Constructing open symplectic manifolds from Lefschetz fibrations

Paul Seidel
Evans lecture
Berkeley,
Feb 2010

Work discussed in this talk:

- Maydanskiy, arXiv 0906.2224
- Maydanskiy – Seidel, arXiv 0906.2230

We will implicitly use results from

- Bourgeois – Ekholm – Eliashberg 0911.0026
  (plus Appendix by Ganatra – Maydanskiy)

This talk also includes "folk" facts I learned from Jeremy van Horn-Morris.

<u>Basic construction in low dimension</u>: the <u>input</u> is

M compact oriented surface, $\partial M \neq \emptyset$

$V_1, \dots, V_m$ simple closed curves in M,

each of them homologically nontrivial

and the <u>output</u> is

E symplectic four-manifold with boundary,

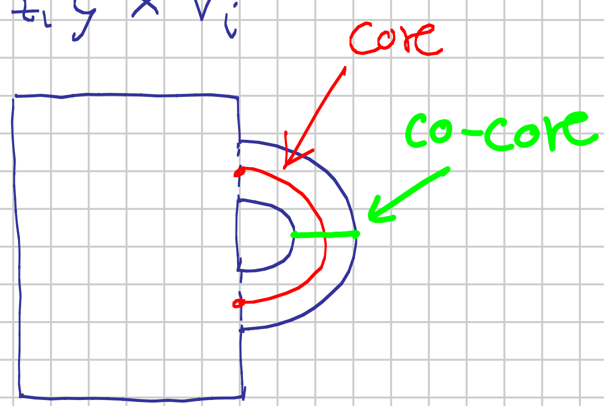
$\omega = d\theta$, $\theta | \partial E$ a contact one-form

All smooth affine algebraic surfaces (and – I believe – Stein four-manifolds) can be constructed in this way.

Two descriptions:

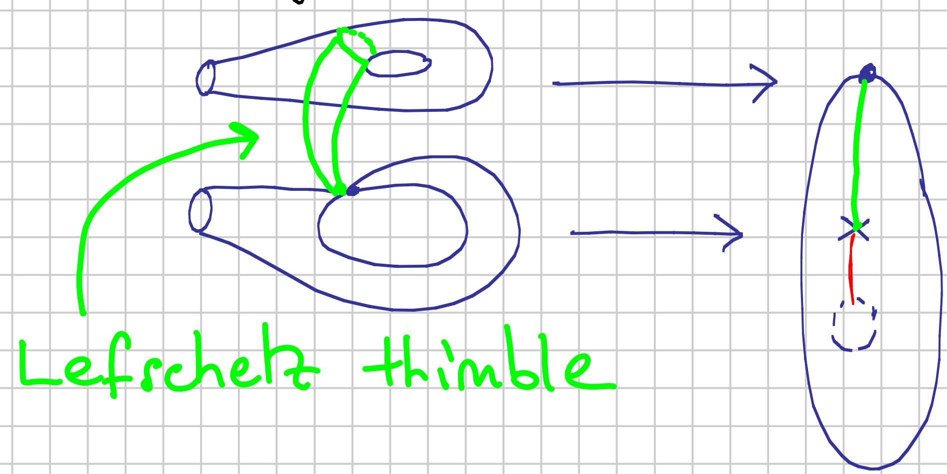
Handle attachment

Take $D^2 \times M$, attach thickened 2-cells to its boundary along $\{z_i\} \times V_i$



Lefschetz fibrations

There is a map $\pi : E \longrightarrow D^2$ with singularities modelled on $\pi(z_1, z_2) = z_1 z_2 : \mathbb{C}^2 \longrightarrow \mathbb{C}$, which has fibre M and vanishing cycles V_i.



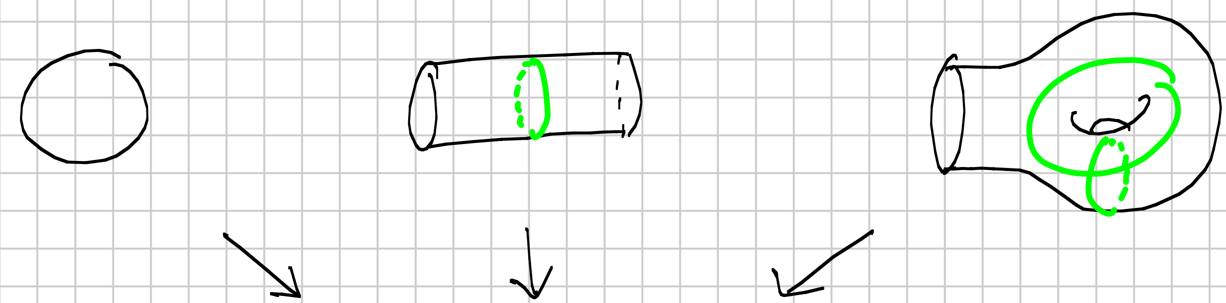
Basic invariant (with higher-dimensional situation in mind): symplectic cohomology $SH^*(E)$, due to Viterbo and Cieliebak-Floer-Hofer-Wysocki.

Thm If E contains a closed exact Lagrangian submanifold ($L^2 \subset E^4$, $\omega|L = 0$, $\theta|L = d\psi_L$) then $SH^*(E) \neq 0$.

Symplectic manifolds with $SH^*(E) = 0$ are called empty. D^4 and $D^2 \times M$ are empty.

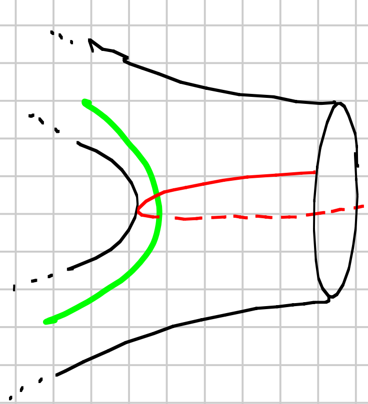
Thm $E' \subset E$ and E empty $\Rightarrow$ E' empty

Ex.



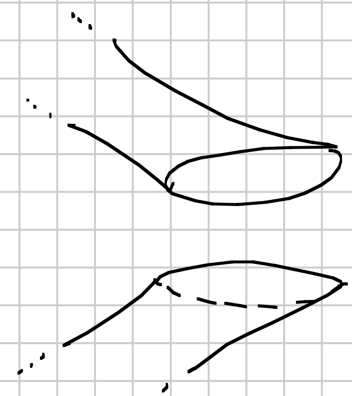
$$E = D^4$$

These are instances of the general <u>handle cancellation rule</u>, which simplifies M and removes V_1:

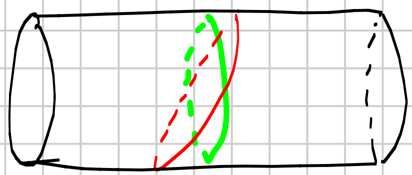


$$C \cap V_1 = \text{point},$$
$$C \cap V_2 = \emptyset,$$
$$\vdots$$

$\longrightarrow$



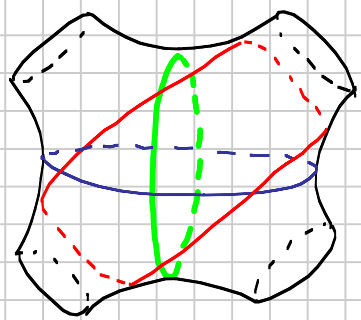
Ex.



$V_1 \simeq V_2$

$\longrightarrow \quad E \cong D^* S^2$

(disc bundle  $_{-2}$)

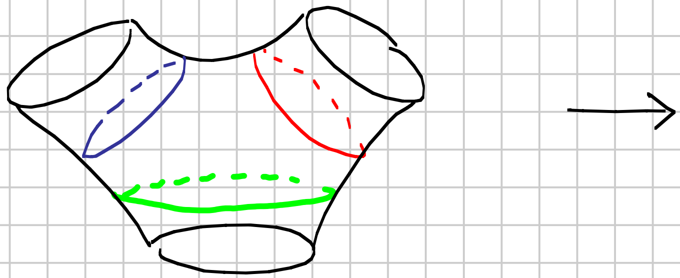
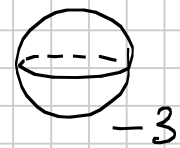


$V_1 \quad V_2 \quad V_3$

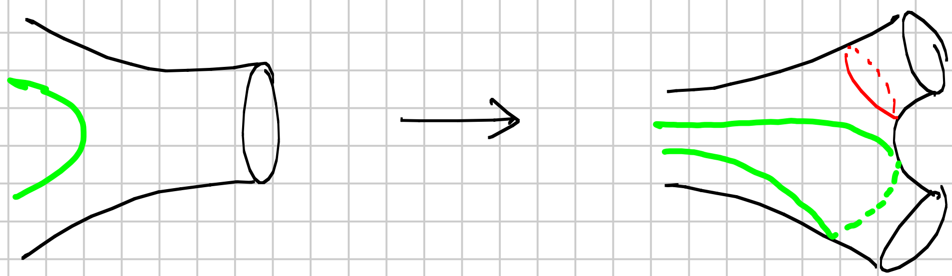
$\longrightarrow \quad E \cong D^* \mathbb{RP}^2$

(exercise: pass to double cover)

Whenever such configurations arise as part of $(V_1, \ldots, V_m)$, we know E is <u>not empty</u>.

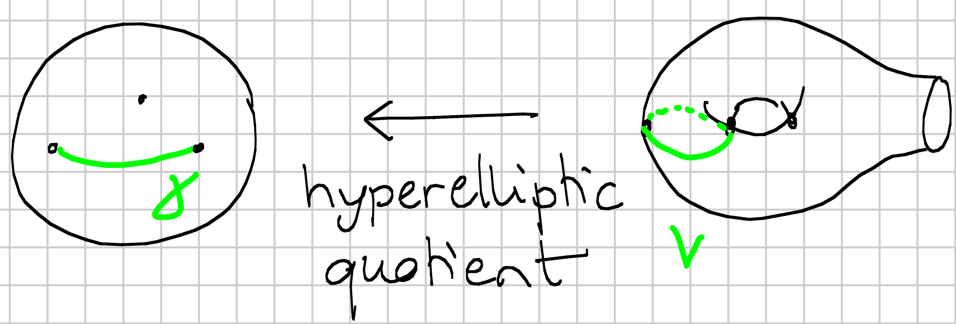
Ex.  E is topologically , and in fact empty

This is an example of a general stabilization process,

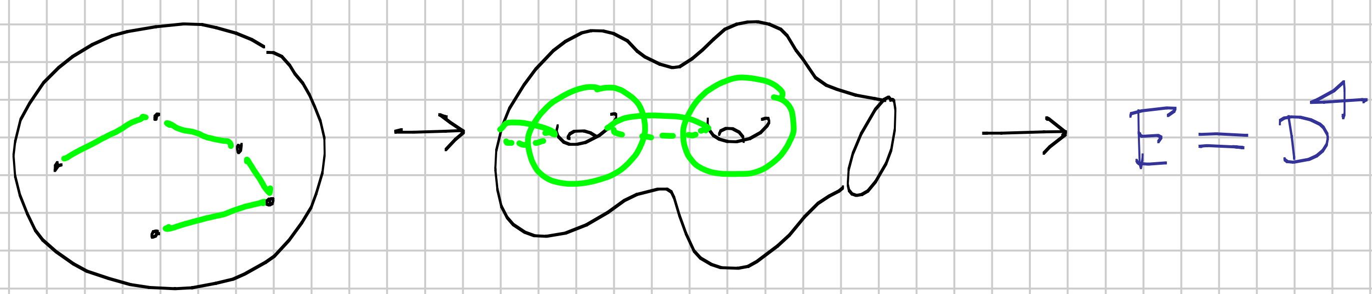


Thm (Seidel, but known to BEE in similar form) stabilizing each V_i kills SH^*.

<u>Hyperelliptic fibrations</u> : take all V_i to be (anti) invariant under the hyperelliptic involution. Then, E is a double cover of D^4 branched along a symplectic surface. Encoding

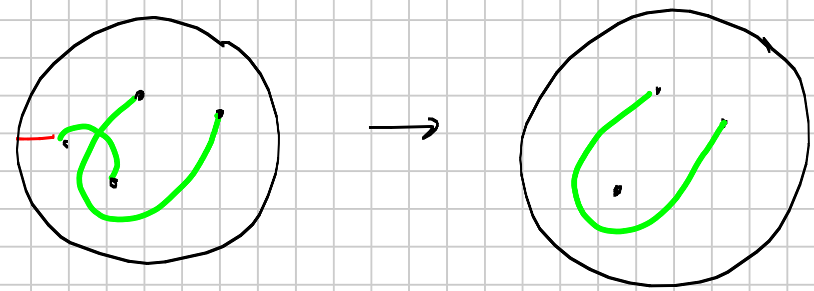


<u>Ex</u>.

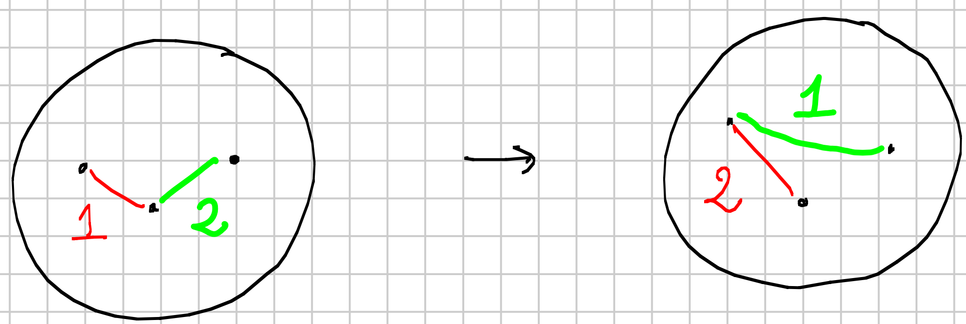


Moves that do not change the output

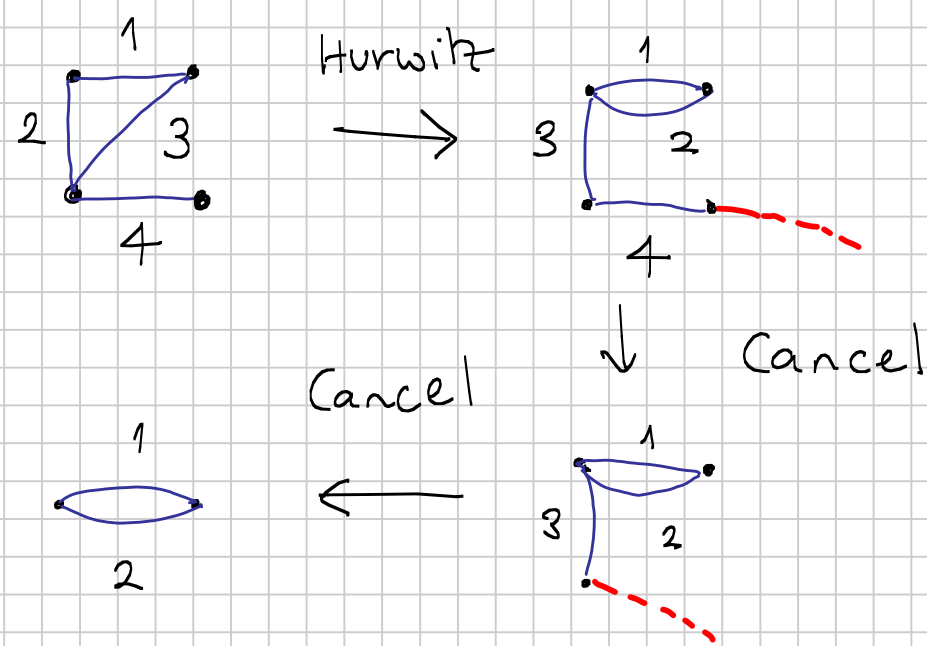
Handle cancellation removes one marked point and a path



Handle slide or **Hurwitz move** changes two paths $(\gamma_i, \gamma_{i+1}) \longmapsto (t_{\gamma_i}(\gamma_{i+1}), \gamma_i)$, where t is the half-twist:



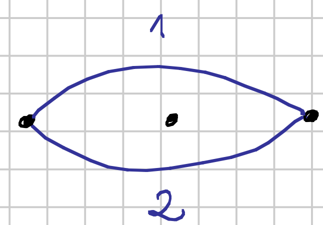
Example



This yields

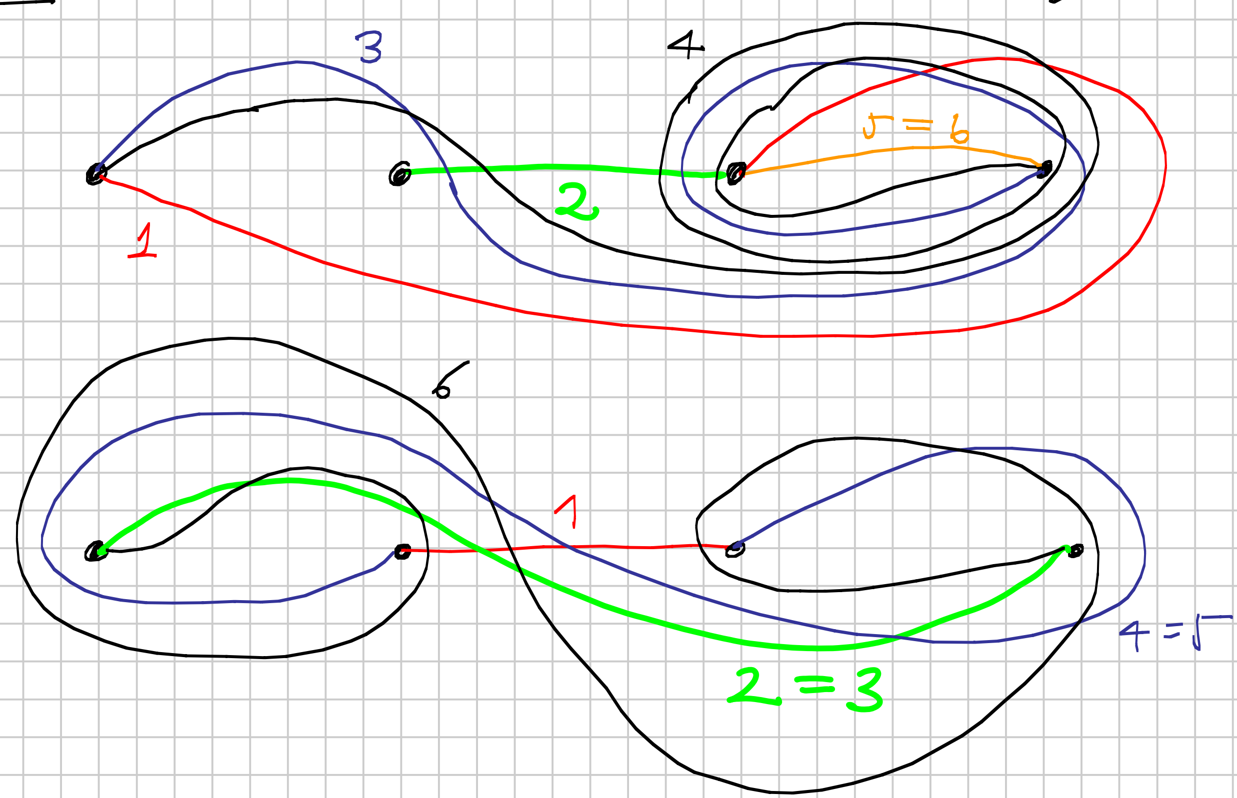
$E \cong T^*S^2$

Example (Maydanskiy)



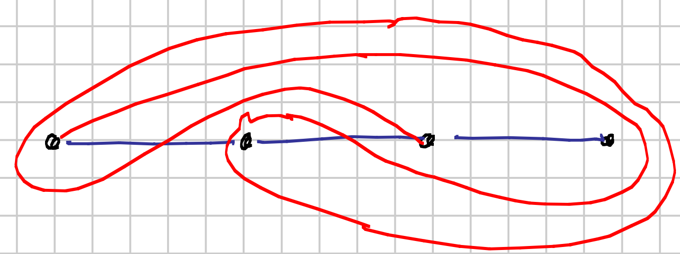
This yields $H_1(E;\mathbb{Z}_2) = H_2(E;\mathbb{Z}_2) = \mathbb{Z}_2$, but the resulting manifold is empty. This is true more generally for any 2 distinct (γ_1, γ_2).

__Example__ (Auroux - Kulikov - Shevchishin)



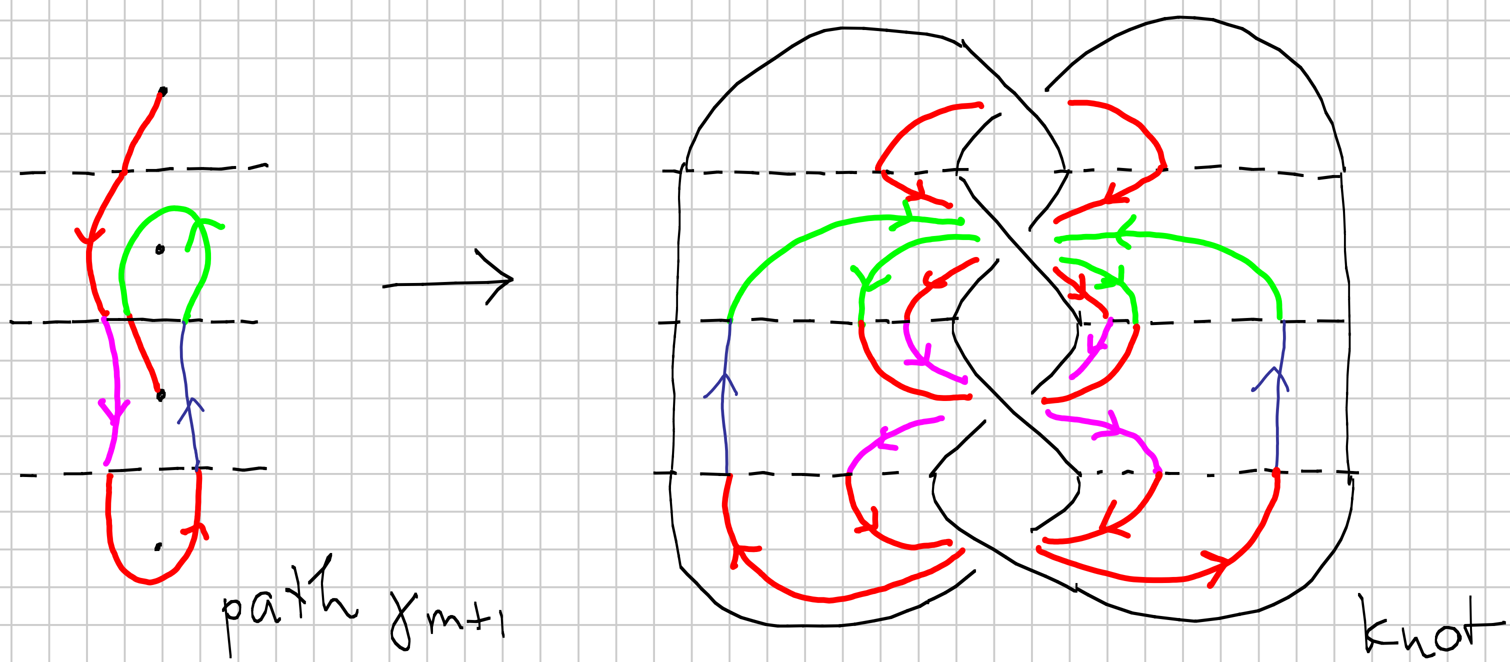
yield E , $\tilde{E}$ with the same boundary (including contact structure) but $H_1(E;\mathbb{Z}_2)=0$, $H_1(\tilde{E};\mathbb{Z}_2)\neq 0$.

Now, take $m+1$ marked points and the same number of paths, where $(\gamma_1, \dots, \gamma_m)$ are fixed and form a chain, but γ_{m+1} is arbitrary.



Possible choices of γ_{m+1} correspond to elements in the braid group Br_{m+1} conjugate to the standard generator.

The resulting E always has $\pi_1(E) = 1$, $H_2(E) = \mathbb{Z}$.
It is obtained from D^4 by attaching a handle to a knot lying on a page of the open book decomposition associated to the $(2, m)$ torus link:



<u>Thm</u> (Maydanskiy - Seidel) There are exactly $\binom{m+1}{2}$ choices of γ_{m+1} such that $E \cong D^*S^2$. In all other cases, E is <u>empty</u>.

<u>Questions</u>

- What happens if I choose the V_i randomly? Is E then typically empty or not? (work of Colin - Honda could be helpful)
- Find explicit descriptions of interesting algebro-geometric examples (Ramanujam surface)
- Find better versions of Auroux - et al's example. Compute the symplectic cohomology.

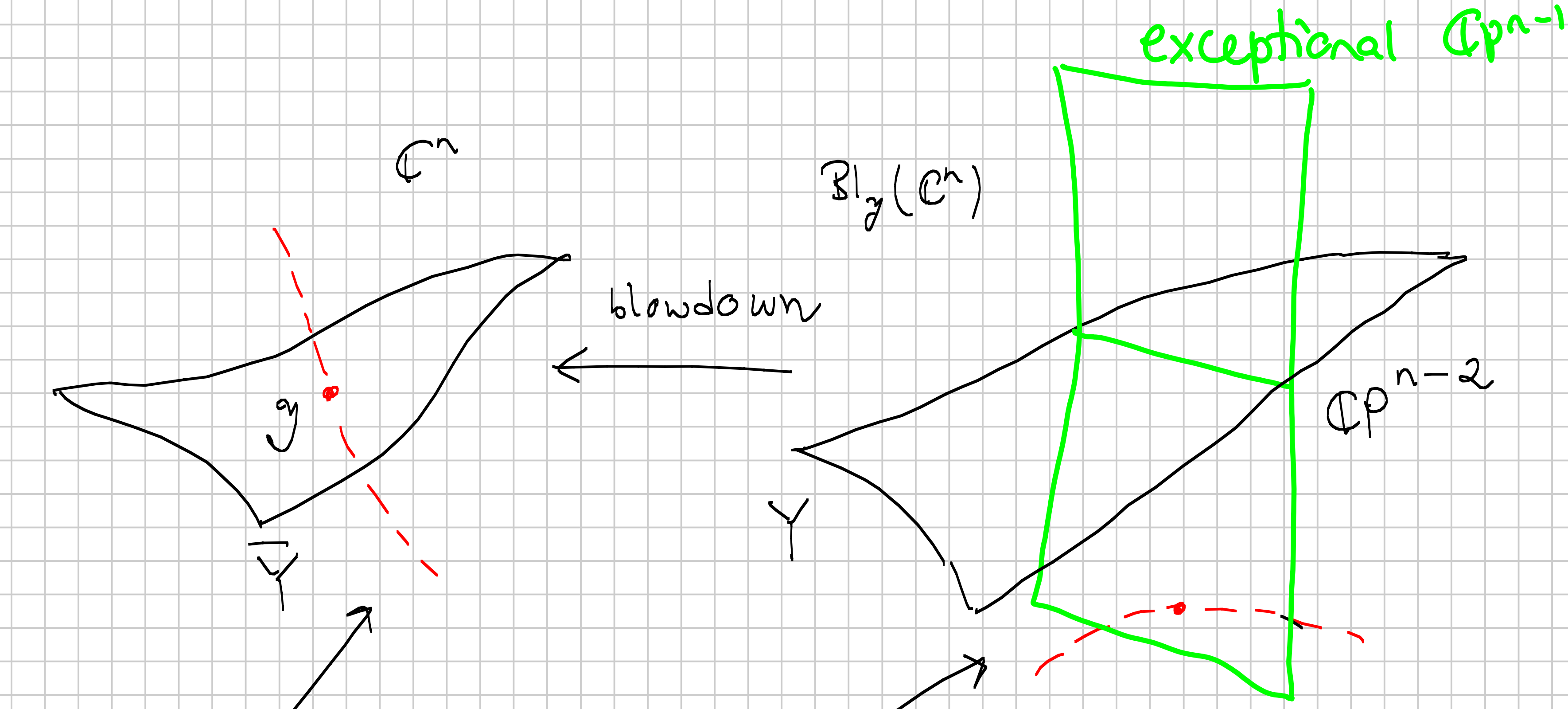
Higher dimensions

Thm (many people) Let E be an exact symplectic 4-manifold with contact type boundary. If E is homeomorphic to D^4, then it is symplectically (deformation equivalent to) D^4.

Thm (McLean) For each $n>2$, there are ∞ many distinct exact symplectic $2n$-manifolds with contact type boundary, which are all diffeomorphic to D^{2n}. They are distinguished by SH^*.

McLean uses an algebro-geometric construction called <u>Kaliman modification</u>.

- Start with $\overline{Y} \subset \mathbb{C}^n$, a singular algebraic hypersurface which is topologically contractible.
- Blow up $\mathbb{C}^n$ at a smooth point $y \in \overline{Y}$. Let $Y \subset Bl_y(\mathbb{C}^n)$ be the <u>proper transform</u> of $\overline{Y}$ and consider $X = Bl_y(\mathbb{C}^n) \setminus Y$. On the level of homology, this is a 2-handle attachment which kills $\pi_1(\mathbb{C}^n \setminus \overline{Y})$.



complex curve transverse to $\bar{Y} \subset \mathbb{C}^n$

proper transform of our curve yields the 2-cell in $X \simeq \bar{X} \cup_{S^1} D^2$.

(A_m) type Milnor fibres

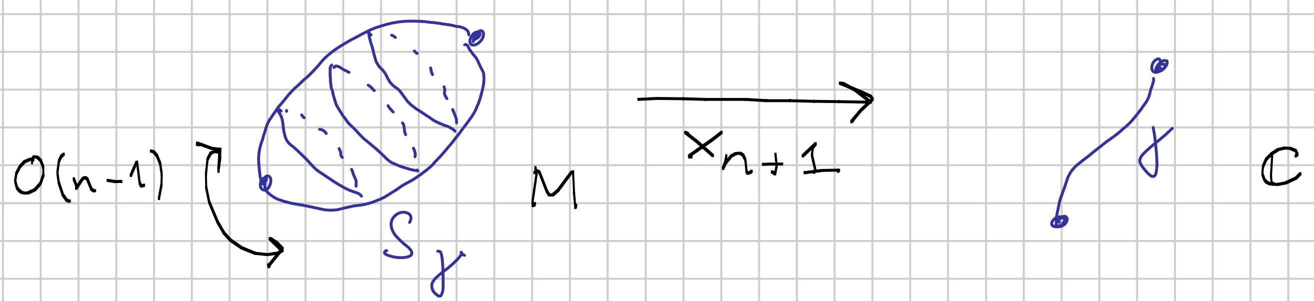
These are higher-dimensional analogues of hyperelliptic curves. Fix $n > 1$ <u>even</u> and a polynomial $p(z) \in \mathbb{C}[z]$ of degree $m+1$. Then

$$M = \{ x \in \mathbb{C}^{n+1} \mid x_1^2 + \cdots + x_n^2 + p(x_{n+1}) = 0 \}$$

Projection $x_{n+1} : M \longrightarrow \mathbb{C}$ has fibre $\cong T^*S^{n-1}$, which degenerates over $p^{-1}(0) \subset \mathbb{C}$. Hence,

$$\boxed{\begin{array}{l}\text{path } \gamma \subset \mathbb{C} \text{ joining} \\ \text{two points of } p^{-1}(0)\end{array}} \longrightarrow \boxed{\begin{array}{l}\text{Lagrangian sphere} \\ S_\gamma \subset M\end{array}}$$

This becomes a bijection $\longleftrightarrow$ if we consider only $O(n-1)$-invariant Lagrangian spheres in M.



<u>Lemma</u> S_γ and S_μ are differentiably isotopic iff γ and μ have the same endpoints.

<u>Lemma</u> (Khovanov-Seidel) S_γ and S_μ are Lagrangian isotopic if and only if γ and μ are isotopic (as paths in the punctured plane)

Again, choose $\gamma_1, \dots, \gamma_m$ in a chain and γ_{m+1} arbitrary. By using M as the fibre and $V_1 = S_{\gamma_1}, \dots, V_{m+1} = S_{\gamma_{m+1}}$ as vanishing cycles, one constructs an exact symplectic manifold with boundary E^{2n+2}.

<u>Lemma</u> E is always diffeomorphic to D^*S^{n+1} (compatibly with the almost complex structure up to homotopy).

<u>Theorem</u> (Maydanski-Seidel) Except for $\binom{m+1}{2}$ (in fact empty) choices, E is always symplectically $\not\cong D^*S^{n+1}$.