Wishful Thinking as a Proof Technique

First example

P: finite p-element poset

$\omega : P \to \{1, 2, \ldots, p\}$: any bijection (**labeling**)

(P, ω)-partition: a map $\sigma : P \to \mathbb{N}$ such that

$$\begin{aligned} s \leq t &\Rightarrow \sigma(s) \geq \sigma(t) \\ s < t,\ \omega(s) > \omega(t) &\Rightarrow \sigma(s) > \sigma(t). \end{aligned}$$

$\mathcal{A}_{P,\omega}$: set of all (P, ω)-partitions σ

An equivalence relation

Define labelings ω, ω' to be **equivalent** if $\mathcal{A}_{P,\omega} = \mathcal{A}_{P,\omega'}$.

How many equivalence classes?

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How many equivalence classes?

Easy result: the number of equivalence classes is the number **ao**(H_P) of **acyclic orientations** of the Hasse diagram $\boldsymbol{H_P}$ of P.

Number of acyclic orientations

For **any** (finite) graph G, we can ask for the number $\mathrm{ao}(G)$ of acyclic orientations.

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No obvious formula.

Deletion-contraction

A function f from graphs to an abelian group (such as $\mathbb{Z}$) is a **deletion-contraction invariant** or **Tutte-Grothendieck invariant** if for any edge e, not a loop or isthmus,

$$f(G) = f(G - e) \pm f(G/e).$$

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Conclusion

Deletion-contraction invariants (of matroids) extensively studied by **Brylawski**. Routine to show that if G has p vertices, then

$$\mathrm{ao}(G) = (-1)^p \chi_G(-1),$$

where χ_G is the **chromatic polynomial** of G.

Second example

$\mathfrak{S}_n$: symmetric group on $\{1, 2, \ldots, n\}$

s_i: the adjacent transposition $(i, i+1)$, $1 \leq i \leq n-1$

$\ell(w)$: length (number of inversions) of $w \in \mathfrak{S}_n$, and the least p such that $w = s_{i_1} \cdots s_{i_p}$

reduced decomposition of w: a sequence $(c_1, c_2, \ldots, c_p) \in [n-1]^p$, where $p = \ell(w)$, such that

$$w = s_{c_1} s_{c_2} \cdots s_{c_p}.$$

More definitions

$\boldsymbol{R(w)}$: set of reduced decompositions of w

$\boldsymbol{r(w)} = \#R(w)$

$\boldsymbol{w_0}$: the longest element $n, n-1, \ldots, 1$ in $\mathfrak{S}_n$, of length $\binom{n}{2}$

Example. $w_0 = 321 \in \mathfrak{S}_3$:

$R(w_0) = \{(1,2,1),(2,1,2)\}$, $r(w_0) = 2$.

A conjecture

$\boldsymbol{f(n)} := r(w_0)$ for $w_0 \in \mathfrak{S}_n$

P. Edelman (∼1983) computed

$$f(3) = 2,\ f(4) = 2^4,\ f(5) = 2^8 \cdot 3.$$

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Conjecture. $f(n) = f^{\delta_n}$, the number of standard Young tableaux (SYT) of the **staircase shape** $\delta_n = (n-1, n-2, \ldots, 1)$.

An explicit formula

Hook length formula $\Rightarrow$

$$f(n) = \frac{\binom{n}{2}!}{1^{n-1}3^{n-2}5^{n-3}\cdots(2n-3)^1}$$

An analogy

Maximal chains in distributive lattices $J(P)$ correspond to linear extensions of P.

Maximal chains in the weak order $W(\mathfrak{S}_n)$ correspond to reduced decompositions of w_0.

A quasisymmetric function

$\mathcal{L}(P)$: set of linear extensions $v = a_1 a_2 \cdots a_p$ of P (regarded as a permutations of the elements $1, 2, \ldots, p$ of P)

Useful to consider

$$F_P = \sum_{v=a_1\cdots a_n \in \mathcal{L}(P)} \sum_{\substack{1 \leq i_1 \leq \cdots \leq i_p \\ i_j < i_{j+1} \text{ if } a_j > a_{j+1}}} x_{i_1} \cdots x_{i_p}.$$

An analogy

Analogously, define for $w \in \mathfrak{S}_n$

$$F_w = \sum_{(c_1,\dots,c_p)\in R(w)} \sum_{\substack{1\le i_1\le\cdots\le i_p \\ i_j<i_{j+1} \text{ if } c_j>c_{j+1}}} x_{i_1}\cdots x_{i_p}.$$

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Compare

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More wishful thinking

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The nicest property

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By considering the coefficient of $x_1 x_2 \cdots x_p$ ($p = \ell(w)$):

Proposition. *If $F_w = \sum_{\lambda \vdash p} c_{w,\lambda} s_\lambda$, then*

$$r(w) = \sum_{\lambda \vdash p} c_{w,\lambda} f^\lambda.$$

Consequences

By a simple argument involving highest and lowest terms in F_w:

Theorem. *There exists a partition* $\lambda \vdash \ell(w)$ *such that* $F_w = s_\lambda$ *if and only if* w *is 2143-avoiding* (**vexillary**).

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Much further work by **Edelman, Greene**, et al. For instance, $c_{w,\lambda} \geq 0$.

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Finally proved by **Noam Elkies**.

Compatible pairs

Elkies' proof is related to the following question:

Let $1 \leq i < j < k \leq n$ and $1 \leq a < b < c \leq n$.

$\{i, j, k\}$ and $\{a, b, c\}$ are **compatible** if there exist integers $x_1 < x_2 < \cdots < x_n$ such that x_i, x_j, x_k is an arithmetic progression and x_a, x_b, x_c is an arithmetic progression.

An example

Example. $\{1, 2, 3\}$ and $\{1, 2, 4\}$ are ***not*** compatible. Similarly 124 and 134 are ***not*** compatible.

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123 and 134 ***are*** compatible, e.g.,

$$(x_1, x_2, x_3, x_4) = (1, 2, 3, 5).$$

Elkies' question

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Example. When $n = 4$ there are eight such subsets $\mathcal{S}$:

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Let $\boldsymbol{M_n}$ be the collection of all such $\mathcal{S} \subseteq \binom{[n]}{3}$, so for instance $\#M_4 = 8$.

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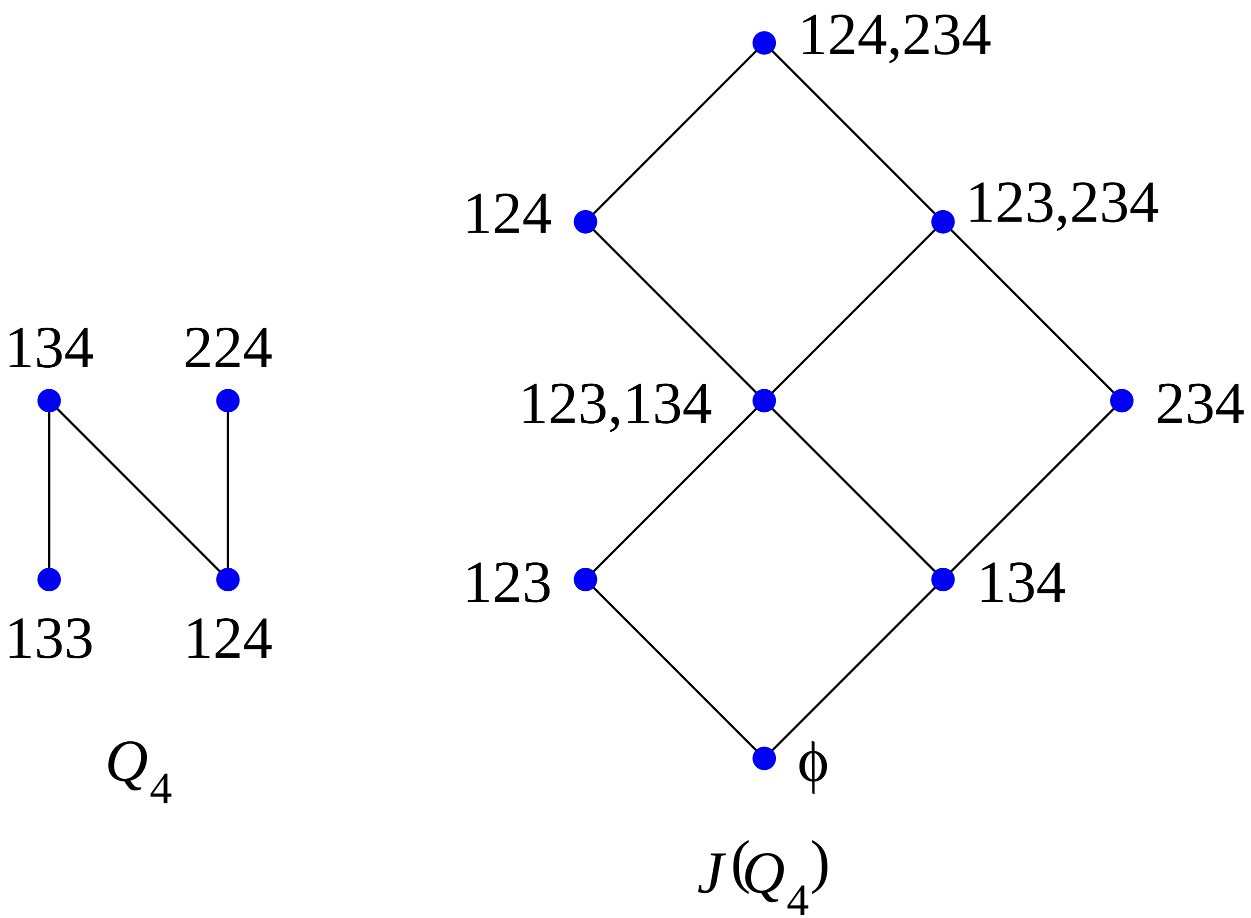
A poset on M_n

Let Q_n be the subposet of $[n] \times [n] \times [n]$ (ordered componentwise) defined by

$$Q_n = \{(i, j, k) \,:\, i + j < n + 1 < j + k\}.$$

Proppposition (**J. Propp**, essentially) *There is a simple bijection from the lattice $J(Q_n)$ of order ideals of Q_n to M_n.*

The case $n = 4$



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Let L_n be a known "reasonable" distributive lattice with $2^{\binom{n-1}{2}}$ elements. Is it true that $J(Q_n) \cong L_n$?

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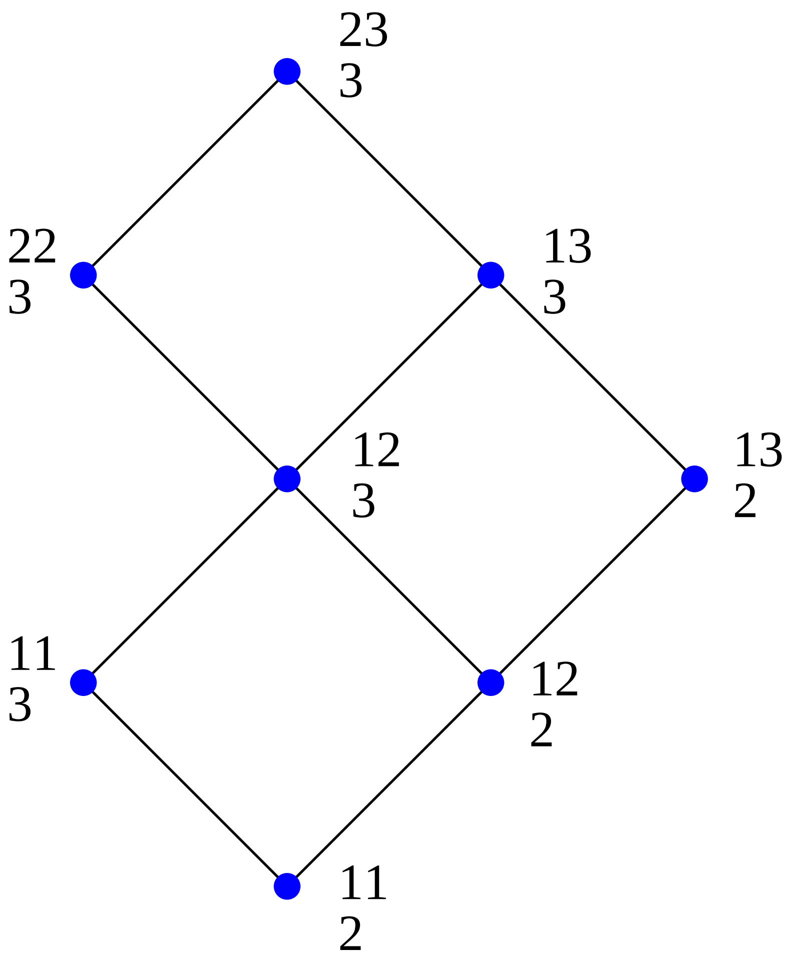
Only **one** possibility for L_n: the lattice of all semistandard Young tableaux of shape $\boldsymbol{\delta_{n-1}} = (n-2, n-1, \dots, 1)$ and largest part at most $n-1$, ordered component-wise.

L_4



L_4

$\#L_n$

$$\#L_n = s_{\delta_{n-2}}(\underbrace{1,\dots,1}_{n-1}) = 2^{\binom{n-1}{2}},$$

by hook-content formula or

$$s_{\delta_{n-2}}(x_1,\dots,x_{n-1}) = \prod_{1\le i<j\le n-1} (x_i + x_j).$$

Proof.

To show $J(Q_n) \cong L_n$, check that their posets of join-irreducibles are isomorphic.

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D: set of all proved theorems.

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Then **Q**$\in$**D**.

The last slide

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The last slide

Thanks!

Karen Collins
Patricia Hersh
Caroline Klivans
Alexander Postnikov
Avisha Lalla
Alejandro Morales
Sergi Elizalde
Clara Chan
Satomi Okazaki
Shan-Yuan Ho