Lagrangian tori and mirror symmetry

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We work in the framework of classical Hamiltonian mechanics:

q = position, p = momentum

Equations of motion for $H = H(p,q)$:

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}$$

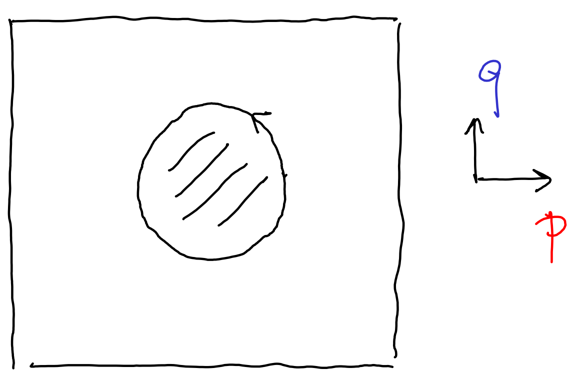
<u>Example</u> The harmonic oscillator (mass 1):

$$H(p,q) = \frac{1}{2}p^2 + \frac{1}{2}q^2$$

$$\dot{q} = p, \quad \dot{p} = -q \quad \Rightarrow \quad \ddot{q} = -q$$

Sommerfeld's quantization condition: orbits that correspond to quantum motion satisfy

$$\int_{\text{orbit}} p \, dq \in \mathbb{Z} \cdot 2\pi\hbar$$



Zum Quantensatz von Sommerfeld und Epstein;

von A. Einstein.

(Vorgetragen in der Sitzung vom 11. Mai.)
(Vgl. oben S. 79.)

§ 1. Bisherige Formulierung. Es unterliegt wohl keinem Zweifel mehr, daß für periodische mechanische Systeme von einem Freiheitsgrad die Quantenbedingung

$$\int p \, dq = \int p \frac{dq}{dt} dt = nh \qquad 1)$$

lautet (Sommerfeld und Debye). Dabei ist das Integral über eine ganze Periode der Bewegung zu erstrecken; q bedeutet die Koordinate, p die zugehörige Impulskoordinate des Systems. Ferner b weisen die spektraltheoretischen Arbeiten Sommerfelds mit Sicherheit, daß bei Systemen mit mehreren Freiheitsgraden an die Stelle dieser einen Quantenbedingung mehrere Quantenbedingungen zu treten haben, im allgemeinen so viele (l), als das System Freiheitsgrade besitzt. Diese l Bedingungen lauten nach Sommerfeld zunächst

$$\int p_i \, dq_i = n_i h. \qquad 2)$$

By Stokes ..

$$\int_{\text{orbit}} p \, dq = \int_{\text{inside of orbit}} dp \wedge dq = \text{area}$$

For a harmonic oscillator,

$$\text{area} = \pi(|p|^2 + |q|^2) \in \mathbb{Z} \cdot 2\pi\hbar$$

$$\Rightarrow \quad H = 0, \hbar, 2\hbar, 3\hbar, \ldots$$

off by $\hbar/2$

Mechanical systems with <u>compact</u> phase space: here, $\int p\, dq$ is usually not single-valued. If we assume that

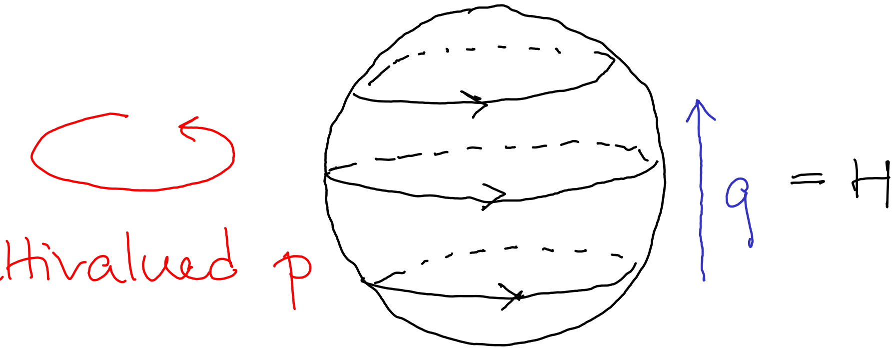
$$\int_{\substack{\text{the entire}\\ \text{phase space}}} dp \wedge dq = 2\pi \qquad (*)$$

Then the quantization condition still makes sense, provided that one sets $\hbar = 1/N$ for $N = 1, 2, 3, \ldots$

$$\int_{\substack{\text{"inside of}\\ \text{orbit"}}} dp \wedge dq \in \frac{2\pi}{N} \mathbb{Z}$$

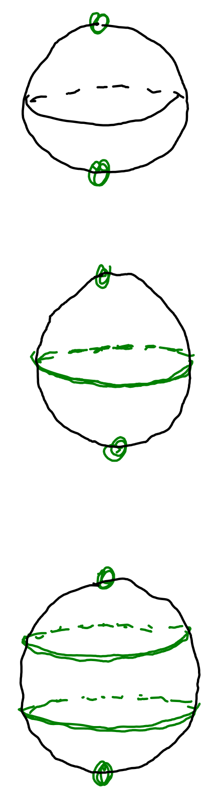
makes sense independently of what we consider the "inside" to be, since the two choices lead to integrals differing by ''

<u>Example</u> Phase space is a sphere:



such that $dp \wedge dq$ is $\frac{1}{2}$ times the spherical area $\Rightarrow$ $(*)$ holds.

Quantizable orbits



<u>N=1</u> 2 orbits

<u>N=2</u> 3 orbits

<u>N=3</u> 4 orbits

for algebraic geometers: sections of the line bundle $\mathcal{O}(N) \to \mathbb{P}^1$

Sommerfeld's attempt at quantization rules for systems with more than one degree of freedom: $H = H(p_1, \ldots, p_n, q_1, \ldots, q_n)$ and one requires that

$$\oint_{\text{orbit}} p_i \, dq_i \in \mathbb{Z} \cdot 2\pi\hbar \qquad i = 1, \ldots, n$$

Dimension-counting: orbits depend on $2n-1$ parameters, we impose n conditions $\Rightarrow$ still have $n-1$ free parameters, this is not discrete. For a general H, one can't expect it to work. As Einstein emphasized, it works for integrable systems, ones with

$$H_1 = H, \quad H_2, \ldots H_n$$

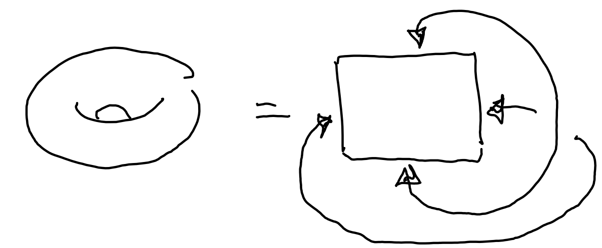
which Poisson-commute:

$$\{H_i, H_j\} = \sum_{k=1}^{n} \frac{\partial H_i}{\partial p_k}\frac{\partial H_j}{\partial q_k} - \frac{\partial H_i}{\partial q_k}\frac{\partial H_j}{\partial p_k} = 0.$$

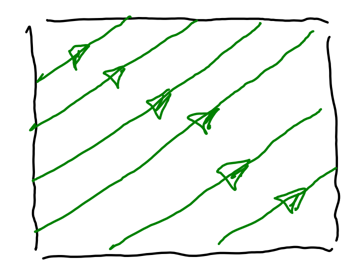
The dynamics of an integrable system - motion is constrained to n-dimensional subspaces

$$H_1 = c_1, \ldots, H_n = c_n$$

If bounded, these are tori

T_c = 

on which motion is quasiperiodic:



The quantizable (Bohr-Sommerfeld) tori are those that satisfy

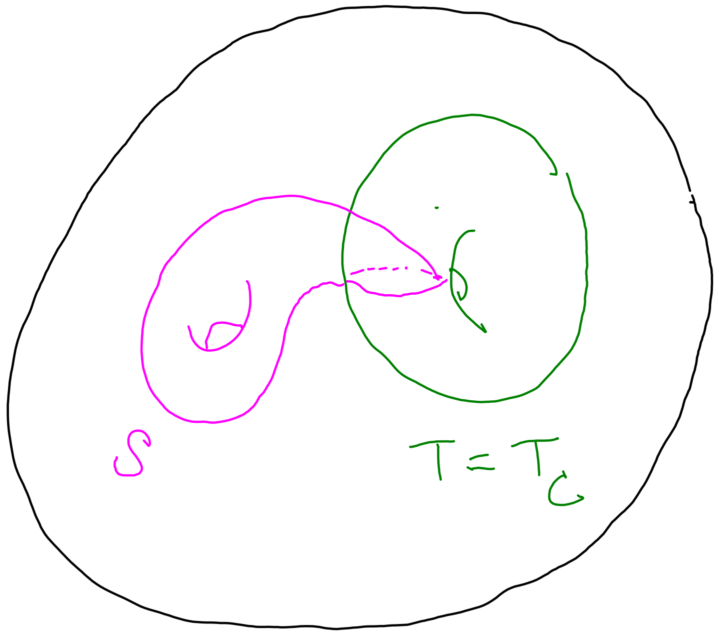
$$\oint_{\text{any loop on } T_c} p_1 \, dq_1 + \ldots + p_n \, dq_n \in \mathbb{Z} \cdot 2\pi\hbar$$

We can also write the quantization condition as

$$\int dp_1 \wedge dq_1 + \cdots + dp_n \wedge dq_n \in \mathbb{Z} \cdot 2\pi\hbar$$

any surface S in phase space with boundary on T.

phase space



Both $\{-,-\}$ and $\omega = dp_1 \wedge dq_1 + \cdots$ are intrinsic invariants of the system (independent of the choice of coordinates; or, invariant under canonical transformations).

Hence, this makes sense even for compact phase spaces.

Modern formulation:

- M = (phase space) = $2n$-dimensional symplectic manifold
  $\{-,-\}$ and $\int_S \omega$ make sense

  such that for each closed S in M,

  $$\int_S \omega \in 2\pi\mathbb{Z}$$

- A Lagrangian torus $T^n \subset M$ is one that can be defined as

  $$T = \{ H_1(x) = \cdots = H_n(x) = 0 \}$$

  where the functions $H_i : M \longrightarrow \mathbb{R}$ are in involution, $\{H_i, H_j\} = 0$

- A Lagrangian torus is a Bohr-Sommerfeld orbit if for every surface S with boundary on T,

  $$\int_S \omega \in \frac{2\pi}{N}\mathbb{Z}, \quad N = 1, 2, 3 \ldots$$

Complex projective space $\mathbb{C}P^2$. Take the phase space with coordinates $p_1, p_2, p_3, q_1, q_2, q_3$. Imagine that we are looking at Hamiltonian systems with symmetry

$$S^1 \circlearrowright \mathbb{R}^6,$$

$$t \cdot \begin{pmatrix} p_k \\ q_k \end{pmatrix} = \begin{pmatrix} \cos(t)\, p_k + \sin(t)\, q_k \\ -\sin(t)\, p_k + \cos(t)\, q_k \end{pmatrix}$$

The associated conserved quantity (Noether's theorem) is

$$\mu = \tfrac{1}{2}\left(p_1^2 + p_2^2 + p_3^2 + q_1^2 + q_2^2 + q_3^2\right).$$

We can study such systems through the "reduced phase space"

$$\mathbb{C}P^2 = \mu^{-1}(1)/S^1.$$

Equivalently, with complex coordinates $z_k = p_k + i q_k$,

$$M = \mathbb{C}^3 \setminus \{(0,0,0)\} / \mathbb{C}^* .$$

$\mathbb{C}P^2$ is a compact symplectic manifold satisfying the properties above. It comes with a map

$$\mathbb{C}P^2 \longrightarrow \{t_1 + t_2 + t_3 = 1,\ t_k \geq 0\} \subset \mathbb{R}^3$$

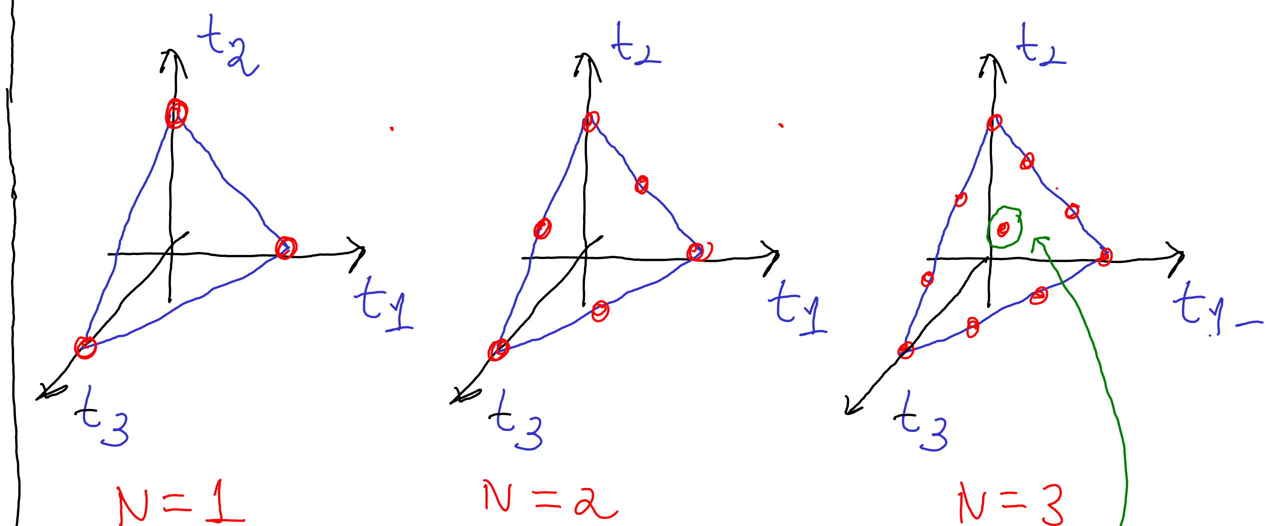
$$t_k = \tfrac{1}{2}\left(p_k^2 + q_k^2\right), \quad \{t_i, t_j\} = 0.$$

Every (nonsingular) fibre

$$T_c = \{t_k = c_k\} \subseteq \mathbb{C}P^2$$

is a Lagrangian torus. The Bohr–Sommerfeld orbits are

compare sections of $\mathcal{O}(N)$



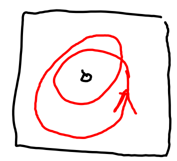
First nonsingular one – Clifford torus

Monotonicity This is a more restrictive version of the Bohr-Sommerfeld condition, at a specific N ($N=3$ for $\mathbb{CP}^2$).
Take first a Lagrangian torus $T^3 \subseteq \mathbb{R}^6 = \mathbb{C}^3$.
Along a loop $\ell \subset T^3$, we can choose a basis of tangent vectors

$$A_1(t), A_2(t), A_3(t) \in T_{\ell(t)}(T^3) \subseteq \mathbb{C}^3$$

Think of these as columns of an invertible complex matrix $A(t)$, and watch

$$t \longmapsto \det_{\mathbb{C}}(A(t)) \in \mathbb{C}^*$$



Let $\mathrm{rot}(\ell) \in \mathbb{Z}$ be the winding number of that loop.

Definition $T \subseteq \mathbb{R}^6$ is monotone if, for all loops ℓ on it,

$$\int_\ell p_1 dq_1 + \cdots p_3 dq_3 = 2\pi \, \mathrm{rot}(\ell)$$

Example $T = \{ |z_k| = 1, \ k = 1, 2, 3 \}$.

Recall our construction

$$\mathbb{R}^6 \supseteq S^5 \longrightarrow \mathbb{CP}^2$$

If $T \subseteq \mathbb{CP}^2$ is a Lagrangian torus, so is its preimage $\tilde{T} \subset \mathbb{R}^6$, we say that T is monotone if $\tilde{T}$ is.

Example The Clifford torus is monotone

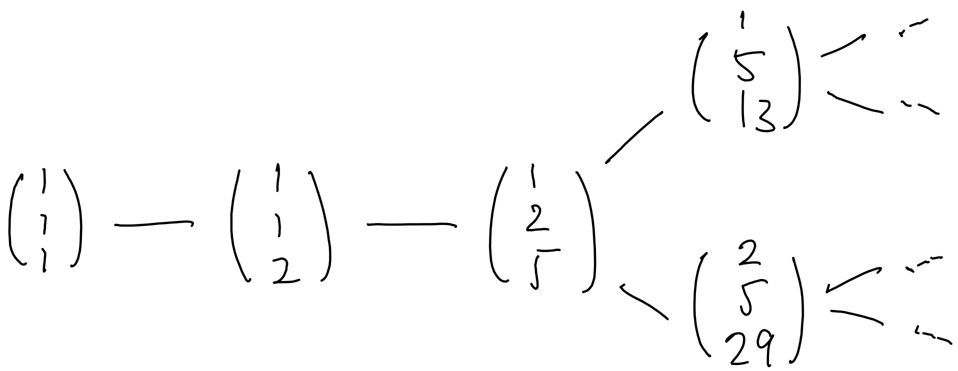
Both the Bohr-Sommerfeld condition are invariant under Hamiltonian deformations ($\approx$ canonical transformations). Hence, it makes sense to ask for their classification up to deformation. We point out that local knotting is impossible:

Theorem (Eliashberg-Polterovich '94)
Any Lagrangian $L \subset \mathbb{R}^4$ which equals $\mathbb{R}^2 = \{p_1 = p_2 = 0\}$ at infinity can be deformed to that $\mathbb{R}^2$.

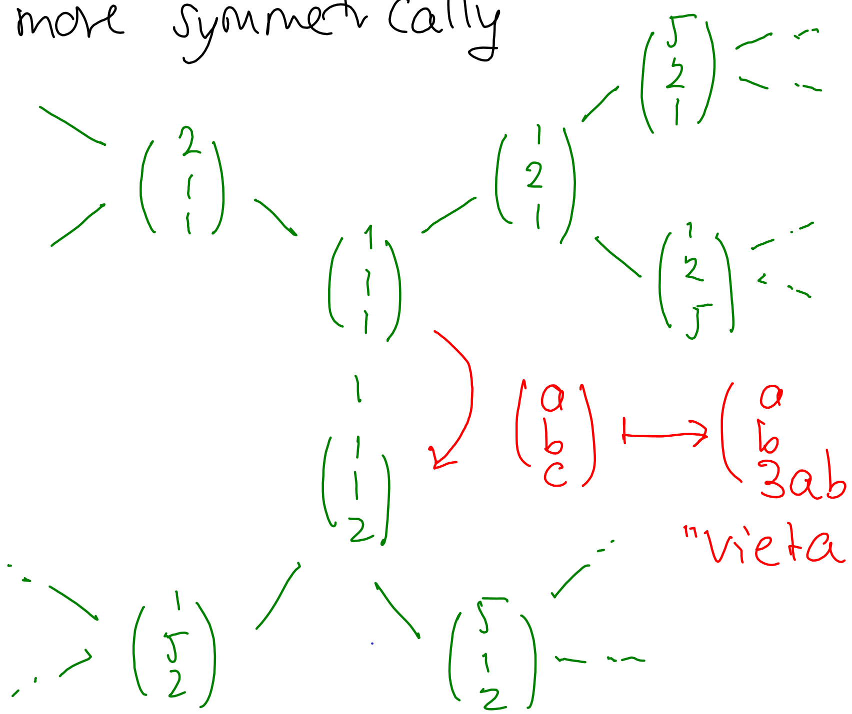
Markov's equation. For natural numbers

$$a^2+b^2+c^2 = 3abc$$

The solutions form an infinite graph



or more symmetrically



$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \longmapsto \begin{pmatrix} a \\ b \\ 3ab-c \end{pmatrix}$$

"Vieta jumping"

There is a relation with representation varieties: if A, B, C lie in $SL_2(\mathbb{Z})$ with $ABC = \mathbb{1}$,

$$\operatorname{tr}(A)\operatorname{tr}(B)\operatorname{tr}(C) + \operatorname{tr}(ABA^{-1}B^{-1}) + 2$$
$$= \operatorname{tr}(A)^2 + \operatorname{tr}(B)^2 + \operatorname{tr}(C)^2$$

If $\operatorname{tr}(A^{-1}B^{-1}AB) = -2$,

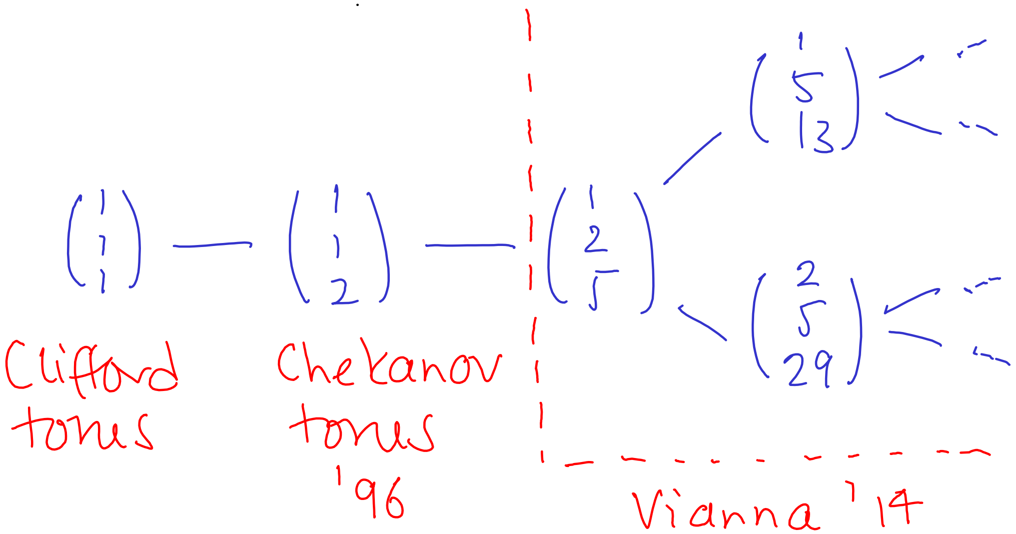
$$(a,b,c) = (\operatorname{tr}(A)/3, \operatorname{tr}(B)/3, \operatorname{tr}(C)/3)$$

is a Markov triple (these are always integers). One can think of them as representations of $\pi_1(\ )$, and the braid group acts on them by pullback, forming the Markov graph.

Conjecture There is a bijection between monotone Lagrangian tori in $\mathbb{CP}^2$ up to deformation, and Markov triples.

Construction by degeneration

(Vianna '14; algebro-geometric background by Hacking-Prokhorov '08)



Consider weighted projective space $\mathbb{P}^2(p,q,r)$, which is obtained by replacing the previous action of S^1 on $\mathbb{C}^3=\mathbb{R}^6$ with

$$t\cdot(z_1,z_2,z_3) = (t^p z_1,\ t^q z_2,\ t^r z_3)$$

This usually has singularities at the points $(1,0,0)$ etc.

There is still a "central torus" in $\mathbb{P}^2(p,q,r)$

$$\begin{array}{ccc}\mathbb{P}^2(p,q,r) & \longrightarrow & \Delta_{p,q,r} = \{ pt_1+qt_2+rt_3=1,\ t_k\geq 0\} \\ \cup & & \cup \\ T_{p,q,r} & \longmapsto & \left(\frac{1}{t_1},\frac{1}{t_2},\frac{1}{t_3}\right)\end{array}$$

If (a,b,c) is a Markov triple, there is a degeneration with generic fibre $\mathbb{CP}^2$ and special fibre $\mathbb{P}^2(a^2,b^2,c^2)$:

$$\mathcal{X} \xrightarrow{\pi} \mathbb{C}$$

$$\mathcal{X}_w = \pi^{-1}(w) \cong \mathbb{CP}^2 \quad \text{for } w\neq 0$$
$$\mathcal{X}_0 = \pi^{-1}(0) \cong \mathbb{P}^2(a^2,b^2,c^2)$$

Here, $\mathcal{X}\subseteq \mathbb{P}^2(a^2,b^2,c,3ab-c)$. The first example is

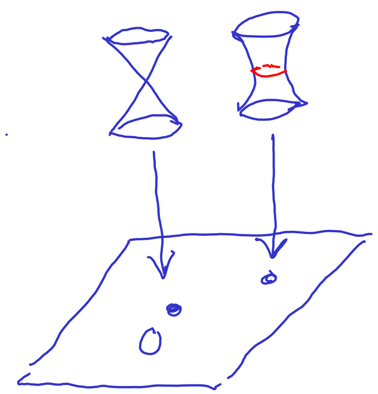
$$\mathcal{X} \subseteq \mathbb{P}^2(1,1,1,2)$$
$$\mathcal{X}_w = \{ z_1z_2-(1-w)z_3^2-wz_4=0\}$$

We use this to move $T_{p,q,r}$ into the generic fibre.

Construction by surgery (Shende – Treumann – Williams '16; Pascaleff – Tonkonog '17)

The first two examples of monotone tori in $\mathbb{CP}^2$ (Clifford and Chekanov) have analogues in $\mathbb{C}^2$, which are related by surgery along a Lagrangian disc.

$$\begin{array}{c} \mathbb{C}^2 \\ \downarrow \; W = z_1 z_2 \\ \mathbb{C} \end{array}$$



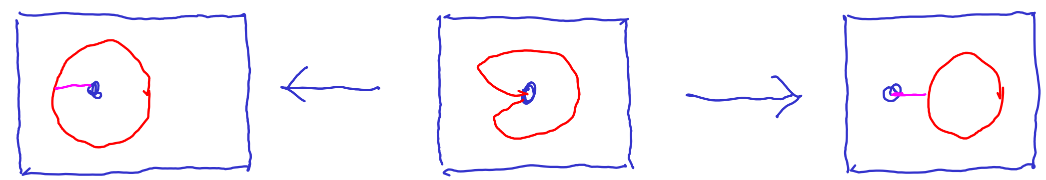
Any fibre $W^{-1}(t)$, $t \neq 0$, contains a "waist circle" $\{ |z_1| = |z_2| = t^{1/2}, \ z_1 z_2 = t \}$. If we draw a loop $\ell \subseteq \mathbb{C} \setminus \{0\}$, the union of waist circles over points of that loop yields a Lagrangian torus $T_\ell \subseteq \mathbb{R}^4$. If we take a path p with one endpoint $\{0\}$, we get a Lagrangian disc $D_p \subset \mathbb{R}^4$.

We have

$\mathbb{C}$

T_ℓ = Clifford $= \{ |z_1| = |z_2| = 1 \}$

T_ℓ = Chekanov



Lagrangian disc with boundary on the Clifford torus

nodal torus

Lagrangian disc with boundary on the Chekanov torus

In $\mathbb{CP}^2$, there are 3 natural Lagrangian discs bounding the Clifford torus. One obtains a "Markov graph of tori" by iterated surgeries.

The mirror geometry: consider the complex manifold $\mathbb{C}^* \times \mathbb{C}^* \ni (z_1, z_2)$ with function

$$W = z_1 + z_2 + \frac{1}{z_1 z_2}$$

We can associate to this its residue integral

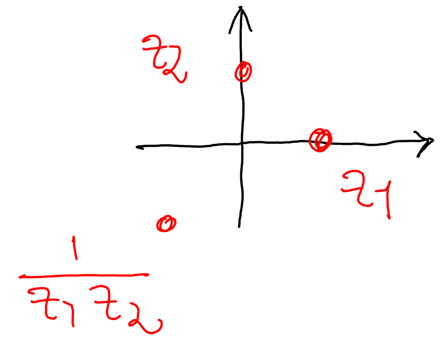
$$\Pi(t) = \int_{\substack{|z_1| = c_1 \\ |z_2| = c_2}} e^{tW} \frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2} =$$

$$= \sum_{k=0}^{\infty} \frac{t^k}{k!} \int_{\substack{|z_1| = c_1 \\ |z_2| = c_2}} W^k \frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2}$$

$$= \sum_{k=0}^{\infty} \frac{t^k}{k!} \left(\text{constant term of } W^k\right)$$

$$= 1 + t^3 + \frac{t^6}{2^3} + \frac{t^9}{6^3} + \dots$$

Combinatorics of the constant term in W^k:



Distribute the factors $\{1, \dots, k\}$ over the tree integral points so that the weighted sum is zero $\Rightarrow$ subsets must be equal. Hence,

$$\text{constant term} = \binom{k}{k/3 \;\; k/3 \;\; k/3}$$

if k is a multiple of 3, and zero otherwise.

$$\Pi(t) = \sum_{\ell=0}^{\infty} \frac{t^{3\ell}}{(\ell!)^3}$$

Mutation of Laurent polynomials

(Galkin - Usnich '12; Fomin - Zelevinsky '02)

Take a rational function $W(z_1, z_2)$ and coprime integers (a_1, a_2). The mutation of W in direction (a_1, a_2) is the change of variables

$$z_1^{m_1} z_2^{m_2} \longmapsto z_1^{m_1} z_2^{m_2} \left(1 + z_1^{a_2} z_2^{-a_1}\right)^{m_1 a_2 - m_2 a_1}$$

We are interested in cases where W and its mutation are both Laurent polynomials $\in \mathbb{C}[z_1^{\pm 1}, z_2^{\pm 1}]$.

Lemma If $W, W' \in \mathbb{C}[z_1^{\pm 1}, z_2^{\pm 1}]$ are mutations of each other,

$$\int_{\substack{|z_1| = c_1 \\ |z_2| = c_2}} e^{-tW} \frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2} = \int_{\substack{|z_1| = c_1 \\ |z_2| = c_2}} e^{-tW'} \frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2}$$

In our case, there is a Markov tree of such mutations

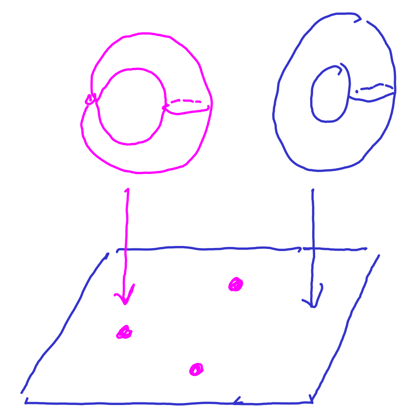
$$W = z_1 + z_2 + z_1^{-1} z_2^{-1}$$

$$\downarrow$$

$$W' = z_1 + z_2 + 2 z_1^{-2} + z_1^{-4} z_2^{-1}$$

Geometrically, what happens is that there is a natural compactification of the fibres of W,

$$\begin{tikzcd} \mathbb{C}^* \times \mathbb{C}^* \arrow[dr, "W"'] & \subset & X \arrow[dl, "\overline{W}"] & \subset X_{9111} \\ & \mathbb{C} & & \end{tikzcd}$$



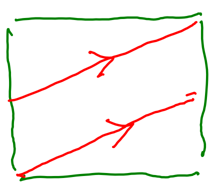
Each mutated superpotential describes the restriction of $\overline{W}$ to a different "cluster coordinate chart" $\mathbb{C}^* \times \mathbb{C}^* \subset X$

Superpotentials of Lagrangian tori
(Cho-Oh '06, Auroux '07)

$T \subseteq \mathbb{CP}^2$ monotone Lagrangian torus. $\mathbb{CP}^2$ is a complex manifold, and T is a totally real submanifold. It therefore makes sense to consider holomorphic maps

$$D = \{ z \in \mathbb{C} \mid |z| \le 1 \} \xrightarrow{u} \mathbb{CP}^2$$
$$u(\partial D) \subseteq T$$

Fix a point $* \in T$. For every u such that $u(1) = *$, and whose boundary winds (m_1, m_2) times around T

$u(\partial D) \rightsquigarrow$  T $\qquad z_1^2 z_2$

we introduce a monomial $\pm z_1^{m_1} z_2^{m_2}$. The outcome is the "superpotential"

$$W_T \in \mathbb{Z}[z_1^{\pm 1}, z_2^{\pm 1}]$$

Fact The superpotential W_T is a deformation invariant of the monotone torus T.

Example For the Clifford torus,

$$W_T = z_1 + z_2 + \frac{1}{z_1 z_2}.$$

Theorem (Pascaleff-Tonkonog '17, based on earlier work of S.) Surgery of monotone tori results in mutation of their superpotentials.

Theorem (Tonkonog '18) For every monotone $T \subseteq \mathbb{CP}^2$,

$$\int_{\substack{|z_1| = \varepsilon \\ |z_2| = \varepsilon}} e^{-tW_T} \frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2} = \Pi(t)$$

is the same! It appears here as a generating function for gravitational descendants $\langle \tau_d(\text{point}) \rangle$.