Catalan Numbers

Richard P. Stanley

March 13, 2024

An OEIS entry

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A000108: $1, 1, 2, 5, 14, 42, 132, 429, \ldots$

$C_0 = 1,\ C_1 = 1,\ C_2 = 2,\ C_3 = 5,\ C_4 = 14, \ldots$

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Comments. ... This is probably the longest entry in OEIS, and rightly so.

Catalan monograph

R. Stanley, *Catalan Numbers*, Cambridge University Press, 2015.

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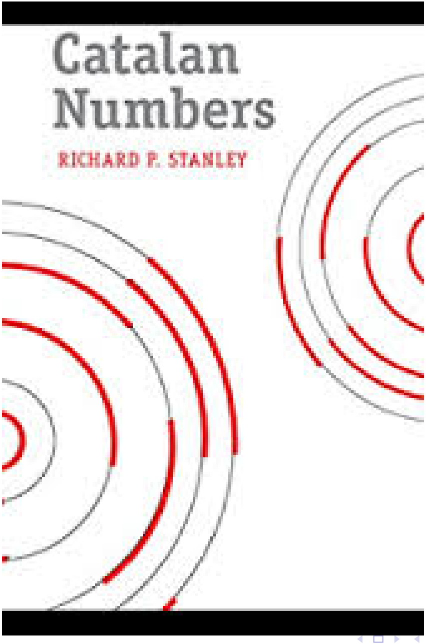
Includes 214 combinatorial interpretations of C_n and 68 additional problems.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History

Sharabiin Myangat, also known as **Minggatu**, **Ming'antu** (明安图), and **Jing An** (c. 1692–c. 1763): a Mongolian astronomer, mathematician, and topographic scientist who worked at the Qing court in China.

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No combinatorics, no further work in China.

Ming'antu

Manuscript of Ming'antu


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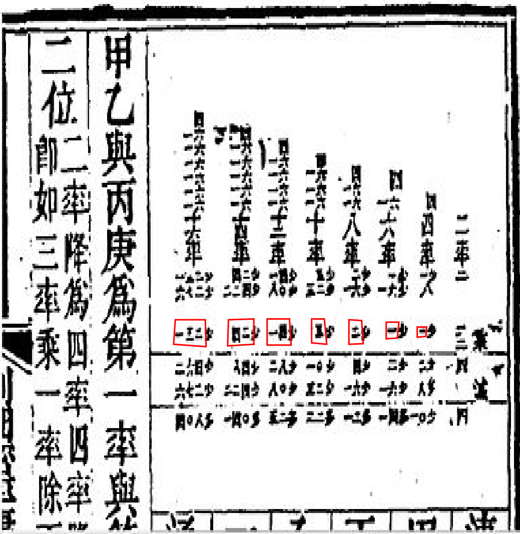
Manuscript of Ming'antu



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More history, via Igor Pak

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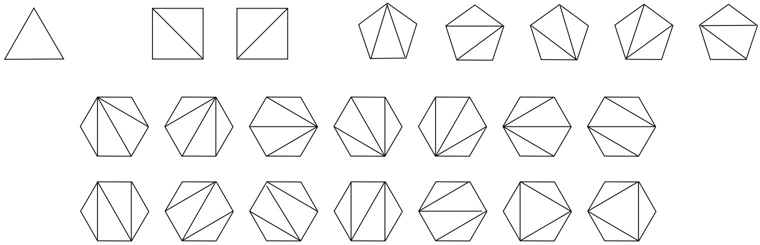

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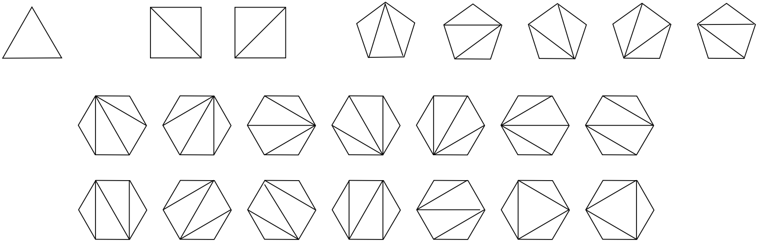

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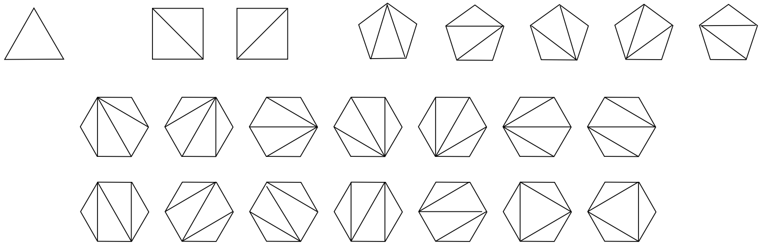

$$1, \quad 2, \quad 5, \quad 14, \quad \ldots$$

We **define** these numbers to be the Catalan numbers C_n.

Completion of proof

- **Goldbach and Segner** (1758–1759): helped Euler complete the proof, in pieces.
- **Lamé** (1838): first self-contained, complete proof.

Catalan

- **Eugène Charles Catalan** (1838): wrote C_n in the form $\frac{(2n)!}{n!\,(n+1)!}$ and showed it counted (nonassociative) **bracketings** (or **parenthesizations**) of a string of $n+1$ letters.

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Born in 1814 in Bruges (now in Belgium, then under Dutch rule). Studied in France and worked in France and Liège, Belgium. Died in Liège in 1894.

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- **Martin Gardner** (1976): used the term in his Mathematical Games column in *Scientific American*. Real popularity began.

The primary recurrence

$$C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}, \quad C_0 = 1$$

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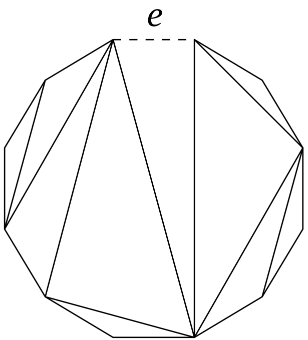

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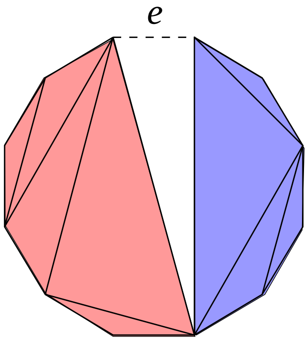

12-gon, $n = 9, k = 5, n - k = 4$

Solving the recurrence

$$C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}, \quad C_0 = 1$$

Let $\boldsymbol{y} = \sum_{n \geq 0} C_n x^n$ (**generating function**).

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Then

$$\begin{aligned} y^2 &= \sum_{n\geq 0}\left(\sum_{k=0}^{n} C_k C_{n-k}\right) x^n \\ &= \sum_{n\geq 0} C_{n+1} x^n \\ &= \frac{y-1}{x}. \end{aligned}$$

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$$\Rightarrow xy^2 - y + 1 = 0$$

Solve this quadratic equation for y!

Solving the quadratic equation

$$xy^2 - y + 1 = 0 \Rightarrow y = \frac{1 - \sqrt{1-4x}}{2x}$$

$$\begin{aligned} \Rightarrow y &= -\frac{1}{2}\sum_{n\geq 1}(-4)^n \binom{1/2}{n} x^{n-1} \\ &= -\frac{1}{2}\sum_{n\geq 1}(-4)^n \frac{\frac{1}{2}(-\frac{1}{2})\cdots(-\frac{2n-3}{2})}{n!} x^{n-1}, \end{aligned}$$

since $\binom{a}{n} = \frac{a\cdot(a-1)\cdot(a-n+1)}{n!}$.

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Simplifying gives

$$C_n = \frac{1}{\boldsymbol{n}+1}\binom{2\boldsymbol{n}}{\boldsymbol{n}} = \frac{(2n)!}{n!\,(n+1)!}$$

Other combinatorial interpretations

$$\begin{aligned}\mathcal{P}_n &:= \{\text{triangulations of convex } (n+2)\text{-gon}\} \\ \Rightarrow \#\mathcal{P}_n &= C_n \quad (\text{where } \#\mathcal{S} = \text{number of elements of } \mathcal{S})\end{aligned}$$

We want other combinatorial interpretations of C_n, i.e., other sets $\mathcal{S}_n$ for which $C_n = \#\mathcal{S}_n$.

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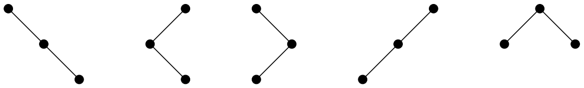
One method: If $D_n = \#\mathcal{S}_n$, then show that

$$D_0 = 1, \quad D_{n+1} = \sum_{k=0}^{n} D_k D_{n-k} \text{ for } n \geq 1.$$

"Transparent" interpretations

4. Binary trees with n vertices (each vertex has a left subtree and a right subtree, which may be empty)



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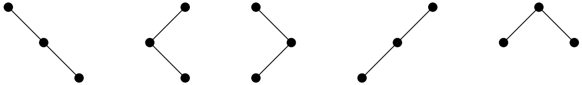

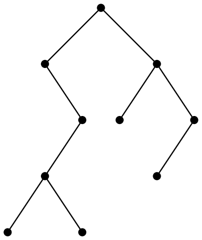

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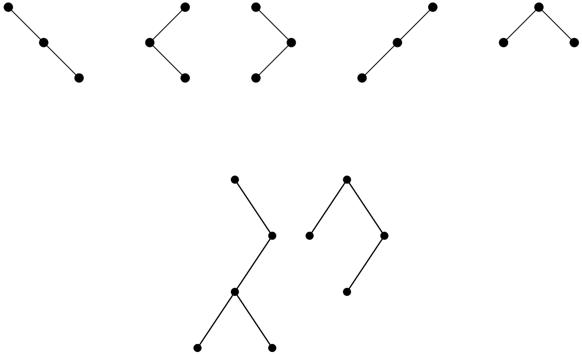

Binary parenthesizations

3. Binary **parenthesizations** or **bracketings** of a string of $n+1$ letters (without assuming the **associative law** $xx \cdot x = x \cdot xx$)

$$(xx \cdot x)x \qquad x(xx \cdot x) \qquad (x \cdot xx)x \qquad x(x \cdot xx) \qquad xx \cdot xx$$

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The ballot problem

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Special case: there are two candidates A and B in an election. Each receives n votes. What is the probability that A will never trail B during the count of votes?

Example. $AABABBBAAB$ is bad, since after seven votes, A receives 3 votes while B receives 4.

Definition of ballot sequence

Encode a vote for A by 1, and a vote for B by -1 (abbreviated $-$). Clearly a sequence $a_1 a_2 \cdots a_{2n}$ of n each of 1 and -1's is allowed if and only if $\sum_{i=1}^{k} a_i \geq 0$ for all $1 \leq k \leq 2n$. Such a sequence is called a **ballot sequence**.

Ballot sequences

77. Ballot sequences, i.e., sequences of n 1's and n -1's such that every partial sum is nonnegative (with -1 denoted simply as $-$ below)

$$111 - - - \qquad 11 - 1 - - \qquad 11 - -1 - \qquad 1 - 11 - - \qquad 1 - 1 - 1-$$

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$$111--- \quad 11-1-- \quad 11--1- \quad 1-11-- \quad 1-1-1-$$

Note. Answer to original problem (probability that a sequence of n each of 1's and -1's is a ballot sequence) is therefore

$$\frac{C_n}{\binom{2n}{n}} = \frac{\frac{1}{n+1}\binom{2n}{n}}{\binom{2n}{n}} = \frac{1}{n+1}.$$

The ballot recurrence

$$1\,1 - 1\,1 - 1 - - - 1 - 1\,1 - 1 - -$$

Consider the first partial sum equal to 0.

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$$\color{blue}{1}\,1 - 1\,1 - 1 - - \color{blue}{-} \color{red}{|} 1 - 1\,1 - 1 - -$$

Remove the first element (which equals 1) of the ballot sequence, and the last element (which equals -1) of this partial sum.

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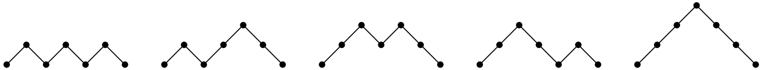
$$1 - 1\,1 - 1 - - \quad \color{red}{|} 1 - 1\,1 - 1 - -$$

Dyck paths

25. **Dyck paths** of length $2n$, i.e., lattice paths from $(0,0)$ to $(2n,0)$ with steps $(1,1)$ and $(1,-1)$, never falling below the x-axis



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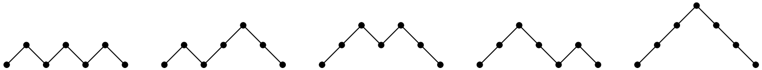

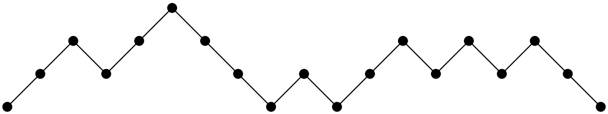

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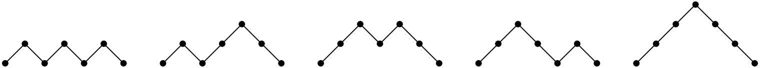

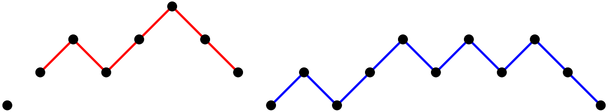

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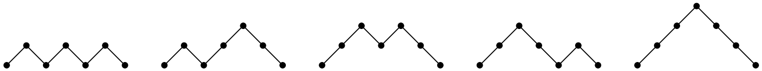
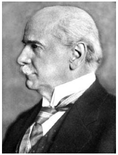


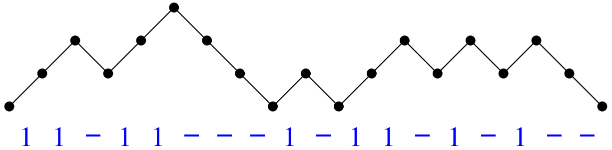
Walther von Dyck (1856–1934) 

Bijective proofs

Suppose we know that $\#\mathcal{S}_n = C_n$ and want to show that $\#\mathcal{T}_n = C_n$.

bijective proof: construct a **bijection** (one-to-one correspondence) between $\mathcal{S}_n$ and $\mathcal{T}_n$.

Bijection between Dyck paths and ballot sequences



For each upstep, record 1.

For each downstep, record -1.

321-avoiding permutations

115. Permutations $a_1a_2\cdots a_n$ of $1,2,\ldots,n$ with longest decreasing subsequence of length at most two (i.e., there does not exist $i<j<k$, $a_i>a_j>a_k$), called **321-avoiding** permutations

$$123 \quad 213 \quad 132 \quad 312 \quad 231$$

321-avoiding permutations

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more subtle: no obvious decomposition into two pieces

Bijection with ballot sequences

$$w = 412573968$$

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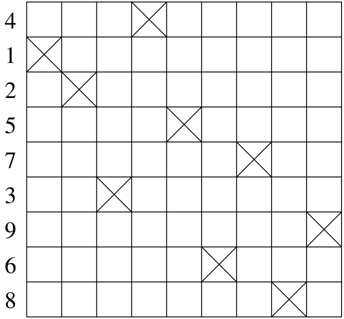

Part of the subject of **pattern avoidance**.

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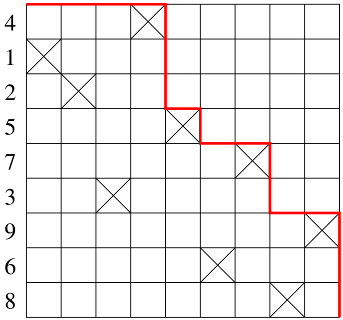

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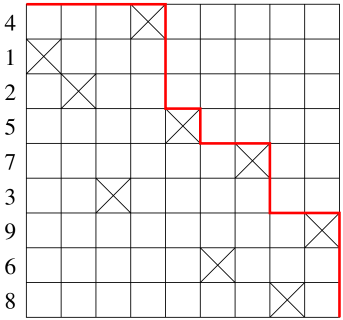

$1111 - - - 1 - 11 - - 11 - - -$

Part of the subject of **pattern avoidance**.

An unexpected interpretation

92. n-tuples $(a_1, a_2, \ldots, a_n)$ of integers $a_i \geq 2$ such that in the sequence $1a_1a_2\cdots a_n1$, each a_i divides the sum of its two neighbors

14321 13521 13231 12531 12341

Bijection with ballot sequences

remove largest; insert bar before the element to its left; continue until only 1's remain; then replace bar with 1 and an original number with -1, except last two

$$1 \quad 2 \quad 5 \quad 3 \quad 4 \quad 1$$

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$$1 \; | 2 \;\; {\color{red}5} \;\; 3 \;\; 4 \;\; 1$$

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|	1	|	|	2	5	|	3	4	1
1	−	1	1	−	−	1	−		

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|	1	|	|	2	5	|	3	4	1
1	−	1	1	−	−	1	−		

tricky to prove

Analysis

A65.(b)

$$\sum_{n\geq 0} \frac{1}{C_n} = ??$$

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$$1 + 1 + \frac{1}{2} + \frac{1}{5} = 2.7$$

Analysis

A65.(b)

$$\sum_{n\geq 0} \frac{1}{C_n} = 2 + \frac{4\sqrt{3}\pi}{27}$$

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$$2 + \frac{4\sqrt{3}\pi}{27} = 2.806133\cdots$$

Why?

A65.(a)

$$\sum_{n\geq 0} \frac{x^n}{C_n} = \frac{2(x+8)}{(4-x)^2} + \frac{24\sqrt{x}\sin^{-1}\left(\frac{1}{2}\sqrt{x}\right)}{(4-x)^{5/2}}.$$

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Based on a (difficult) calculus exercise: let

$$y=2\left(\sin^{-1}\frac{1}{2}\sqrt{x}\right)^2.$$

Then $y=\sum_{n\geq 1}\frac{x^n}{n^2\binom{2n}{n}}$.

Completion of proof

Recall $y = \sum_{n\geq 1} \frac{x^n}{n^2\binom{2n}{n}}$. Note that:

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$$= -1 + \sum_{n\geq 0}\frac{x^n}{C_n},$$

etc.

The final slide

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Encore: odd Catalan numbers

$$C_0 = 1,\ C_1 = 1,\ C_3 = 5,\ C_7 = 429,\ C_{15} = 9694845$$

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There will always be a carry at the first digit unless $n = (111\dots1)_2$ (binary expansion with k 1's for some k). This equals $2^k - 1$. Conversely, there are no carries when $n = 2^k - 1$.