Increasing and decreasing subsequences

$$\mathbf{3}\ 1\ 8\ \mathbf{4}\ 9\ \mathbf{6}\ \mathbf{7}\ 2\ 5 \quad \text{(i.s)}$$

$$3\ 1\ \mathbf{8}\ \mathbf{4}\ 9\ 6\ 7\ \mathbf{2}\ 5 \quad \text{(d.s)}$$

$$\mathbf{is}(w) = |\text{longest i.s.}| = 4$$

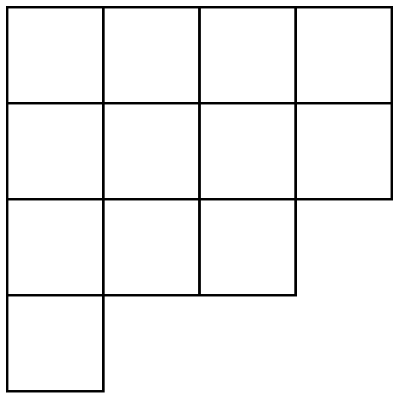
$$\mathbf{ds}(w) = |\text{longest d.s.}| = 3$$

partition $\lambda \vdash n$: $\lambda = (\lambda_1, \lambda_2, \ldots)$

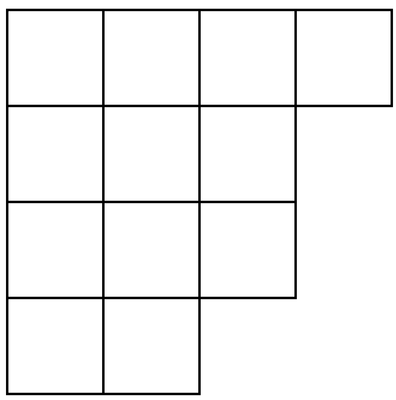
$$\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$$

$$\sum \lambda_i = n$$

(Young) diagram of $\lambda = (4, 4, 3, 1)$:



Young diagram of the **conjugate** partition $\lambda' = (4, 3, 3, 2)$:



standard Young tableau (SYT) of shape $\lambda \vdash n$, e.g., $\lambda = (4, 4, 3, 1)$:

$<$

	1	2	7	10
	3	5	8	12
$\wedge$	4	6	11	
	9			

$$\boldsymbol{f^{\lambda}} = \# \text{ of SYT of shape } \lambda$$

E.g., $f^{(3,2)} = 5$:

1 2 3	1 2 4	1 2 5	1 3 4	1 3 5
4 5	3 5	3 4	2 5	2 4

$\exists$ simple formula for f^{λ} (Frame-Robinson-Thrall **hook-length formula**)

Note. $f^\lambda = \dim(\text{irrep. of } \mathfrak{S}_n)$, where $\boldsymbol{\mathfrak{S}_n}$ is the **symmetric group** of all permutations of $1, 2 \ldots, n$.

RSK algorithm: a bijection

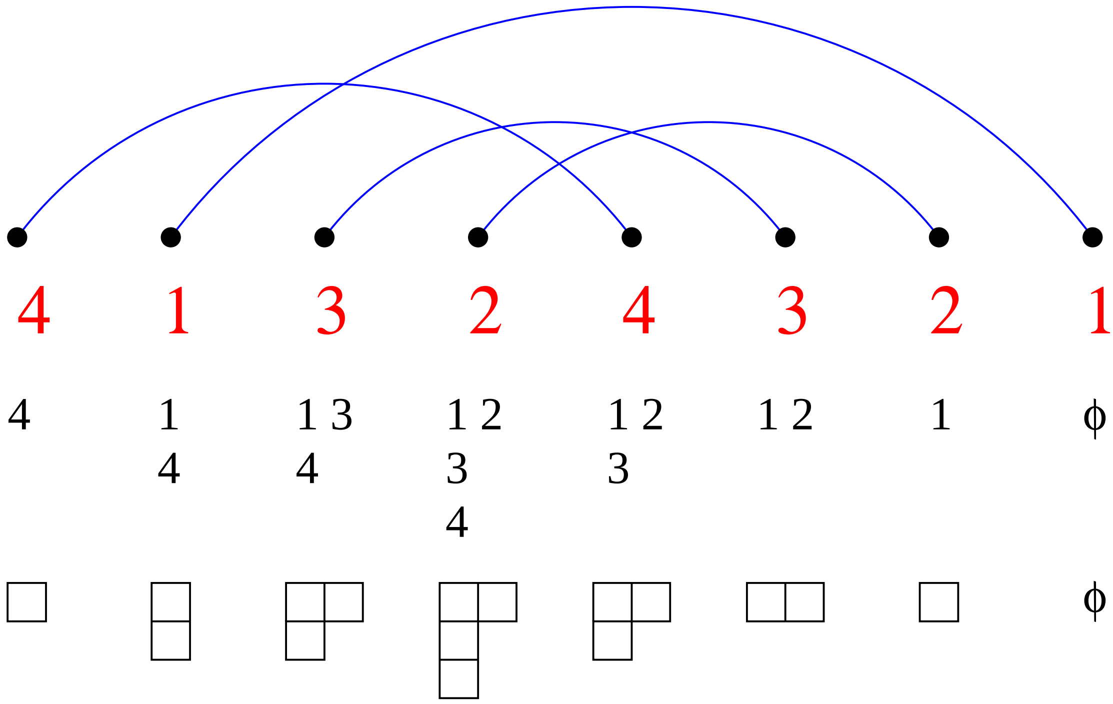
$$w \xrightarrow{\text{rsk}} (P, Q),$$

where $w \in \mathfrak{S}_n$ and P, Q are SYT of the same shape $\lambda \vdash n$.

Write $\lambda = \mathbf{sh}(w)$, the **shape** of w.

R = Gilbert de Beauregard Robinson
S = Craige Schensted (= Ea Ea)
K = Donald Ervin Knuth

$w = 4132$:



$$(P,Q) = \begin{pmatrix} \begin{matrix} 1\,2 \\ 3 \\ 4 \end{matrix} & , & \begin{matrix} 1\,3 \\ 2 \\ 4 \end{matrix} \end{pmatrix}$$

Schensted's theorem: *Let* $w \xrightarrow{\text{rsk}} (P, Q)$, *where* $sh(P) = \text{sh}(Q) = \lambda$. *Then*

$$\begin{aligned} \text{is}(w) &= \text{longest row length} = \lambda_1 \\ \text{ds}(w) &= \text{longest column length} = \lambda'_1. \end{aligned}$$

Corollary (Erdős-Szekeres, Seidenberg). *Let* $w \in \mathfrak{S}_{pq+1}$. *Then either* $\text{is}(w) > p$ *or* $\text{ds}(w) > q$.

Proof. Let $\lambda = \text{sh}(w)$. If $\text{is}(w) \le p$ and $\text{ds}(w) \le q$ then $\lambda_1 \le p$ and $\lambda'_1 \le q$, so $\sum \lambda_i \le pq$. $\square$

Corollary. *Say $p \le q$. Then*

$$\#\{w \in \mathfrak{S}_{pq} \,:\, \mathrm{is}(w) = p,\ \mathrm{ds}(w) = q\}$$

$$= \left(f^{(p^q)}\right)^2$$

By hook-length formula, this is

$$\left(\frac{(pq)!}{1^1 2^2 \cdots p^p (p+1)^p \cdots q^p (q+1)^{p-1} \cdots (p+q-1)^1}\right)^2.$$

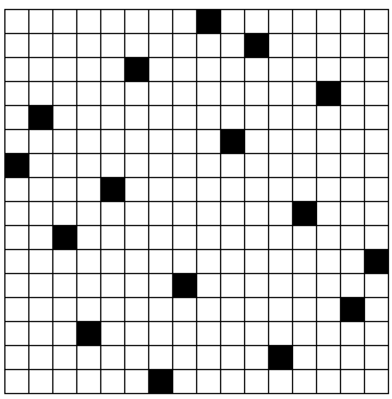
Romik: let

$$w \in \mathfrak{S}_{n^2}, \ \text{is}(w) = \text{ds}(w) = n.$$

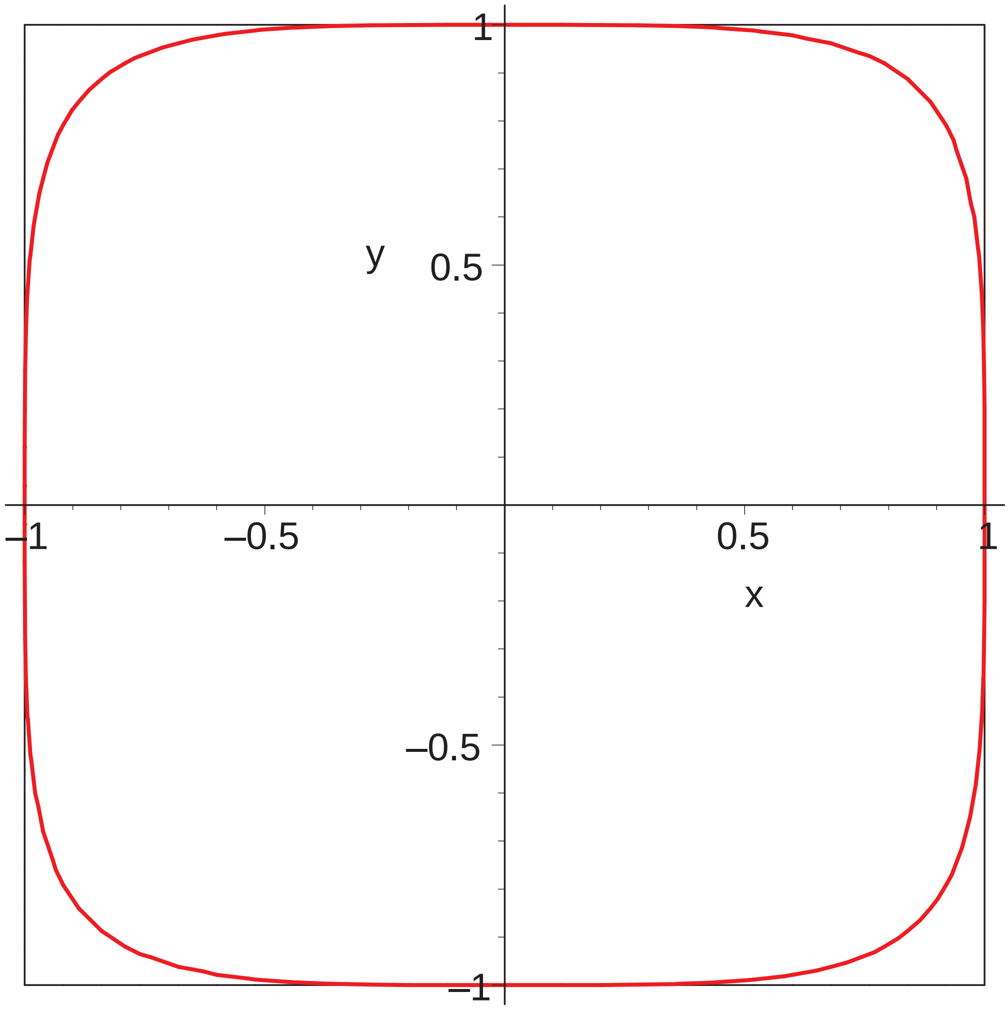
Let P_w be the permutation matrix of w with corners $(\pm 1, \pm 1)$. Then (informally) as $n \to \infty$ almost surely the 1's in P_w will become dense in the region bounded by the curve

$$(x^2 - y^2)^2 + 2(x^2 + y^2) = 3,$$

and will remain isolated outside this region.



$w = 9, 11, 6, 14, 2, 10, 1, 5, 13, 3, 16, 8, 15, 4, 12, 17$



$$(x^2 - y^2)^2 + 2(x^2 + y^2) = 3$$

Area enclosed by curve:

$$\alpha = 8\int_0^1 \frac{1}{\sqrt{(1-t^2)(1-(t/3)^2)}}dt$$
$$-6\int_0^1 \sqrt{\frac{1-(t/3)^2}{1-t^2}}dt$$
$$= 4(0.94545962\cdots)$$

Distribution of is(w)

$$\boldsymbol{E(n)} = \text{expectation of is}(w),\ w \in \mathfrak{S}_n$$
$$= \frac{1}{n!} \sum_{\lambda \vdash n} \lambda_1 \left(f^\lambda\right)^2$$

Ulam: what is distribution of is(w)? rate of growth of $E(n)$?

Hammersley (1972):

$$\exists\ c = \lim_{n\to\infty} n^{-1/2} E(n),$$

and

$$\frac{\pi}{2} \leq c \leq e.$$

Conjectured $c = 2$.

Logan-Shepp, Vershik-Kerov (1977): $c = 2$

Idea of proof.

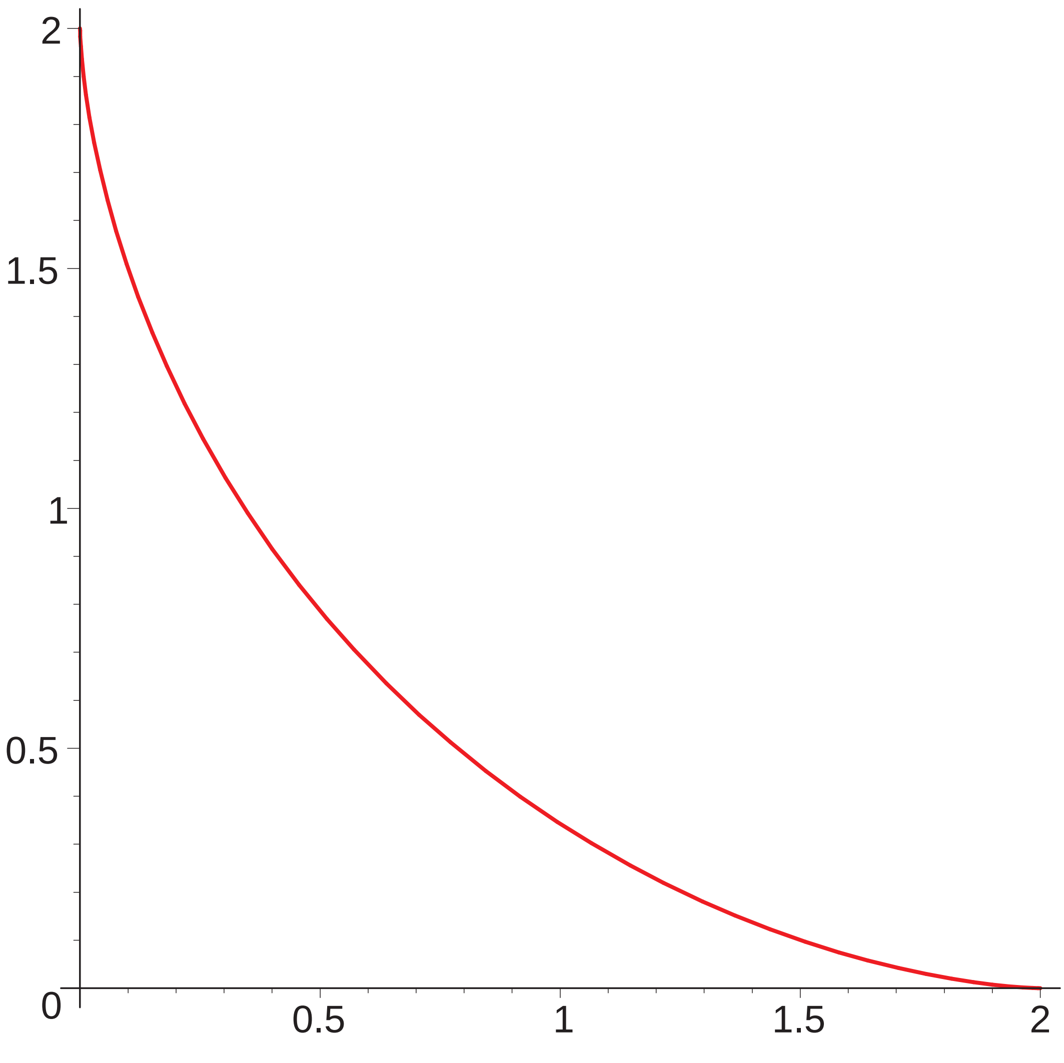
$$\begin{aligned} E(n) &= \frac{1}{n!}\sum_{\lambda \vdash n} \lambda_1 \left(f^\lambda\right)^2 \\ &\approx \frac{1}{n!}\max_{\lambda \vdash n} \lambda_1 \left(f^\lambda\right)^2. \end{aligned}$$

Find "limiting shape" of $\lambda \vdash n$ maximizing λ as $n \to \infty$ using hook-length formula.

$$x = y + 2\cos\theta$$

$$y = \frac{2}{\pi}(\sin\theta - \theta\cos\theta)$$

$$0 \le \theta \le \pi$$



$\boldsymbol{u_k(n)} := \#\{w \in \mathfrak{S}_n \,:\, \mathrm{is}_n(w) \leq k\}.$

J. M. Hammersley (1972):

$$u_2(n) = C_n = \frac{1}{n+1}\binom{2n}{n},$$

a **Catalan number**.

For $\geq$130 combinatorial interpretations of C_n, see

`www-math.mit.edu/~rstan/ec`

I. Gessel (1990):

$$\sum_{n\geq 0} u_k(n)\frac{x^{2n}}{n!^2} = \det\left[I_{|i-j|}(2x)\right]_{i,j=1}^{k},$$

where

$$I_m(2x) = \sum_{j\geq 0}\frac{x^{m+2j}}{j!(m+j)!},$$

a **hyperbolic Bessel function** of the first kind of order m.

Example. $\displaystyle\sum_{n\geq 0} u_2(n)\frac{x^{2n}}{n!^2}$

$$= I_0(2x)^2 - I_1(2x)^2$$

$$= \sum_{n\geq 0} C_n \frac{x^{2n}}{n!}.$$

Corollary. For fixed k, $u_k(n)$ is **P-recursive**, e.g.,

$$\begin{aligned}&(n+4)(n+3)^2u_4(n)\\&=(20n^3+62n^2+22n-24)u_4(n-1)\\&\quad -64n(n-1)^2u_4(n-2)\end{aligned}$$

$$\begin{aligned}&(n+6)^2(n+4)^2u_5(n)\\&=(375-400n-843n^2-322n^3-35n^4)u_5(n-1)\\&\quad +(259n^2+622n+45)(n-1)^2u_5(n-2)\\&\quad -225(n-1)^2(n-2)^2u_5(n-3).\end{aligned}$$

Conjectures on form of recurrence due to Bergeron, Favreau, and Krob.

Baik-Deift-Johansson:

Define $\boldsymbol{u(x)}$ by

$$\frac{d^2}{dx^2}u(x) = 2u(x)^3 + xu(x) \quad (*),$$

with certain initial conditions.

$(*)$ is the **Painlevé II** equation (roughly, the branch points and essential singularities are independent of the initial conditions).

Paul Painlevé

1863: born in Paris.

1890: Grand Prix des Sciences Mathématiques

1908: first passenger of Wilbur Wright; set flight duration record of one hour, 10 minutes.

1917, 1925: Prime Minister of France.

1933: died in Paris.

Tracy-Widom distribution:

$$\boldsymbol{F(t)}$$

$$= \exp\left(-\int_t^\infty (x-t)u(x)^2\,dx\right)$$

Let $\boldsymbol{\chi}$ be a random variable with distribution F, and let $\boldsymbol{\chi_n}$ be the random variable on $\mathfrak{S}_n$:

$$\chi_n(w) = \frac{\mathrm{is}_n(w) - 2\sqrt{n}}{n^{1/6}}.$$

Theorem. *As* $n \to \infty$,

$$\chi_n \to \chi \quad \text{in distribution},$$

i.e.,

$$\lim_{n\to\infty} \mathrm{Prob}(\chi_n \le t) = F(t).$$

Corollary.

$$
\begin{aligned}
E(n) &= 2\sqrt{n} + \left(\int t \, dF(t) \right) n^{1/6} + o(n^{1/6}) \\
&= 2\sqrt{n} - (1.7711\cdots) n^{1/6} + o(n^{1/6})
\end{aligned}
$$

Gessel's theorem reduces the problem to "just" analysis, viz., the **Riemann-Hilbert problem** in the theory of integrable systems, and the **method of steepest descent** to analyze the asymptotic behavior of integrable systems.

Where did the Tracy-Widom distribution $F(t)$ come from?

$$F(t)$$

$$= \exp\left(-\int_t^\infty (x-t)u(x)^2\,dx\right)$$

$$\frac{d^2}{dx^2}u(x) = 2u(x)^3 + xu(x) \quad (*),$$

Gaussian Unitary Ensemble (GUE):

Consider an $n \times n$ hermitian matrix $\boldsymbol{M = (M_{ij})}$ with probability distribution

$$Z_n^{-1} e^{-\mathrm{tr}(M^2)} dM,$$

$$dM = \prod_i dM_{ii} \cdot \prod_{i<j} d(\Re M_{ij}) d(\Im M_{ij}),$$

where $\boldsymbol{Z_n}$ is a normalization constant.

Tracy-Widom (1994): let $\boldsymbol{\alpha_1}$ denote the largest eigenvalue of M. Then

$$\lim_{n\to\infty} \operatorname{Prob}\left(\left(\alpha_1 - \sqrt{2n}\right)\sqrt{2}n^{1/6} \le t\right) = F(t).$$

Is the connection between is(w) and GUE a coincidence?

The proof of Okounkov provides a connection, via the theory of **random topologies on surfaces**.

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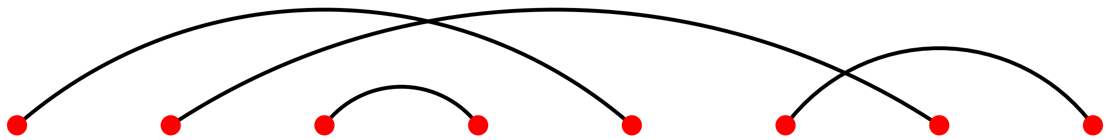
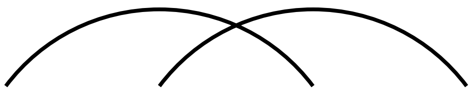
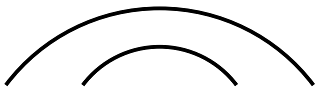
(complete) matching:



crossing: 

nesting: 

total number of matchings on $[2n] := \{1, 2, \ldots, 2n\}$ is

$\mathbf{(2n-1)!!} := 1 \cdot 3 \cdot 5 \cdots (2n-1)$.

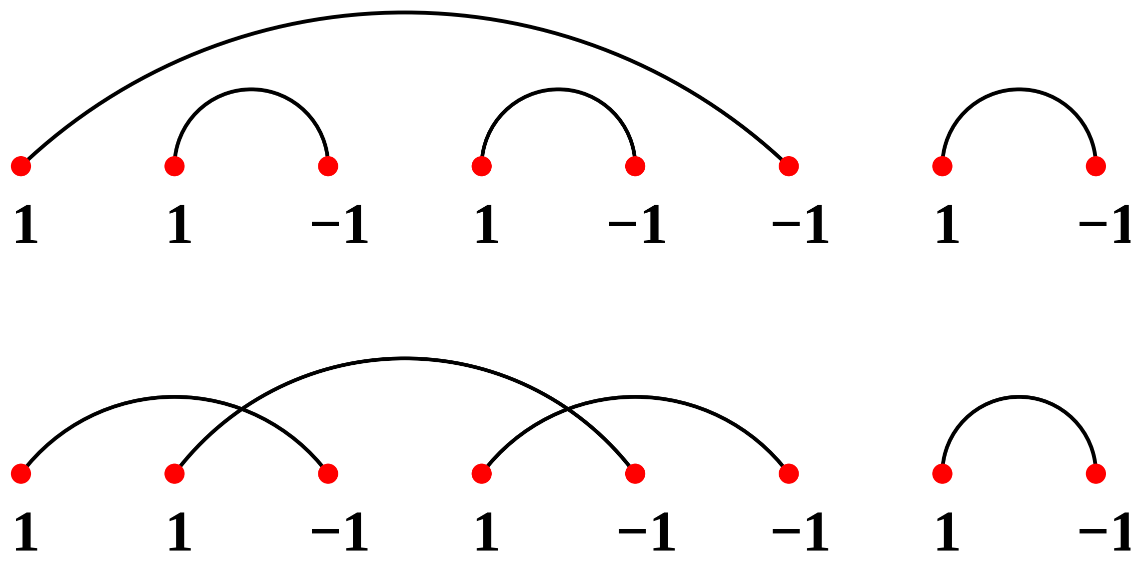
Theorem. *The number of matchings on $[2n]$ with no crossings (or with no nestings) is*

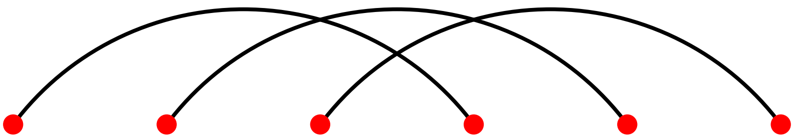
$$\boldsymbol{C_n} := \frac{1}{n+1}\binom{2n}{n}.$$

Well-known:

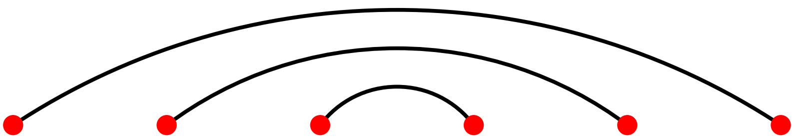
$$C_n = \#\{a_1 \cdots a_{2n} \,:\, a_i = \pm 1,$$
$$a_1 + \cdots + a_i \ge 0,\ \sum a_i = 0\}$$

(**ballot sequence**).





3−crossing



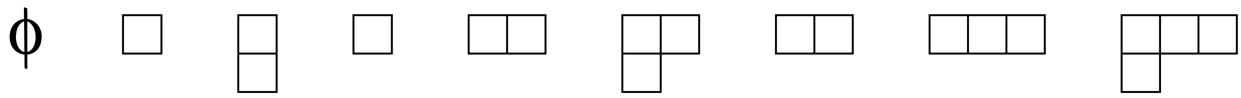
3−nesting

$$\boldsymbol{M} = \text{matching}$$
$$\mathbf{cr}(M) = \max\{k : \exists\ k\text{-crossing}\}$$
$$\mathbf{ne}(M) = \max\{k : \exists\ k\text{-nesting}\}.$$

Theorem. *Let $f_n(i,j) = \#$ matchings M on $[2n]$ with $\mathrm{cr}(M) = i$ and $\mathrm{ne}(M) = j$. Then* $\boldsymbol{f_n(i,j) = f_n(j,i)}$.

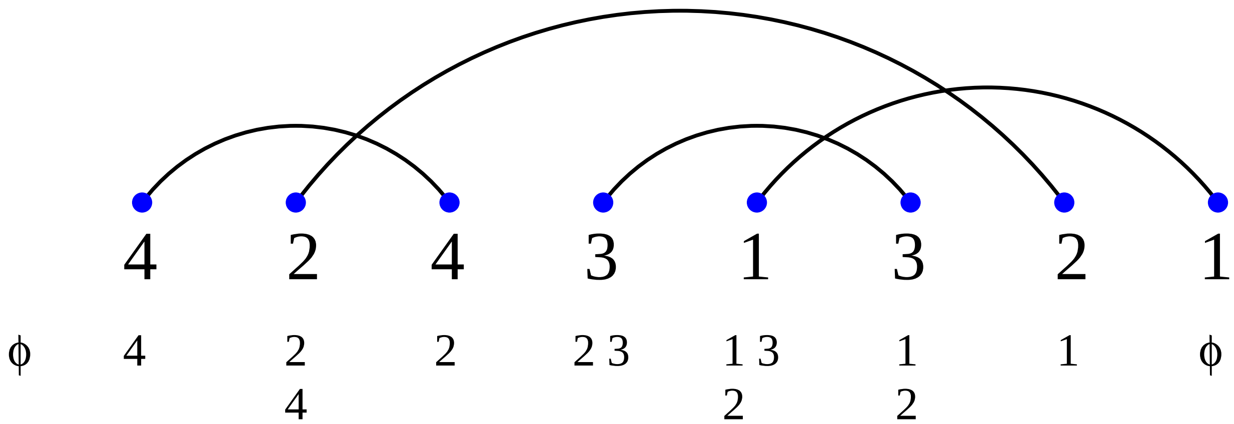
Corollary. *$\#$ matchings M on $[2n]$ with $\mathrm{cr}(M) = k$ equals $\#$ matchings M on $[2n]$ with $\mathrm{ne}(M) = k$.*

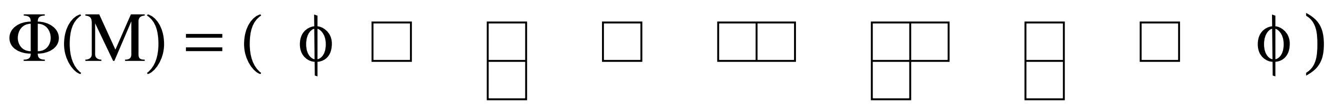
Main tool: oscillating tableaux.



shape $(3,1)$, length 8

$$\tilde{f}_n^{\lambda} := \#\{\text{osc. tab. of shape } \lambda, \text{ length } n\}$$





Φ is a bijection from matchings on $1, 2, \ldots, 2n$ to oscillating tableaux of length $2n$, shape $\emptyset$.

Corollary.

$$\sum_{\lambda} \left(\tilde{f}_n^{\lambda}\right)^2 = (2n-1)!!$$

Proof. Number of oscillating tableaux

$$(\emptyset = \lambda^0, \lambda^1, \ldots, \lambda^{2n} = \emptyset)$$

of length $2n$, shape $\emptyset$, and with $\lambda^n = \lambda$ is $\left(\tilde{f}_n^{\lambda}\right)^2$. Sum on all λ to get the total number of matchings on $[2n]$, viz., $(2n-1)!!$. □

Brauer algebra $\mathcal{B}_n$: a complex semisimple algebra (depending on a parameter x) of dimension $(2n-1)!!$.

Dimensions of irreducible representations of $\mathcal{B}_n$: $\tilde{f}_n^\lambda$, confirming

$$\sum_\lambda \left(\tilde{f}_n^\lambda\right)^2 = (2n-1)!!$$

Schensted's theorem for matchings. *Let*

$$\Phi(M) = (\emptyset = \lambda^0, \lambda^1, \ldots, \lambda^{2n} = \emptyset).$$

then

$$\begin{aligned} \mathrm{cr}(M) &= \max\{(\lambda^i)'_1 : 0 \le i \le n\} \\ \mathrm{ne}(M) &= \max\{\lambda^i_1 : 0 \le i \le n\}. \end{aligned}$$

Proof. Reduce to ordinary RSK.

Enumeration of k-noncrossing matchings (or nestings).

Recall: The number of matchings M on $[2n]$ with no crossings, i.e., $\mathrm{cr}(M) = 1$, (or with no nestings) is $\boldsymbol{C_n} = \frac{1}{n+1}\binom{2n}{n}$.

What about the number with $\mathrm{cr}(M) \leq k$?

Assume $\mathrm{cr}(M) \leq k$. Let

$\Phi(M) = (\emptyset = \lambda^0, \lambda^1, \ldots, \lambda^{2n} = \emptyset)$.

Regard $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{N}^k$.

Corollary. *The number* $\boldsymbol{f_k(n)}$ *of matchings* M *on* $[2n]$ *with* $\mathrm{cr}(M) \leq k$ *is the number of lattice paths of length* $2n$ *from* $\mathbf{0}$ *to* $\mathbf{0}$ *in the region*

$$\mathcal{C}_n := \{(a_1, \ldots, a_k) \in \mathbb{N}^k \,:\, a_1 \leq \cdots \leq a_k\}$$

with steps $\pm e_i$ *(*e_i *= ith unit coordinate vector).*

$\mathcal{C}_n \otimes \mathbb{R}_{\geq 0}$ is a fundamental chamber for the Weyl group of type B_k.

Grabiner-Magyar: applied **Gessel-Zeilberger reflection principle** to solve this lattice path problem (not knowing connection with matchings).

Theorem. *Define*

$$\boldsymbol{H_k(x)} = \sum_n f_k(n) \frac{x^{2n}}{(2n)!}.$$

Then

$$H_k(x) = \det \left[I_{|i-j|}(2x) - I_{i+j}(2x) \right]_{i,j=1}^{k}$$

where

$$I_m(2x) = \sum_{j \geq 0} \frac{x^{m+2j}}{j!(m+j)!}$$

as before.

Example. $k = 1$ (noncrossing matchings):

$$H_1(x) = I_0(2x) - I_2(2x) = \sum_{j \geq 0} C_j \frac{x^{2j}}{(2j)!}.$$

Compare:

$\boldsymbol{u_k(n)} := \#\{w \in \mathfrak{S}_n \ : \ \text{longest increasing subsequence of length } \leq k\}.$

$$\sum_{n \geq 0} u_k(n) \frac{x^{2n}}{n!^2} = \det \left[I_{i-j}(2x) \right]_{i,j=1}^{k}.$$

Baik-Rains (implicitly):

$$\lim_{n\to\infty} \text{Prob}\left(\frac{\text{cr}_n - \sqrt{2n}}{(2n)^{1/6}} \le \frac{t}{2}\right) = F_1(t),$$

where

$$\boldsymbol{F_1(t)} = \sqrt{F(t)}\exp\left(\frac{1}{2}\int_t^\infty u(s)ds\right),$$

where $F(t)$ is the Tracy-Widom distribution and $u(t)$ the Painlevé II function.

$$F(t) = \exp\left(-\int_t^\infty (x-t)u(x)^2\,dx\right)$$

$$\frac{d^2}{dx^2}u(x) = 2u(x)^3 + xu(x)$$

$$\boldsymbol{g_{j,k}(n)} := \#\{\text{matchings } M \text{ on } [2n],$$
$$\text{cr}(M) \le j,\ \text{ne}(M) \le k\}$$

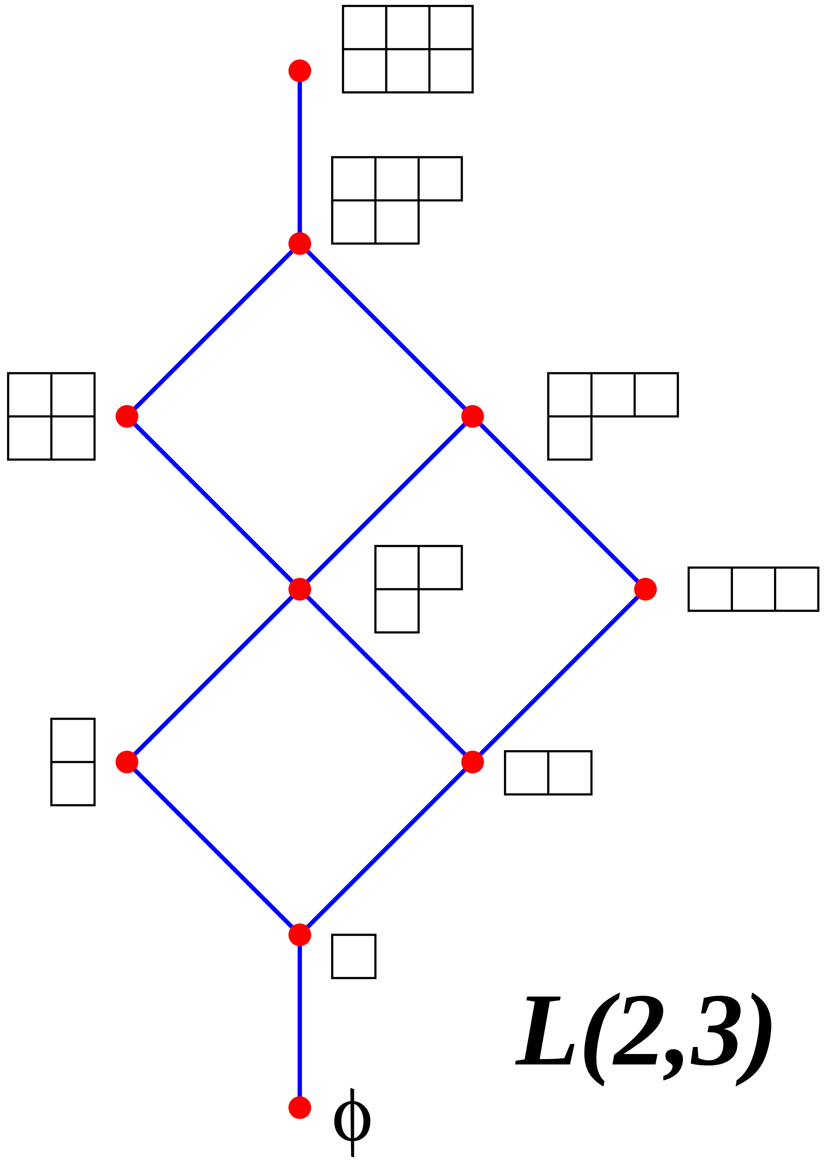
Now

$$g_{j,k}(n) = \#\{(\emptyset = \lambda^0, \lambda^1, \ldots, \lambda^{2n} = \emptyset)\ :$$
$$\lambda^{i+1} = \lambda^i \pm \square,\ \lambda^i \subseteq j \times k \text{ rectangle}\},$$

a walk on the Hasse diagram $\boldsymbol{\mathcal{H}(j,k)}$ of

$$\boldsymbol{L(j,k)} := \{\lambda \subseteq j \times k \text{ rectangle}\},$$

ordered by inclusion.



ϕ
L(2,3)

$$\begin{aligned} \boldsymbol{A} &= \text{adjacency matrix of } \mathcal{H}(j,k) \\ \boldsymbol{A_0} &= \text{adjacency matrix of } \mathcal{H}(j,k) - \{\emptyset\}. \end{aligned}$$

Transfer-matrix method $\Rightarrow$

$$\sum_{n\geq 0} g_{j,k}(n)x^{2n} = \frac{\det(I - xA_0)}{\det(I - xA)}.$$

Theorem. *Every factor of* $\det(I - xA)$ *over* $\mathbb{Q}$ *has degree dividing*

$$\frac{1}{2}\phi(2(p+q+1)),$$

where ϕ *is the Euler phi-function.*

Proof based on determinantal formula of Grabiner for walks in affine Weyl chambers.

Example.

$j = 2$, $k = 5$, $\frac{1}{2}\phi(16) = 4$:

$$\det(I - xA) = (1 - 2x^2)(1 - 4x^2 + 2x^4)$$
$$(1-8x^2+8x^4)(1-8x^2+8x^3-2x^4)$$
$$(1 - 8x^2 - 8x^3 - 2x^4)$$

$j = k = 3$, $\frac{1}{2}\phi(14) = 3$:

$$\det(I-xA) = (1-x)(1+x)(1+x-9x^2-x^3)$$
$$(1-x-9x^2+x^3)(1-x-2x^2+x^3)^2$$
$$(1 + x - 2x^2 - x^3)^2$$

Alternating Subsequences

A sequence $b_1 b_2 \cdots b_k$ is **alternating** if

$$b_1 > b_2 < b_3 > b_4 < \cdots b_k.$$

$\boldsymbol{E_n}$: number of alternating $w \in \mathfrak{S}_n$ (**Euler number**)

$E_4 = 5$: 2134, 3142, 3241, 4132, 4231

Desirée André (1879):

$$\sum_{n \geq 0} E_n \frac{x^n}{n!} = \sec x + \tan x.$$

$\mathbf{as}_{\boldsymbol{n}}(w)$: length of longest alternating subsequence of $w \in \mathfrak{S}_n$

$$\mathrm{as}_9(3\mathbf{86}41\mathbf{9257}) = 5$$

$$\boldsymbol{b_k(n)} = \#\{w \in \mathfrak{S}_n \ : \ \mathrm{as}_n(w) \leq k\}.$$

$$\begin{aligned} b_1(n) &= 1 \quad (w = 12 \cdots n) \\ b_n(n) &= n! \\ b_n(n) - b_{n-1}(n) &= E_n \end{aligned}$$

Define

$$\boldsymbol{B(x,t)} = \sum_{k,n\geq 0} b_k(n) t^k \frac{x^n}{n!},$$

and set $\boldsymbol{\rho} = \sqrt{1-t^2}$.

Theorem. *We have*

$$B(x,t) = \frac{1+\rho+2te^{\rho x}+(1-\rho)e^{2\rho x}}{1+\rho-t^2+(1-\rho-t^2)e^{2\rho x}}.$$

Corollary (with I. Gessel).

$$b_k(n) = \frac{1}{2^{k-1}} \sum_{\substack{i+2j\leq k \\ i\equiv k \,(\mathrm{mod}\, 2)}} (-2)^j \binom{k-j}{(k+i)/2} \binom{n}{j} i^n.$$

E.g.,

$$\begin{aligned} b_2(n) &= 2^{n-1} \\ b_3(n) &= \frac{1}{4}(3^n - 2n + 3) \end{aligned}$$

$$b_4(n) = \frac{1}{8}(4^n - 2(n-2)2^n).$$

Corollary.

$$\frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \mathrm{as}_n(w) = \frac{4n+1}{6}, \ \ n \geq 2.$$

$$\mathrm{Var}(\mathrm{as}_n) = \frac{32n-3}{180}, \ \ n \geq 4.$$

Open (so far): limiting distribution of as_n

Key lemma: *Some longest alternating subsequence of $w \in \mathfrak{S}_n$ contains n.*

Leads to recurrence for

$$\begin{aligned} \boldsymbol{c_k(n)} &= b_k(n) - b_{k-1}(n) \\ &= \#\{w \in \mathfrak{S}_n \,:\, \mathrm{as}_n(w) = k\}, \end{aligned}$$

namely,

$$c_k(n) = \sum_{j=1}^{n} \binom{n-1}{j-1} \sum_{2r+s=k-1} (c_{2r}(j-1)+c_{2r+1}(j-1))c_s(n-j).$$