LEFSCHETZ PENCILS AND TOPOLOGICAL QUANTUM FIELD THEORY

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Panorama of Mathematics

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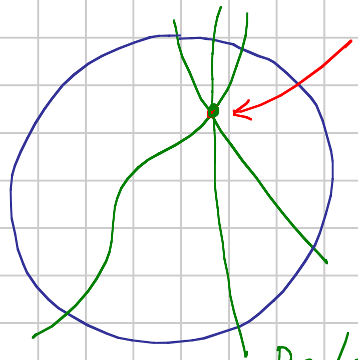
total of 10 slides

Constructing the example

On $\mathbb{CP}^2$, take a pencil of cubics



$p_0 = 0$ & $p_\infty = 0$
(9 points)

$p_0 / p_\infty = z$

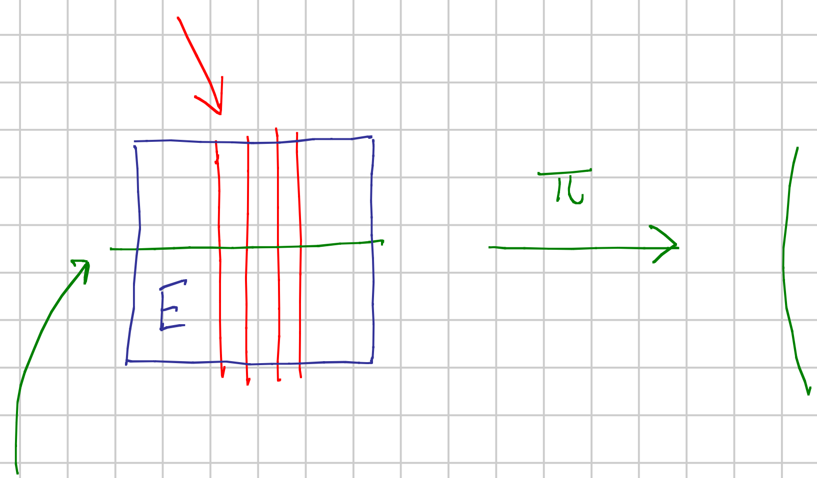
$$\frac{p_0(x_0, x_1, x_2)}{p_\infty(x_0, x_1, x_2)} = z, \quad z \in \mathbb{C} \cup \{\infty\} = \mathbb{CP}^1$$

where p_0, p_∞ are cubic homogeneous polynomials. This gives a map

$$p_0/p_\infty : \mathbb{CP}^2 \setminus \{9 \text{ points}\} \longrightarrow \mathbb{CP}^1$$

We can blow up the 9 points of indeterminacy, replacing each of them by a $\mathbb{CP}^1$. This yields a rational elliptic surface E. For topologists, $E = \mathbb{CP}^2 \# 9 \overline{\mathbb{CP}}^2$, so $H^*(\mathbb{CP}^2) \cong \mathbb{C}^{12}$.

$D = D_1 \cup \cdots \cup D_9, \quad D_i \cong \mathbb{CP}^1$



generic fibre $M \cong T^2$

Let's look at the enumerative geometry of E (counting rational curves). In any class $A \in H_2(E)$ with $A \cdot M = 1$, one expects a finite number of such curves ("holomorphic section of π"), $\langle\rangle_A \in \mathbb{Z}$.

Example Each component $D_j \subset D$ is such a curve, and the only one in its homology class, $\langle\rangle_{D_j} = 1$.

Properly defined, this count (a Gromov-Witten invariant) is independent of the choice of p_0, p_∞.

There are infinitely many A, but we can add up all the $\langle\rangle_A$ after introducing a formal parameter q:

$$z_q^{(1)} = \sum_A q^{A \cdot D} \langle\rangle_A \, A = q^{-1} D + \dots \in H^2(E)((q))$$

Theorem (Bryan-Leung ... Zagier, late '90s)

$$z_q^{(1)} = \frac{1}{q\,\psi(q)} D + \frac{\psi'(q)}{\psi(q)^2} F,$$

where

$$\psi(q) = q^{-\frac{1}{2}} \Delta(q^3)^{\frac{1}{6}}, \quad \Delta = \text{Dirichlet fn.}$$

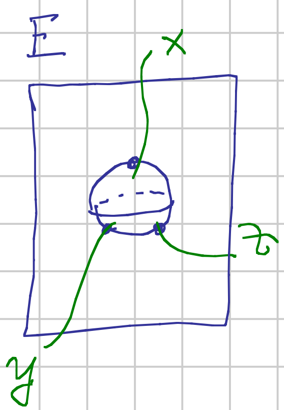
Modularity is surprising (not explained by the occurrence of the E_8 lattice).

More enumerative geometry — the quantum product $*$ on $H^*(E)((q))$:

$$\int_E (x*y)\cup z = \sum_{A\in H_2(E)} q^{A\cdot D}\langle x,y,z\rangle_A$$

where

$\langle x,y,z\rangle_A =$



counts rational A-curves with incidence conditions

Theorem (Kontsevich–Manin, ..., 90's)
$*$ is associative.

We want to consider

$$Q_q(x) = F*x : H^*(E)((q)) \circlearrowleft$$

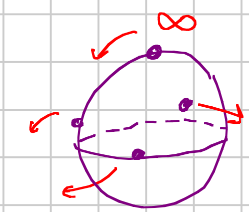
if x has degree 2

$$Q_q(x) = q^{-1}\sum_{i=1}^{9} (x\cdot D_i)\, D_i + \cdots$$

Q_q has 12 eigenvalues $\lambda_1(q),\dots,\lambda_{12}(q)$,

$\lambda_1(0),\lambda_2(0),\lambda_3(0)\in 3\sqrt[3]{1}$

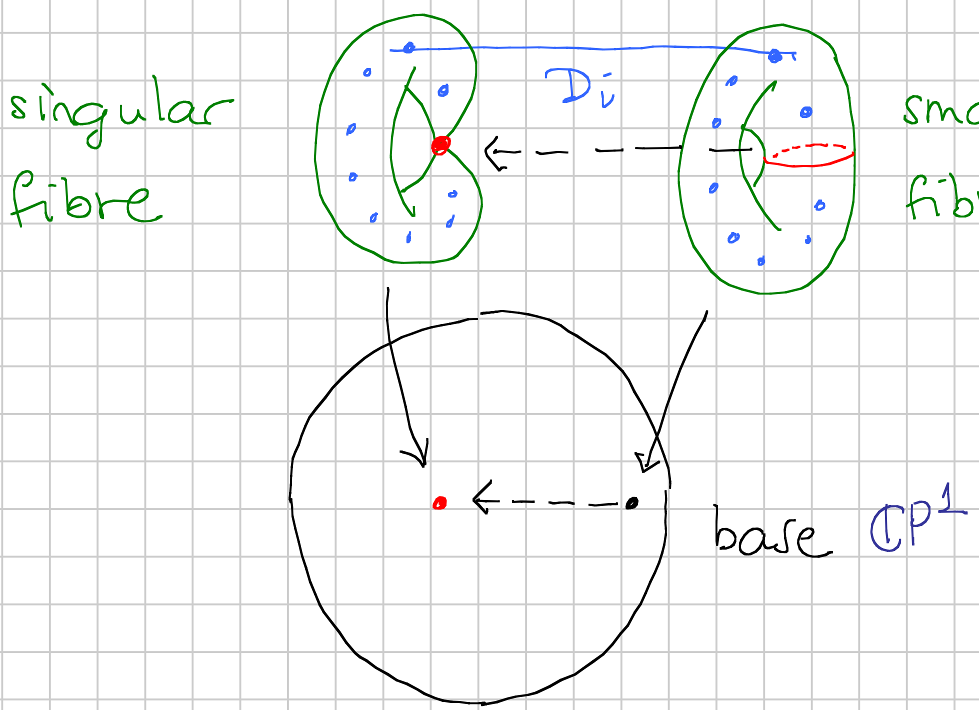
$\lambda_4(0),\dots,\lambda_{12}(0)=\infty$

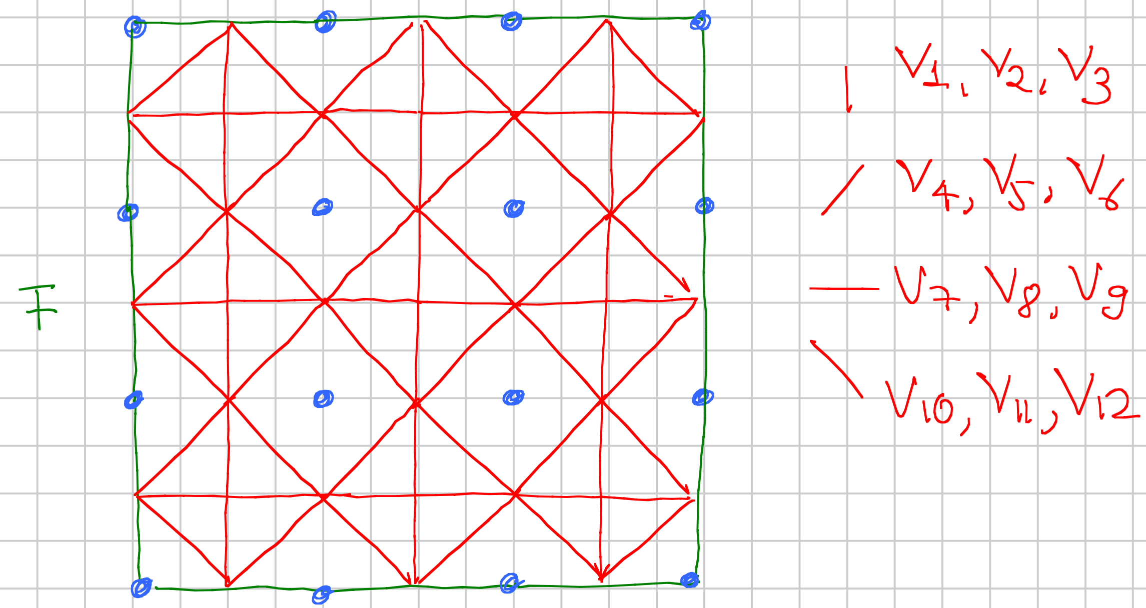


Lemma As q varies, the eigenvalues move by a common conformal transformation (not preserving 0)

<u>Picard-Lefschetz theory</u> $\pi : E \longrightarrow \mathbb{CP}^1$

has 12 singular fibres. Each fibre is a nodal elliptic curve, and contributes a circle, a vanishing cycle V_i in the fibre F.





To this we associate a $\mathbb{C}[[q]]$-linear category $\mathcal{B}_q$ (a <u>Fukaya category</u>).

<u>Lemma</u> $\mathcal{B}_q$ is polynomial in $\xi = \xi(q)$, where

(modular) j-function

$$\xi^3 \frac{(\xi^3-24)^3}{\xi^3-27} = j(q^9) = q^{-9} + 744 + \cdots$$

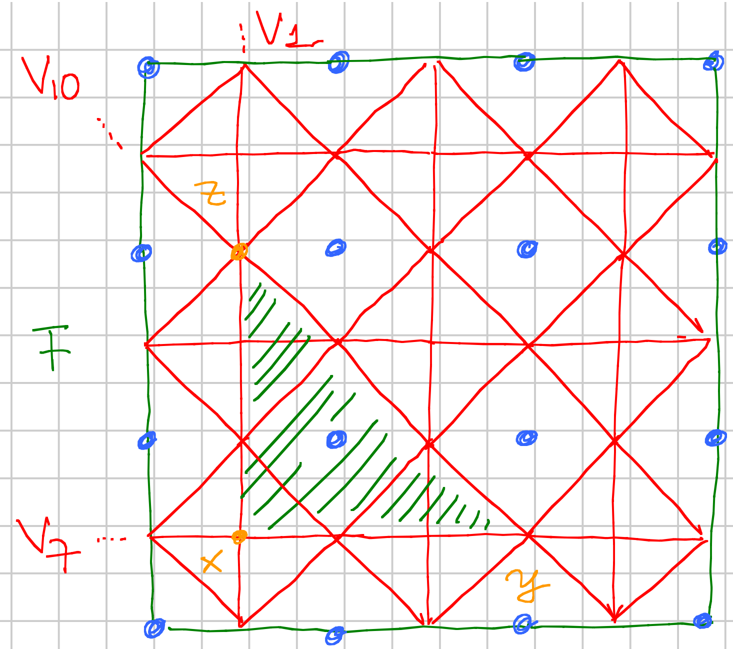
Defining $\mathcal{B}_q$:

- Objects = 12 vanishing cycles V_i
- Morphisms = intersection points
- Composition = counts triangles with sides V_i, with

$\pm q^{\#\text{ marked points in interior}}$

Remark The vanishing cycles are not unique (they involve choices of paths),

and that affects $\mathcal{B}_q$, but not its derived equivalence class.



$$\begin{tikzcd} V_1 \arrow[r, "x"] \arrow[rr, bend right, "z"'] & V_7 \arrow[r, "y"] & V_{10} \end{tikzcd}$$

$$y \cdot x = \pm q z + \cdots$$

So far, we have ignored the ordering of the V_i. We can define another category $\mathcal{A}_q$ by only allowing morphisms which increase the order.

$$\boxed{\text{Lemma } \mathcal{A}_q \text{ depends trivially on } q}$$

LIST OF PHENOMENA

Enumerative geometry of the total space E, no boundary conditions

Enumerative geometry of the fibre F, boundary conditions on vanishing cycles

Rigidity Eigenvalues of Q_q remain constant in q up to conformal transf.

Rigidity A_q is constant in q, up to isomorphism.

Special dependence $z_q^{(i)} \in H^2(E)((q))$ can be expressed in terms of modular forms

not a general phenomenon

Special dependence Up to isomorphism, the q-dependence of B_q can be expressed in terms of modular forms

Inspiration from string theory –
The mirror geometry is another fibration (again a rational elliptic surface, but a very degenerate one)

$$\pi_q^\vee : E_q^\vee \longrightarrow \mathbb{P}^1(\mathbb{C}[[q]])$$

- $\mathcal{A}_q$ describes the geometry of vector bundles on $E_q^\vee$ (its derived category of coherent sheaves)
- $\mathcal{B}_q$ does the same for the fibre $F_q^\vee$ over $\infty \in \mathbb{P}^1$.

- The eigenvalues $\lambda_i(q)$ are the critical values of $\pi_q^\vee$.
- $z^{(i)}$ corresponds to a section of $H^*(E^\vee, \Omega^*_{E^\vee})$ (presumably, with specific Hodge-theoretic meaning ?)

Rigidity $E_q^\vee$ is in fact independent of q

Special dependence $\pi_q^\vee = \Theta_q \circ \pi_0^\vee$, where Θ_q is a family of conformal transformations – an automorphism of $\mathbb{P}^1(\mathbb{C}[[q]])$

But, no general theory of "mirrors"...

Noncommutative geometry approach: we define $\mathcal{A}_q$ itself to be the mirror "space". We have to translate all of the geometric notions into that language — $\mathcal{A}_q$ must carry a "noncommutative pencil" of which $\mathcal{B}_q$ is the "fibre at infinity".

Target situation Instead of $\mathbb{CP}^2$, we start with a smooth projective (Fano) variety with a pencil of anticanonical (Calabi-Yau) hypersurfaces.

After blowing up, we get $\pi : E \longrightarrow \mathbb{CP}^1$ with Calabi-Yau fibres. Counting sections yields

$$z_q^{(1)} \in H^2(E)((q))$$

Assumption There are functions $\psi(q), \eta(q)$ such that

$$z_q^{(1)} = \frac{1}{q\psi(q)} \underbrace{D}_{\text{blowup}} + \frac{\eta(q)}{\psi(q)} \underbrace{F}_{\text{fibre}}$$

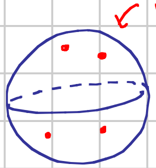
Theorem This implies that $\mathcal{A}_q$ depends trivially on q

This is the start of a whole program, including phenomena such as "special dependence": $\mathcal{B}_q$ is defined over

$$\mathbb{C}[\xi] \subseteq \mathbb{C}[[q]], \qquad \xi = \xi(q)$$

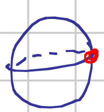
Why the assumption? Consider a simplified form $z^{(1)} = q^{-1} D$.

Gromov-Witten invariants

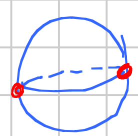


$$\langle \cdots \rangle : H^*(E)^{\otimes n} \longrightarrow \mathbb{C}((q))$$

Ex.

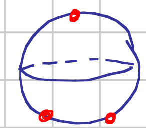
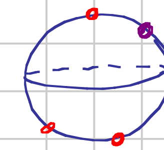


$$\text{"="}\ z_q^{(1)},$$

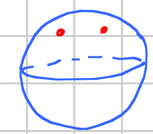
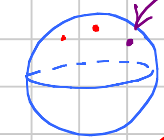


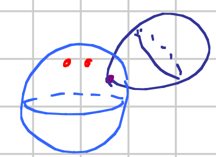
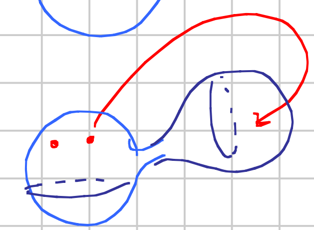
$$\text{"="}\ Q_q$$

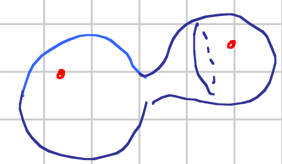
Divisor equation

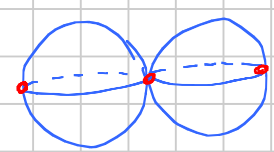
∂_q  $= -$ 

Therefore,

$\partial_q \Phi_q = \partial_q$  $=$ 

$=$  $=$ 

$=$ 

$=$  $= Q_q\, Q_q$