# VOLUMES OF CONVEX POLYTOPES

Richard P. Stanley
Department of Mathematics
M.I.T. 2-375
Cambridge, MA 02139
rstan@math.mit.edu
http://www-math.mit.edu/~rstan

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$\mathcal{P}$ = convex polytope in $\mathbb{R}^n$

**integer polytope**: vertices $\in \mathbb{Z}^n$

$\mathbf{V}(\mathcal{P})$ = volume of $\mathcal{P}$

If $\mathcal{P}$ is an integer polytope, let

$$\widetilde{\mathbf{V}}(\mathcal{P}) = n!\, V(\mathcal{P}) \in \mathbb{Z},$$

the **normalized volume** of $\mathcal{P}$.

## Why compute $V(\mathcal{P})$?

- Let

$$\mathcal{P} = \text{integer polytope}$$
$$\Sigma = \text{the normal fan of } \mathcal{P}$$
$$X_\Sigma = \text{toric variety corresponding to } \Sigma$$
$$\Rightarrow \deg(X_\Sigma) = \widetilde{V}(\mathcal{P})$$

- (Gelfand, Kapranov, Zelevinsky) The number of linearly independent solutions to a generic hypergeometric system with Newton polytope $\mathcal{P}$ is $\widetilde{V}(\mathcal{P})$.

# Two Refinements of Volume

Let $\mathcal{P}$ be an integer polytope and let $r \geq 1$. Define

$$\mathbf{r}\mathcal{P} = \{rv \,:\, v \in \mathcal{P}\}$$

$$\mathbf{i}(\mathcal{P}, \mathbf{r}) = \#\left(r\mathcal{P} \cap \mathbb{Z}^n\right),$$

the **Ehrhart polynomial** of $\mathcal{P}$.

- $i(\mathcal{P}, r)$ is a polynomial in $r$
- $i(\mathcal{P}, 0) = 1$
- If $r > 0$, then $i(\mathcal{P}, -r) = (-1)^{\dim \mathcal{P}} \#(\mathrm{int}(r\mathcal{P}) \cap \mathbb{Z}^n)$
- $i(\mathcal{P}, r) = V(\mathcal{P}) r^n + O(r^{n-1})$.

- Let $\dim \mathcal{P} = n$ and

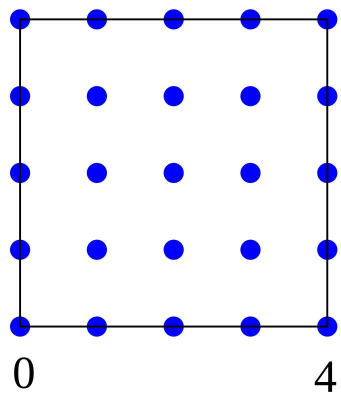
$$\sum_{r \geq 0} i(\mathcal{P}, r) x^r = \frac{h_0 + h_1 x + \cdots + h_n x^n}{(1-x)^{n+1}}.$$

Then $h_j \in \mathbb{Z}$, $h_j \geq 0$, and

$$\sum_j h_j = \widetilde{V}(\mathcal{P}).$$

**Example.** $\mathcal{P}$ = unit square:

$$0 \le x \le 1, \quad 0 \le y \le 1$$




$$\begin{aligned} \widetilde{V}(\mathcal{P}) &= 2! \cdot 1 = 2 \\ i(\mathcal{P}, r) &= (r+1)^2 \\ i(\mathcal{P}, -r) &= (r-1)^2 \\ \sum_{r \ge 0} i(\mathcal{P}, r) x^r &= \frac{1+x}{(1-x)^3} \end{aligned}$$

Let $\mathcal{P}_1, \ldots, \mathcal{P}_k$ be convex polytopes (or any convex bodies) in $\mathbb{R}^n$. Let

$$t_1, \ldots, t_k \in \mathbb{R}_{\geq 0}.$$

Define the **Minkowski sum**

$$t_1\mathcal{P}_1 + \cdots + t_k\mathcal{P}_k = \{t_1 v_1 + \cdots + t_k v_k \,:\, v_i \in \mathcal{P}_i\}.$$

**Theorem** (Minkowski) *There exist* $V(\mathcal{P}_1^{a_1}, \ldots, \mathcal{P}_k^{a_k}) \geq 0$ (**mixed volumes**) *such that*

$$V(t_1\mathcal{P}_1 + \cdots + t_k\mathcal{P}_k) = \sum_{a_1+\cdots+a_k=n} \binom{n}{a_1, \ldots, a_k} \cdot V(\mathcal{P}_1^{a_1}, \ldots, \mathcal{P}_k^{a_k}) t_1^{a_1} \cdots t_k^{a_k}.$$

Write

$$V(s\mathcal{P}+t\mathcal{Q}) = \sum_{j=0}^{n} \binom{n}{j} V_j(\mathcal{P}, \mathcal{Q}) s^{n-j} t^j,$$

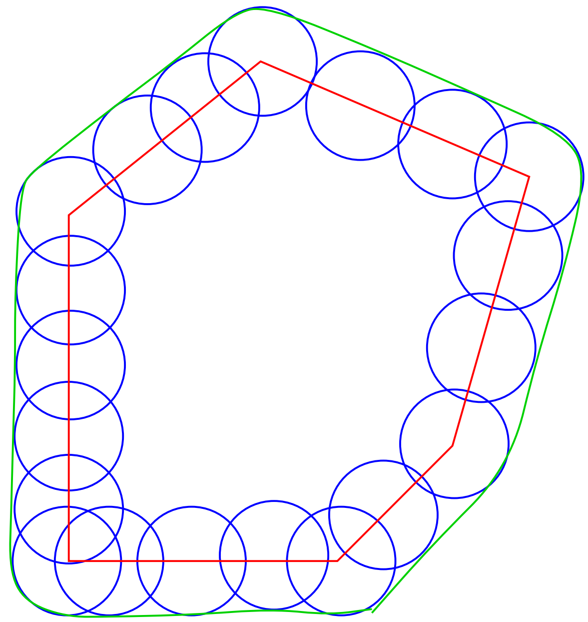
so

$$V_0(\mathcal{P}, \mathcal{Q}) = V(\mathcal{P}),\ \ V_n(\mathcal{P}, \mathcal{Q}) = V(\mathcal{Q}).$$

Minkowski proved $V_1^n \geq V_0^{n-1} V_n$ and conjectured $V_i^2 \geq V_{i-1} V_{i+1}$ (Alexandrov-Fenchel inequalities).

Let $\mathbb{B}^n$ be the unit ball in $\mathbb{R}^n$. Then

$$V_1(\mathcal{P}, \mathbb{B}^n) = \frac{1}{n}(\text{surface area of } \mathcal{P}).$$



$$V_1(\mathcal{P},\mathbb{B})^n \geq V_0(\mathcal{P},\mathbb{B})^{n-1} V_n(\mathcal{P},\mathbb{B})$$

$$\Rightarrow \left(\frac{\text{area}(\mathcal{P})}{n}\right)^n \geq V(\mathcal{P})^{n-1} V(\mathbb{B}^n)$$

$\rightarrow$ **isoperimetric inequality**.

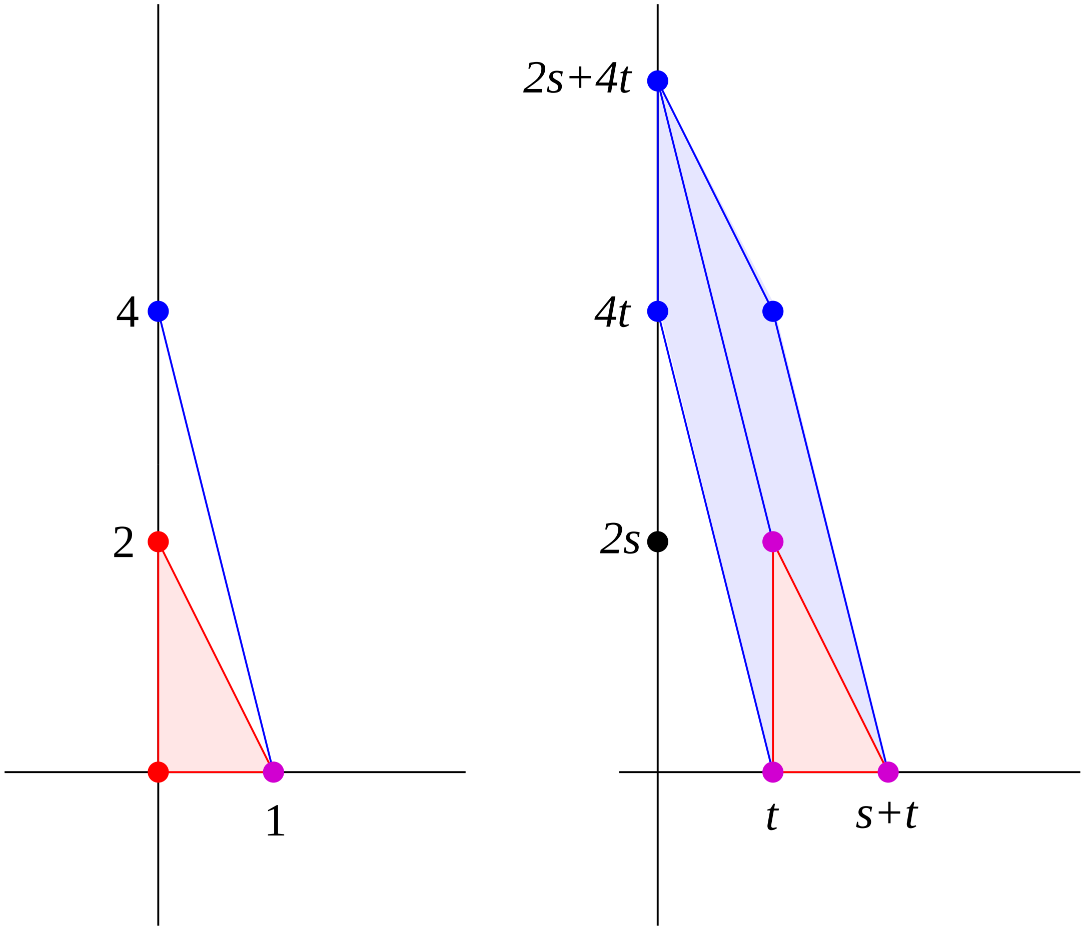
**Theorem** (Bernstein). *Let* $f_1, \ldots, f_n$ *be complex polynomials in the variables* $x_1, \ldots, x_n$. *Let* $New(f_j)$ *be the Newton polytope (convex hull of exponent vectors) of* $f_j$. *If* $f_1, \ldots, f_n$ *are otherwise generic, then the number* $Z(f_1, \ldots, f_n)$ *of solutions to*

$$f_1 = \cdots = f_n = 0$$

*with no* $x_i = 0$ *is*

$$Z(f_1, \ldots, f_n) = n!\, V(\mathrm{New}(f_1), \ldots, \mathrm{New}(f_n)).$$

**Example.** $f_1(x, y) = 1 + \alpha x + \beta y^2$, $f_2(x, y) = x + \gamma y^4$.




$$V(s\,\mathrm{New}(f_1)+t\,\mathrm{New}(f_2)) = s^2+\binom{2}{1}2st$$

$$\Rightarrow V(\mathrm{New}(f_1),\mathrm{New}(f_2)) = 2!\cdot 2 = 4$$

## A common generalization of Ehrhart polynomials and mixed volumes

(McMullen). For any convex body $\mathcal{P}$, let

$$\mathbf{N}(\mathcal{P}) = \#\left(\mathcal{P} \cap \mathbb{Z}^n\right).$$

**Theorem.** *Let* $\mathcal{P}_1, \ldots, \mathcal{P}_k$ *be integer polytopes, and let*

$$t_1, \ldots, t_k \in \mathbb{N} = \{0, 1, \ldots\}.$$

*Then*

$$N(t_1\mathcal{P}_1 + \cdots + t_k\mathcal{P}_k) \in \mathbb{Q}[t_1, \ldots, t_k]$$

(**mixed lattice point enumerator** of $\mathcal{P}_1, \ldots, \mathcal{P}_k$).

The degree $n$ part of

$$N(t_1\mathcal{P}_1 + \cdots + t_k\mathcal{P}_k)$$

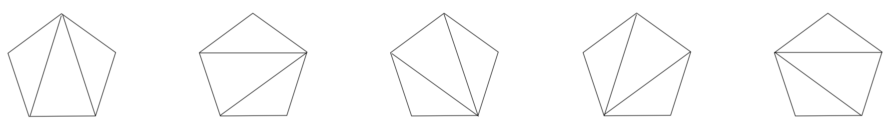
is given by

$$N(t_1\mathcal{P}_1 + \cdots + t_k\mathcal{P}_k)|_n = V(t_1\mathcal{P}_1 + \cdots + t_k\mathcal{P}_k).$$

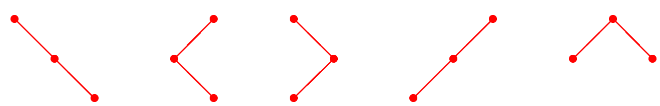
# Catalan Numbers

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

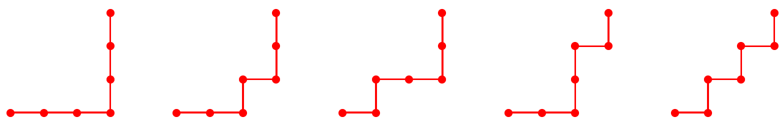
- triangulations of a convex $(n+2)$-gon into $n$ triangles by $n-1$ diagonals that do not intersect in their interiors



- binary trees with $n$ vertices



- lattice paths from $(0,0)$ to $(n,n)$ with steps $(0,1)$ or $(1,0)$, never rising above the line $y = x$



- sequences of $n$ 1's and $n$ $-1$'s such that every partial sum is nonnegative (with $-1$ denoted simply as $-$ below)

$$111---\qquad 11-1--\qquad 11--1-$$

$$1-11--\qquad 1-1-1-$$

For 62 additional combinatorial interpretations of $C_n$, see Exercise 6.19 of R. Stanley, *Enumerative Combinatorics*, volume 2, Cambridge University Press (just published).

## Flow Polytopes and Kostant's Partition Function

(with A. Postnikov)

Let $\mathcal{B}_{\mathbf{m}}$ denote the **Birkhoff polytope** of all $m \times m$ doubly stochastic matrices $\left(a_{ij}\right)$, i.e.,

$$\begin{aligned} a_{ij} &\geq 0 \\ \sum_i a_{ij} &= 1 \\ \sum_j a_{ij} &= 1. \end{aligned}$$

**Open:** $V(\mathcal{B}_m) =$ **???** (as a polytope of dimension $(m-1)^2$).

Chan and Robbins (1998) defined the **Chan-Robbins polytope** $\mathrm{CR}_m$ by

$$\mathrm{CR}_{\mathbf{m}} = \{(a_{ij}) \in \mathcal{B}_m \ : \ a_{ij} = 0 \text{ if } j > i+1\}$$

(a face of $\mathcal{B}_m$).

$$\begin{array}{cccc} * & * & 0 & 0 \\ * & * & * & 0 \\ * & * & * & * \\ * & * & * & * \end{array}$$

$$\dim \mathrm{CR}_m = \binom{m}{2}$$

Chan and Robbins conjectured that

$$\widetilde{V}(\mathrm{CR}_m) = C_1 C_2 \cdots C_{m-2}.$$

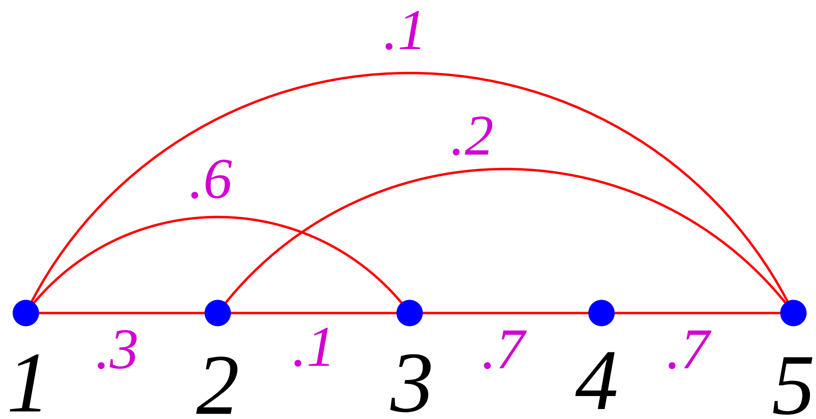
**Flow polytopes.** Let $G$ a directed graph with vertices $1, \ldots, m+1$ and edge set $E$ such that if $i \to j$ is an edge, then $i < j$. Call $G$ a **flow graph**. Define the **flow polytope** $\mathcal{F}_G$ to be the set of all $f \in \mathbb{R}^E_{\geq 0}$ satisfying

$$\sum_{(1,j) \in E} f(1,j) = 1$$

$$\sum_{(j,m+1) \in E} f(j,m+1) = 1$$

$$\sum_{i\,:\,(i,j) \in E} f(i,j) = \sum_{k\,:\,(j,k) \in E} f(j,k),$$

for $2 \leq j \leq m$.




total flow out of 1 and into $m + 1$ is 1

flow into an internal vertex = flow out

**Fact:** If

$$E = \{(i,j) \,:\, 1 \le i < j \le m+1\},$$

then $\mathcal{F}_G$ is "unimodularly equivalent" to $\mathrm{CR}_m$ (so same volume and Ehrhart polynomial). Call $G$ the **complete** flow graph on $m+1$ vertices.

**Kostant's partition function for $A_{n-1}$.** Let

$\mathbf{e_i}$ = $i$th unit coordinate vector in $\mathbb{R}^n$.

Write $e_{ij} = e_i - e_j$. Let

$$\mathbf{A^+_{n-1}} = \{e_{ij} \, : \, 1 \leq i < j \leq n\}.$$

$$v \in \mathbb{N}A_{n-1}^{+}$$
$$= \left\{ \sum_{1\le i<j\le n} a_{ij}e_{ij} \ : \ a_{ij} \in \mathbb{N} \right\}.$$

$$\mathbf{K(v)} =$$
$$\#\left\{ \left(a_{ij}\right)_{1\le i<j\le n} \ : \ v = \sum a_{ij}e_{ij} \right\}.$$

If $v = (v_1, \ldots, v_n)$ and $x^v = x_1^{v_1} \cdots x_n^{v_n}$, then

$$K(v) = \text{coef. of } x^v \text{ in } \frac{1}{\prod_{1\le i<j\le n}\left(1 - x_i/x_j\right)}.$$

**Example.** $K(2,-1,0,-1) = 4$, since

$$\begin{aligned}(2,-1,0,-1) &= 2e_{12} + e_{23} + e_{34} \\ &= 2e_{12} + e_{24} \\ &= e_{12} + e_{13} + e_{34} \\ &= e_{12} + e_{14}.\end{aligned}$$

**Gelfand:** Every subject has one "transcendental aspect." For the representation theory of semisimple Lie algebras, it is Kostant's partition function.

More generally, if $S \subseteq A_{n-1}^{+}$, then define the **restricted Kostant's partition function** by

$$\mathbf{K_S(v)} = \#\Big\{ (a_{ij})_{e_{ij}\in S} : v = \sum_{e_{ij}\in S} a_{ij} e_{ij} \Big\}.$$

Let $G$ be a flow graph on the vertex set $1, \ldots, m+1$ with edge set $E$. Let

$\mathbf{G_i}$ = restriction of $G$ to vertices $i, \ldots, m+1$.

Let $t_i \geq 0$ and

$\mathcal{F}_{\mathbf{G}}(\mathbf{t_1}, \ldots, \mathbf{t_{m-1}}) = t_1 \mathcal{F}_{G_1} + \cdots + t_{m-1} \mathcal{F}_{G_{m-1}}$.

**Note.** $\mathcal{F}_G(t_1, \ldots, t_{m-1})$ is the polytope of all $f \in \mathbb{R}^E_{\geq 0}$ satisfying

$$\sum_{k\,:\,(j,k) \in E} f(j,k) - \sum_{i\,:\,(i,j) \in E} f(i,j) = t_j,$$

for $1 \leq j \leq m-1$. In other words, there is an **excess flow** of $t_j$ out of vertex $j$.

Let $S = S_G = \{e_{ij} : (i,j) \text{ is an edge of G}\}$.

**Theorem.** *If each $t_j \in \mathbb{N}$ then*

$$N(\mathcal{F}_G(t_1, \ldots, t_{m-1}))$$

$$= K_{S_G}(t_1+\cdots+t_{m-1}, 0, -t_{m-1}, \ldots, -t_1).$$

**Note:** $K_{S_G}(t_1+\cdots+t_m, -t_m, \ldots, -t_1) = K_S(t_1+\cdots+t_{m-1}, 0, -t_{m-1}, \ldots, -t_1)$ if $e_{12} \in S_G$. (Otherwise $K_{S_G}(t_1 + \cdots + t_m, -t_m, \ldots, -t_1) = 0$ unless $t_m = 0$.)

**Corollary.** *For $t_i \in \mathbb{N}$ we have*

$$K_S(t_1 + \cdots + t_{m-1}, 0, -t_{m-1}, \ldots, -t_1)$$
$$\in \mathbb{Q}[t_1, \ldots, t_{m-1}].$$

**Example:** $G$ = complete flow graph on $\{1,\dots,m+1\}$, so $\mathcal{F}_G \cong \mathrm{CR}_m$ and

$$\mathcal{F}_{G_i} \cong \mathrm{CR}_{m-i+1}.$$

Then

$$N(\mathcal{F}_G(t_1,\dots,t_{m-1})) =$$

$$K(t_1+\cdots+t_{m-1}, 0, -t_{m-1},\dots,-t_1).$$

E.g.,

$$K(a+b,0,-b,-a) = \frac{1}{6}(a+1)(a+2)(a+3b+3)$$

$$K(a+b+c,0,-c,-b,-a) =$$

$$\frac{1}{360}(a+1)(a+2)(a+3)(a+b+3c+3)$$

$$\cdot(a^2+5ab+10b^2+9a+30b+20).$$

**Theorem.**

$$K(t_1+\cdots+t_{m-1},0,-t_{m-1},\ldots,-t_1)$$

*is divisible by* $(t_1+1)\cdots(t_1+m-1)$.

**Conjecture.**

$$K(t_1+\cdots+t_{m-1},0,-t_{m-1},\ldots,-t_1)$$

is divisible by

$$t_1+t_2+\cdots+t_{m-2}+3t_{m-1}+3.$$

More strongly,

$$3K(t_1+\cdots+t_{m-1},0,-t_{m-1},\ldots,-t_1)=$$

$$(t_1+\cdots+t_{m-2}+3t_{m-1}+3)$$

$$\cdot K_{\text{no }\mathbf{e_{23}}}(t_1+\cdots+t_{m-2},0,0,-t_{m-2},\ldots,-t_1).$$

**Problem** (not carefully looked at). Are the coefficients of the polynomial $K_S(t_1+\cdots+t_{m-1}, 0, -t_{m-1}, \ldots, -t_1)$ nonnegative?

**Recall:** Let $q = \#E(G)$, so

$$\mathcal{F}_G(t_1, \ldots, t_{m-1}) \subset \mathbb{R}^q.$$

Then

$$K_{S_G}(t_1 + \cdots + t_{m-1}, 0, -t_{m-1}, \ldots, -t_1)\Big|_q = V(t_1\mathcal{F}_{G_1} + \cdots + t_{m-1}\mathcal{F}_{G_{m-1}}).$$

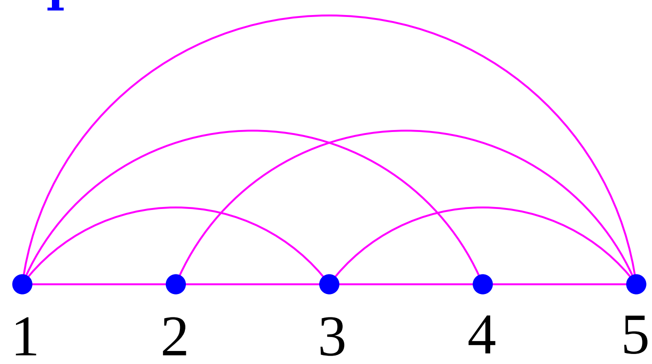
**Theorem.** *Let* $a_1 + \cdots + a_{m-1} = q$, $a_i \geq 0$. *Let*

$$\alpha_i = \operatorname{outdeg}(i) - 1.$$

*Then*

$$\widetilde{V}(\mathcal{F}_{G_1}^{a_1}, \ldots, \mathcal{F}_{G_{m-1}}^{a_{m-1}})$$
$$= K_{S_G}(a_1 - \alpha_1, \ldots, a_{m-1} - \alpha_{m-1}).$$

**Example.**




$$(\alpha_1, \alpha_2, \alpha_3) = (3, 1, 1)$$

$$S = \{e_{12}, e_{13}, e_{14}, e_{23}, e_{34}\}$$

$$V(s\mathcal{F}_{G_1} + t\mathcal{F}_{G_2} + u\mathcal{F}_{G_3}) =$$

$$\mathbf{1}\frac{s^3t^2}{3!\,2!} + \mathbf{1}\frac{s^3tu}{3!\,1!\,1!} + \mathbf{2}\frac{s^4t}{4!\,1!} + \mathbf{1}\frac{s^4u}{4!\,1!} + \mathbf{2}\frac{s^5}{5!}.$$

$$
\begin{aligned}
K(0,1,-1) &= \mathbf{1} \\
K(0,0,0) &= \mathbf{1} \\
K(1,0,-1) &= \mathbf{2} \\
K(1,-1,0) &= \mathbf{1} \\
K(2,-1,-1) &= \mathbf{2}
\end{aligned}
$$

**Corollary.**

$$\widetilde{V}(\mathcal{F}_G) = K_{S_G}(q-m+1-\alpha_1, -\alpha_2, \ldots, -\alpha_{m-2})$$

For the complete flow graph $G$ with $m+1$ vertices,

$$\widetilde{V}(\mathcal{F}_G) = \widetilde{V}(\mathrm{CR}_m) =$$

$$K(\binom{m-1}{2}, -(m-2), -(m-3), \ldots, -1).$$

E.g.,

$$\begin{aligned} K(6, -3, -2, -1) &= C_1 C_2 C_3 \\ &= 1 \cdot 2 \cdot 5 \\ &= 10. \end{aligned}$$

**Corollary.** *Let CT denote the constant term of a Laurent series. Then*

$$\widetilde{V}(\mathrm{CR}_{n+2}) =$$

$$\mathrm{CT}\prod_{i=1}^{n}(1-x_i)^{-2}\prod_{1\leq i<j\leq n}(x_j-x_i)^{-1}.$$

**Theorem** (Morris).

$$\mathrm{CT}\prod_{i=1}^{n}(1-x_i)^{-a}\prod_{i=1}^{n}x_i^{-b}\prod_{1\leq i<j\leq n}(x_j-x_i)^{-2c}$$

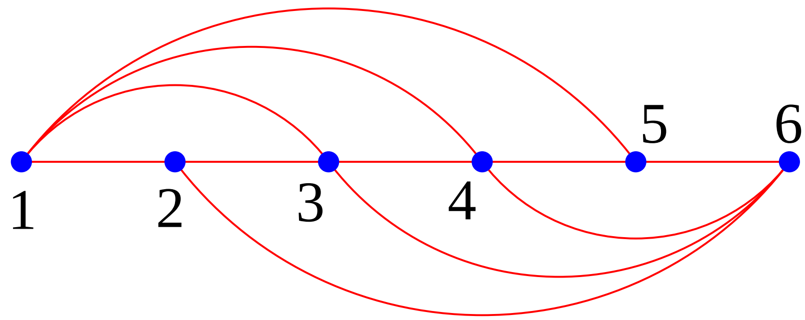
$$=\frac{1}{n!}\prod_{j=0}^{n-1}\frac{\Gamma(a+b+(n-1+j)c)\Gamma(c)}{\Gamma(a+jc)\Gamma(c+jc)\Gamma(b+jc+1)}.$$

**Corollary** (Zeilberger) $\widetilde{V}(\mathrm{CR}_n) = C_1\cdots C_{n-2}$.

**Note:** Let $0 \leq a \leq b$. $\exists$ simple product formula for

$$K\left(\binom{b+1}{2} - \binom{a}{2}, -b, -b+1, \ldots, -a\right).$$

# One Further Flow Graph

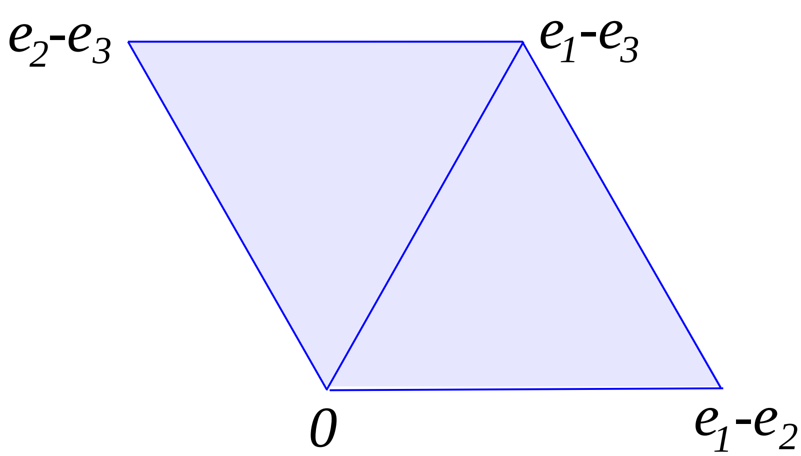



$$
\begin{aligned}
E(G) = \; & \{(i, i+1) \; : \; 1 \le i \le m\} \\
& \cup \; \{(1, i) \; : \; 2 \le i \le m\} \\
& \cup \; \{(i, m+1) \; : \; 2 \le i \le m\}.
\end{aligned}
$$

**Theorem.** $\widetilde{V}(\mathcal{F}_G) = C_{m-2}$.

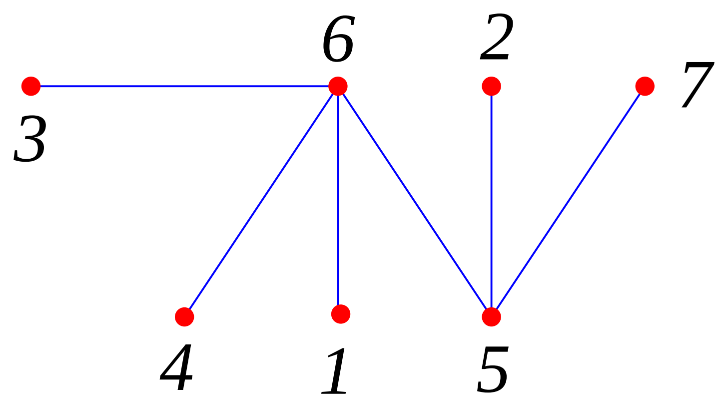
# The Catalanotope

$$\mathcal{C}_{\mathbf{n}} = \mathrm{conv}(A_n^+ \cup \{0\}) \subset \mathbb{R}^{n+1}.$$



Let $T$ be a tree with vertex set $\{1, \ldots, n+1\}$ and edge set $E$. Define the simplex

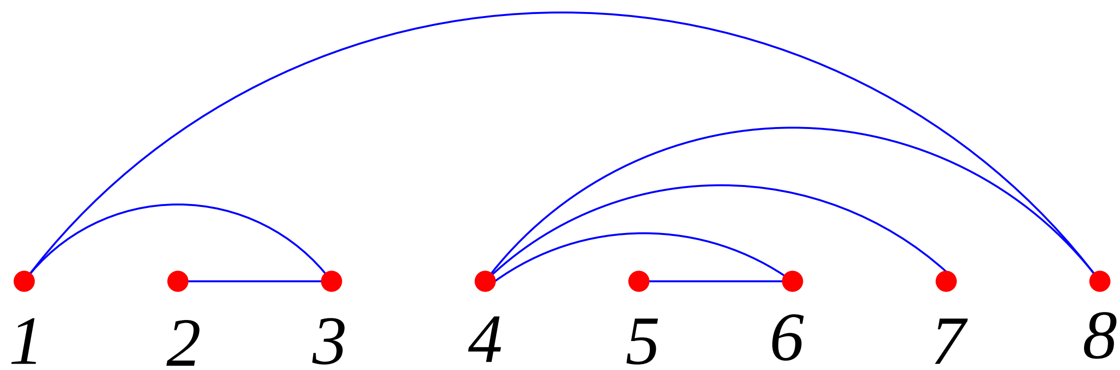
$$\sigma_T = \operatorname{conv}\left(\{e_{ij} \,:\, ij \in E,\ i < j\} \cup \{0\}\right).$$



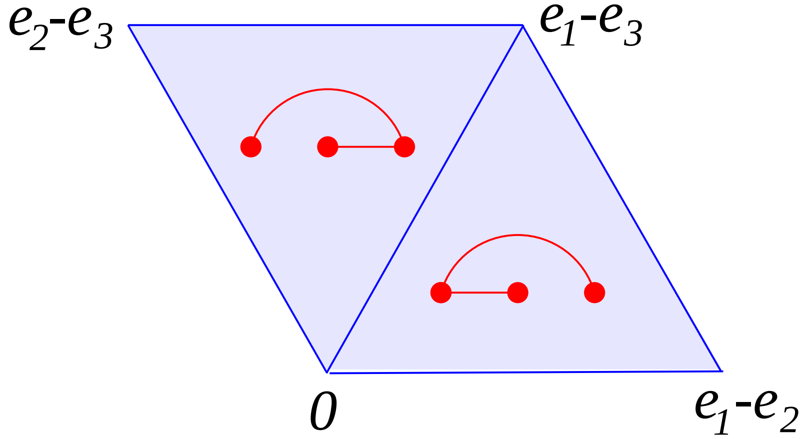

$$\sigma_T = \operatorname{conv}\{e_{16}, e_{25}, e_{36}, e_{46}, e_{56}, e_{57}, 0\}$$

$T$ is **alternating** if either every neighbor of vertex $i$ is less than $i$ or is greater than $i$.

$T$ is **noncrossing** if there are not edges $ik$ and $jl$ where $i < j < k < l$.

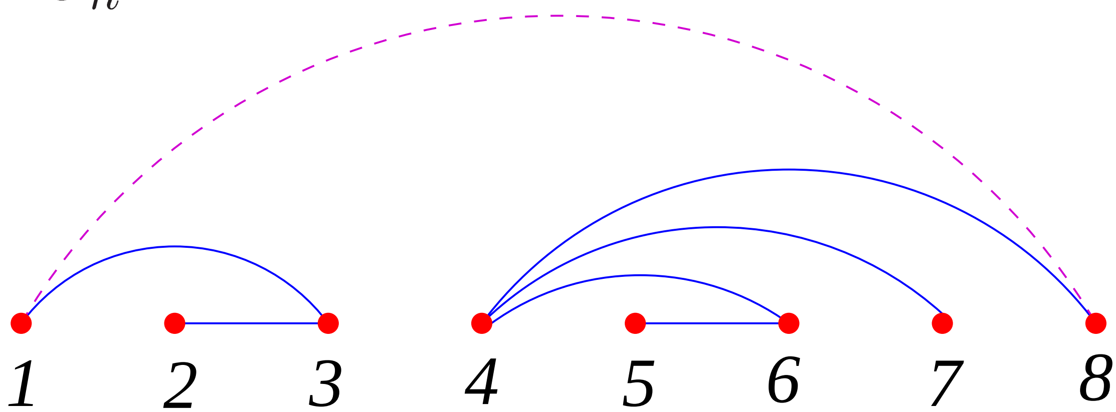


**Theorem** (A. Postnikov). *The simplices $\sigma_T$, where $T$ ranges over all noncrossing alternating trees with vertex set $\{1, \ldots, n+1\}$, are the maximal faces of a triangulation of $\mathcal{C}_n$.*




**Easy:** $\widetilde{V}(\sigma_T) = 1$.

**Lemma.** *The number of noncrossing alternating trees with vertex set* $\{1, \ldots, n+1\}$ *is the Catalan number* $C_n$.




**Corollary.** $\widetilde{V}(\mathcal{C}_n) = C_n$.

**Theorem.**

$$\sum_{r\geq 0} i(\mathcal{C}_n, r)x^r = \frac{\sum_{j=0}^{n-1} \frac{1}{n}\binom{n}{j}\binom{n}{j+1}x^j}{(1-x)^{n+1}}$$

Here $\frac{1}{n}\binom{n}{j}\binom{n}{j+1}$ is a **Narayana number**.

**Example** (an order polytope). Let $\mathcal{O}_{mn}$ be the set of all points

$$\left(a_{ij}\right)_{\substack{1\leq i\leq m\\ 1\leq j\leq n}} \subset \mathbb{R}^{m\times n}$$

satisfying

$$0 \leq a_{ij} \leq 1$$

$$\begin{aligned} a_{ij} &\geq a_{i-1,j} && (1)\\ a_{ij} &\geq a_{i,j-1}. && (2)\end{aligned}$$

Vertices are the $(0,1)$-matrices in $\mathcal{O}_{mn}$.

**Note:** $i(\mathcal{O}_{mn}, r)$ is the number of matrices $\left(a_{ij}\right)_{\substack{1\leq i\leq m\\ 1\leq j\leq n}}$ satisfying (1), (2), and $a_{ij} \in \{0, 1, \ldots, r\}$ (**plane partition** with $\leq m$ rows, $\leq n$ columns, and largest part $\leq r$).

$$\begin{matrix} 0 & 0 & 1 & 2 \\ 1 & 3 & 3 & 3 \\ 3 & 3 & 4 & 5 \\ 5 & 7 & 7 & 9 \end{matrix}$$

$\mathcal{O}_{mn}$ has a triangulation whose facets (maximal faces) $\sigma_T$ are indexed by standard Young tableaux $T$ of shape $(n, \ldots, n)$ ($m$ $n$'s).

**Example.** $T = \begin{matrix} 1 & 2 & 4 \\ 3 & 5 & 6 \end{matrix}$

$\sigma_T : 0 \le a_{11} \le a_{12} \le a_{21} \le a_{13} \le a_{22} \le a_{23}$

Each $\sigma_T$ is **primitive**, i.e.,

$$\widetilde{V}(\sigma_T) = 1.$$

**Corollary.** *We have*

$$\widetilde{V}(\mathcal{O}_{mn}) = \text{number of SYT of shape } (n,\ldots,n)$$
$$= \frac{n!}{\prod_{i=1}^{m}\prod_{j=1}^{n}(i+j-1)}$$

**Theorem** (MacMahon). *We have*

$$i(\mathcal{O}_{mn}, r) = \prod_{i=1}^{m}\prod_{j=1}^{n}\frac{r+i+j-1}{i+j-1}.$$

**Special case:** $m = 2$. Then

$$\widetilde{V}(\mathcal{O}_{2n}) = C_n = \frac{1}{n+1}\binom{2n}{n},$$

the $n$th Catalan number.

$$\sum_{r\geq 0} i(\mathcal{O}_{2n}, r)x^r = \frac{\sum_{j=0}^{n-1} \frac{1}{n}\binom{n}{j}\binom{n}{j+1}x^j}{(1-x)^{2n+1}},$$

where $\frac{1}{n}\binom{n}{j}\binom{n}{j+1}$ is a Narayana number as above.