mathrm{Sym}^2(V) \oplus uV[[u]]$$

$u$ sends $v\otimes v \longmapsto v$

$v\otimes w + w\otimes v \mapsto 0$

$$\boxed{\hat{H}^*(\mathbb{Z}/2; V\otimes V) \cong V((u)) \qquad (*)}$$

Note that $\hat{H}^*(\mathbb{Z}/2; V\otimes V)$ is __additive__ in $V$, but the underlying chain complex __isn't__.

Consider the Tate map

$$V[[u]] \longrightarrow C^*(\mathbb{Z}/2; V\otimes V)$$

$$v \longmapsto v\otimes v$$

This is __not additive__, but the induced map on cohomology becomes additive after multiplying with $u$:

$$(v+w)\otimes(v+w) - v\otimes v - w\otimes w =$$

$$= v\otimes w + w\otimes v = d_{\mathbb{Z}/2}(u^{-1}(v\otimes w)).$$

After tensoring with $\mathbb{K}((u))$, it induces $(*)$.

## Chain complexes with involution

Take $(V, d_V)$ chain complex with involution $\iota$. Define $C^*(\mathbb{Z}/2; V)$ and $\hat{C}^*(\mathbb{Z}/2; V)$ as before, with

$$d_{\mathbb{Z}/2} = d_V + u(\mathrm{id} - \iota)$$

The resulting $H^*(\mathbb{Z}/2; V)$, $\hat{H}^*(\mathbb{Z}/2; V)$ are quasi-isomorphism invariants.

Example $(V, d_V)$ any chain complex,

$$H^*(V)((u)) \xrightarrow[\cong]{\text{not graded}} \hat{H}^*(\mathbb{Z}/2; V \otimes V)$$

by the Tate map on cocycles.

Filtration by powers of $u$ yields a spectral sequence

$E_1 = C^*(\mathbb{Z}/2; H^*(V))$

$E_2 = H^*(\mathbb{Z}/2; H^*(V))$

...

Assuming $V$ is bounded, the finite filtration by degrees in $V$ yields a spectral sequence

$E_1 = H^*(\mathbb{Z}/2; V$ as graded vector space$)$

Example $V$ bounded, each $V^i$ has a basis freely acted on by $\iota$ $\Rightarrow$ $\hat{H}^*(\mathbb{Z}/2; V^i) = 0$ $\Rightarrow$ $\hat{H}^*(\mathbb{Z}/2; V) = 0$.

## Algebraic topology

$M$ a manifold with a (smooth) involution $\rho$ ($\Rightarrow$ fixed point set $M^{\mathbb{Z}/2}$ is a submanifold). $C^*(M)$ = cochains with $\mathbb{K}$-coefficients. Equivariant cohomology

$$H^*_{eq}(M) \overset{def}{=} H^*(\mathbb{Z}/2; C^*(M))$$

Similarly, Tate version.

First spectral sequence

$$E^1 = H^*(M)[[u]] \Rightarrow H^*_{eq}(M)$$

differential on $E^1$ is $id - \rho^*$

$$\Downarrow$$

$$\boxed{\dim_{\mathbb{K}((u))} \hat{H}^*_{eq}(M) \leq \dim_{\mathbb{K}} H^*(M)^{\mathbb{Z}/2}}$$

Second spectral sequence implies: if $\rho$ is fixed-point free, $\hat{H}^*_{eq}(M) = 0$ (in fact, $H^*_{eq}(M) \cong H^*(M/\mathbb{Z}/2)$).

Example Given any $M$, take $M \times M$ with the involution exchanging the two factors. Equivariant Eilenberg-Zilber:

$$C^*(\mathbb{Z}/2; C^*(M \times M)) \simeq C^*(\mathbb{Z}/2; C^*(M) \otimes C^*(M))$$

$$\Downarrow$$

$$H^*_{eq}(M \times M) \cong H^*(\mathbb{Z}/2; H^*(M) \otimes H^*(M))$$

$$\Downarrow$$

$$\hat{H}^*_{eq}(M \times M) \cong H^*(M)((u))$$

not degree-preserving (Tate map)

## Steenrod squares

Consider the diagonal $\Delta \subset M \times M$. This is the fixed point set of the involution, hence $H^*_{eq}(\Delta) \cong H^*(M)[[u]]$.

$$\begin{tikzcd}
H^i(M) \arrow[r, "\text{Tate}"] \arrow[ddr, "Sq"'] & H^{2i}(\mathbb{Z}/2; C^*(M) \otimes C^*(M)) \arrow[d, "\cong"', "\text{Eilenberg-Zilber}"] \\
& H^{2i}_{eq}(M \times M) \arrow[d, "\text{restrict to } \Delta"] \\
& H^{2i}_{eq}(\Delta) \cong H^{2i}(M) \oplus u H^{2i-1}(M) \oplus \cdots
\end{tikzcd}$$

Total Steenrod operation

More precisely, for $x \in H^i(M)$

cup-square

$$Sq(x) = x^2 + u\, Sq^{i-1}(x) + \cdots + u^{i-1} Sq^1(x) + u^i x$$

Bockstein identity

The induced map

$$\widehat{Sq} : H^*(M)((u)) \longrightarrow H^*(M)((u))$$
$$x \longmapsto u^i(x + u^{-1} Sq^1(x) + \cdots)$$

is an isomorphism. Alternatively, this is a consequence of the <u>localization theorem</u> applied to $M \times M$.

## Fixed point Floer cohomology

Variational interpretation:

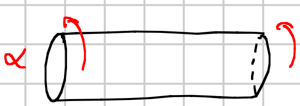
**Setup** $M$ is a compact symplectic manifold with $\omega = d\theta$ (hence $\partial M \neq \emptyset$). $\varphi : M \xrightarrow{\cong} M$ satisfies $\varphi^*\theta = \theta + dG$ ($\Rightarrow \varphi^*\omega = \omega$); and has no fixed points on $\partial M$ (+ some convexity type conditions near $\partial M$).

One can define a $\mathbb{Z}/2$-graded $\mathbb{K}$-vector space $HF^*(\varphi)$, fixed point Floer cohomology, such that

- $\chi(HF^*(\varphi)) =$ Lefschetz number
- $|Fix(\varphi)| \geq \dim HF^*(\varphi)$ if the fixed points are nondegenerate
- $HF^*(\varphi)$ is invariant under isotopies (in the class of permitted $\varphi$)

**Example** $H \in C^\infty(M, \mathbb{R})$ with $H|\partial M = const.$ $dH|\partial M$ positive in outwards direction. Take the flow $\phi^t$ for small $t > 0$. Then $HF^*(\phi^t) \cong H^*(M)$.

**Example** $k$-fold Dehn twist on annulus:




$HF^*(\varphi) \cong \bigoplus_k H^*(S^1)$, hence $2k$ fixed points persist.

**Twisted loop space**

$$\mathcal{L}_\varphi = \{ x : \mathbb{R} \to M \mid x(t) = \varphi(x(t+1)) \}$$

**Action functional** $A_\varphi : \mathcal{L}_\varphi \to \mathbb{R}$

$$A_\varphi(x) = -\int_0^1 x^* \theta + G(x(1))$$

Critical points $\Leftrightarrow \frac{dx}{dt} = 0 \Leftrightarrow$ fixed points

Assume from now on that the fixed points are nondegenerate ($\Leftrightarrow A_\varphi$ is formally Morse). Define

filtered chain complex

$$CF^*(\varphi) = \bigoplus_{\varphi(x)=x} \mathbb{K} \cdot x$$

where the $\mathbb{Z}/2$-grading is determined by the sign of $\det(\mathbb{1} - D\varphi_x)$.

The differential $d_\varphi : CF^*(\varphi) \to CF^{*+1}(\varphi)$ is defined by considering gradient flow lines of $A_\varphi$ with respect to a metric

$$\langle \xi, \eta \rangle = \int_0^1 \omega(\xi, J_t \eta)\, dt$$

family of almost complex structures needs to satisfy a genericity (Palais-Smale) condition

More concretely, these are solutions of

$$\begin{cases} u : \mathbb{R}^2 \longrightarrow M, \quad u(s,t) = \varphi(u(s,t+1)), \\ \partial_s u + J_t \cdot \partial_t u = 0 \\ \lim_{s \to \pm\infty} u(s, \cdot) = \text{fixed points} \end{cases}$$

$$\boxed{\text{Note: Differential } d_\varphi \text{ increases } A_\varphi}$$

## Equivariant Floer cohomology

The space $\mathcal{L}_{\varphi^2}$ carries an involution, the $\frac{1}{2}$-rotation of loops:

$$\rho(x)(t) = \varphi(x(t+\tfrac{1}{2}))$$

This preserves $A_\varphi$, hence induces an involution $\rho^*$ on $CF^*(\varphi^2)$. Because of the Palais–Smale condition, this is not necessarily a chain map. But one has a different chain map

$$r : CF^*(\varphi^2) \longrightarrow CF^*(\varphi^2)$$

$$r = \rho^* + (\text{terms that strictly increase the action})$$

$r$ is not an involution, but instead satisfies

$$r^2 - \mathbb{1} = (r+\mathbb{1})^2 = d_\varphi d_2 + d_2 d_\varphi$$

for some chain homotopy $d_2$, which strictly increases the action. There is an infinite sequence

$$d_0 = d_{\varphi^2}, \quad d_1 = \mathbb{1} + r, \quad d_2, \ldots$$

which constitutes a homotopy action of $\mathbb{Z}/2$ on $CF^*(\varphi^2)$. One defines

$$CF^*_{eq}(\varphi^2) = CF^*(\varphi^2)[[u]]$$

$$d_{eq} = d_0 + u d_1 + u^2 d_2 + \cdots$$

and its cohomology $HF^*_{eq}(\varphi^2)$.

**Example** $\varphi = \varphi^t$, $t>0$ small, as before.

Then $HF^*_{eq}(\varphi^2) \cong H^*(M)[[u]]$.

Filtration by powers of $u$ gives a spectral sequence

$$E_1 = HF^*(\varphi^2)[[u]] \Rightarrow HF^*_{eq}(\varphi^2)$$

differential is $\mathbb{1} - r^*$

$E_2 = H^*(\mathbb{Z}/2; HF^*(\varphi^2))$, hence

$$\boxed{\dim HF^*_{eq}(\varphi^2) \otimes_{\mathbb{K}[[u]]} \mathbb{K}((u)) \leq \dim HF^*(\varphi^2)^{\mathbb{Z}/2}}$$

Tate version $\widehat{HF}^*_{eq}(\varphi^2)$

Filtration by action gives a spectral sequence with

$$E_1 = H^*(\mathbb{Z}/2; CF^*(\varphi^2))$$

with $\rho^*$ acting

If one passes to the Tate version,

$$E_1 = \hat{H}^*(\mathbb{Z}/2; CF^*(\varphi^2))$$
$$\cong CF^*(\varphi)((u))$$

$\mathbb{Z}/2$-grading issues are nontrivial, and the differentials are a priori mysterious.

## Steenrod operations in Floer cohomology

First approach, Floer stable homotopy type (Cohen–Jones–Segal). Impose

**Stable homotopical triviality condition**

$TM$ is stably trivial $(*)$ as a symplectic vector bundle, and with respect to that trivialization, $D\varphi : M \to Sp(\infty)$ is nullhomotopic $(*)$.

Then, can define a stable homotopy type underlying $HF^*(\varphi)$, hence Steenrod squares. Advantage: satisfy standard axioms. Drawback: depend on $(*)$

Second approach (Fukaya, Betz–Cohen–Norbury): there is an equivariant product operation

$$H^*(\mathbb{Z}/2;\, CF^*(\varphi) \otimes CF^*(\varphi)) \xrightarrow{\text{product}} HF^*_{eq}(\varphi^2)$$

combine it with Tate

$$HF^*(\varphi)[[u]]$$

becomes linear after multiplying with $u$

Advantage: no homotopy condition, canonical. Drawback: may not satisfy standard axioms.

For $\varphi = \varphi^t$ ($t > 0$ small) as before, both approaches should recover classical Steenrod squares on $H^*(M)$.

Relation? Hendricks (based on earlier work of Seidel–Smith, using Lagrangian Floer cohomology) defines a localization map $HF^*_{eq}(\varphi^2) \longrightarrow HF^*(\varphi)$. Using that, we have

$$\begin{tikzcd}
HF^*(\varphi) \arrow[r, dashed, "\text{Floer homotopy type}", "\text{Steenrod squares}"', "\text{requires homotopical triviality}" below=12pt] \arrow[d, "\text{Tate + equivariant product (additive after multiplying by } u)"'] & HF^*(\varphi) \\
HF^*_{eq}(\varphi^2) \arrow[ur, dashed, "\text{Localization map (defined after multiplying by } u^k,\ k \gg 0\text{); also requires homotopical triviality (*) for somewhat different reasons}"']
\end{tikzcd}$$

**Conjecture** Diagram commutes.

**Consequence** If homotopical triviality holds, the equivariant product has special properties (e.g. contains Bockstein)

__Localization theorem__ (Borel) Let $M$ be a manifold with $\mathbb{Z}/2$-action. Then the map induced by $M^{\mathbb{Z}/2} \underset{i}{\subseteq} M$,

$$H^*_{\mathbb{Z}/2}(M) \xrightarrow{i^*} H^*(M^{\mathbb{Z}/2})[[u]]$$

becomes an isomorphism over $\mathbb{K}((u))$ (i.e. in the __Tate version__).

$$\boxed{\begin{array}{l}\text{\underline{Corollary} (Smith inequality)} \\ \dim H^*(M^{\mathbb{Z}/2}) \leq \dim H^*(M)^{\mathbb{Z}/2}\end{array}}$$

__Example__ A smooth real algebraic curve of degree $d$ (in $\mathbb{R}P^2$) has at most $\frac{1}{2}d^2 - \frac{3}{2}d + 2$ components.

__First proof__ Using a suitable choice (e.g. a triangulation) ensure that $C^*(M) \to C^*(M^{\mathbb{Z}/2})$ is __onto__, and that the kernel $C^*(M, M^{\mathbb{Z}/2})$ has a basis freely acted on by $\mathbb{Z}/2$. This implies $\hat{H}^*(\mathbb{Z}/2; C^*(M, M^{\mathbb{Z}/2})) = 0$.

not independent

"__Second proof__" Assume $M$ closed. We have

$$H^*(M^{\mathbb{Z}/2})[[u]] \xrightarrow{i_!} H^{*+c}_{\mathbb{Z}/2}(M)$$

and composition of the two is cup product with the (equivariant) Thom or Euler class of the normal bundle. But

$$e_{\mathbb{Z}/2}(\nu) = \sum_i u^{c-i} w_i(\nu) = u^c + \dots$$

becomes invertible over $\mathbb{K}((u))$.

__Thm__ (Hendricks 2014) Suppose the homotopical triviality condition is satisfied, so that the localization map

$$HF^*_{eq}(\varphi^2) \xrightarrow{(*)} HF^*(\varphi)[[u]]$$

is defined. This map becomes an isomorphism over $\mathbb{K}((u))$.

__Cor__ $\dim HF^*(\varphi^2) \geq \dim HF^*(\varphi)$

__Applications__ To isotopy classification of symplectic diffeomorphisms.

__Remark__ The cokernel of $(*)$ is a potentially interesting object

__Thm__ (S) The equivariant product

$$H^*(\mathbb{Z}/2;\, CF^*(\varphi) \otimes CF^*(\varphi)) \longrightarrow HF^*_{eq}(\varphi^2)$$

becomes an isomorphism over $\mathbb{K}((u))$

$\widehat{HF}^*_{eq}(\varphi^2) \cong HF^*(\varphi)((u))$, removes the homotopical triviality condition

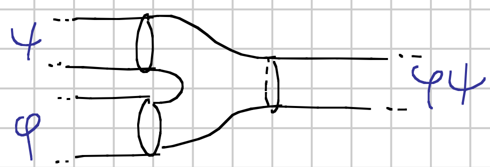
__Sketch of proof__ uses action filtration spectral sequence; the chain map

$$(CF^*(\varphi) \otimes CF^*(\varphi))((u)) \longrightarrow CF^*(\varphi^2)((u))$$

$$x \otimes x \longmapsto u^{\kappa(x)} x + \text{higher action terms}$$

where $\kappa(x)$ = Krein index (local invariant).

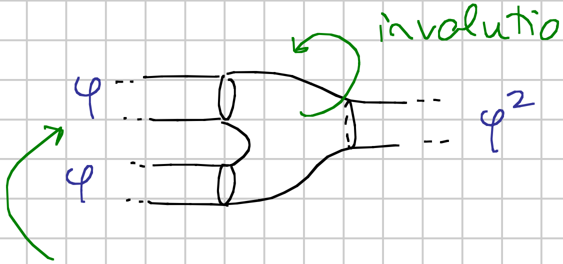
Topological quantum field theory (TQFT) is a formalism for operations on Floer cohomology groups. Product:




$HF^*(\varphi) \otimes HF^*(\psi) \longrightarrow HF^*(\varphi\psi)$

Version with symmetry:

involution

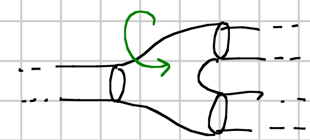



$\leftarrow$ acts by $\frac{1}{2}$-rotation

acts by exchanging the ends

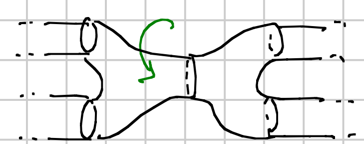
$H^*(\mathbb{Z}/2; CF^*(\varphi)^{\otimes 2}) \longrightarrow HF^*_{eq}(\varphi^2)$

The proof that we should have, modelled on the "second proof" of the classical localization theorem:

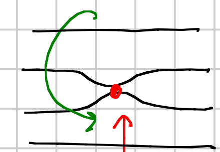


map in inverse direction (degree n)

Composition

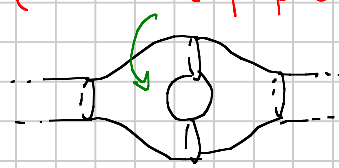


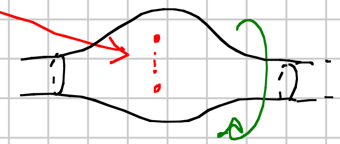
degenerate



"Cap product" with the equivariant diagonal class

Two points identified (also a cap product)





**Noncommutative geometry** In certain situations, fixed point Floer cohomology can be interpreted (via the Fukaya category) as Hochschild (co)homology.

Geometry – algebra dictionary:

| | |
|---|---|
| manifold $M$ | dg algebra $\mathcal{A}/\mathbb{K}$ (smooth and proper) |
| $\varphi : M \to M$ | $\mathcal{A}$-bimodule $\mathcal{P}$ |
| $\varphi^2$ | $\mathcal{P} \otimes_{\mathcal{A}} \mathcal{P}$ (derived tensor product) |
| $HF^*(\varphi)$ | $HH_*(\mathcal{A}, \mathcal{P})$ |

This works well formally, for instance $HH_*(\mathcal{A}, \mathcal{P} \otimes_{\mathcal{A}} \mathcal{P})$ has a $\mathbb{Z}/2$-action.

**Conjecture** For all such $(\mathcal{A}, \mathcal{P})$,

$$\dim HH_*(\mathcal{A}, \mathcal{P} \otimes_{\mathcal{A}} \mathcal{P})^{\mathbb{Z}/2} \geq \dim HH_*(\mathcal{A}, \mathcal{P})$$

Treumann + Lipshitz (2011): There is a spectral sequence

$$HH_*(\mathcal{A}, \mathcal{P})((u)) \Rightarrow \hat{H}^*(\mathbb{Z}/2; CC_*(\mathcal{A}, \mathcal{P} \otimes_{\mathcal{A}} \mathcal{P}))$$

Hochschild chain complex

whose degeneration implies the conjecture. This roughly speaking corresponds to our previous action action filtration.

More precisely, one would want to have a product structure

$$\begin{array}{c} H^*(\mathbb{Z}/2;\, CC_*(\mathcal{A},\mathcal{P}) \otimes CC_*(\mathcal{A},\mathcal{P})) \\ \downarrow \\ H^*(\mathbb{Z}/2;\, CC_*(\mathcal{A}, \mathcal{P} \otimes_{\mathcal{A}} \mathcal{P})) \end{array} \qquad (*)$$

One can't expect this in general, but if $\mathcal{A}$ is <u>Calabi-Yau</u>, work of Costello-Hopkins-Lurie should provide a TQFT. One would try to prove the conjecture by applying a spectral sequence comparison argument to $(*)$

<u>Slight letdown</u> A different product structure is already in Treumann-Lipshitz

Namely, they use them to show that it is sufficient to establish degeneration in the "universal case" $\mathcal{P} = \mathcal{A}^!$, which has

$$HH_*(\mathcal{A}, \mathcal{A}^!) \cong HH^*(\mathcal{A},\mathcal{A}).$$ ring

From this viewpoint, considering general $\mathcal{P}$ is not really necessary.

<u>Remark</u> The case $\mathcal{P} = \mathcal{A}^!$ is a toy model for the degeneration of the Hodge-de Rham spectral sequence (Kontsevich, Kaledin).