Two Analogues of Pascal's Triangle

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Pascal's triangle

rows 0–4:

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}
$$

kth entry in row n, beginning with $k = 0$: $\binom{n}{k}$

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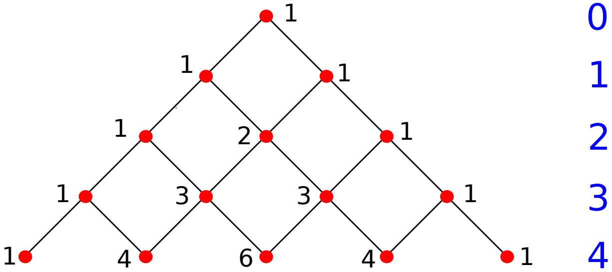
$$\sum_k \binom{n}{k}^3 = ??$$

Even worse! Generating function is not algebraic.

A diagram (poset) associated with Pascal's triangle



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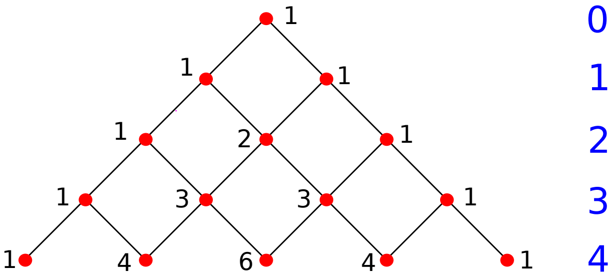

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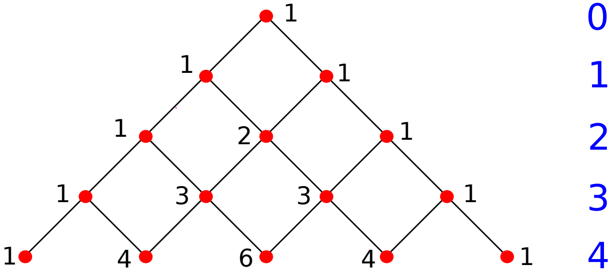

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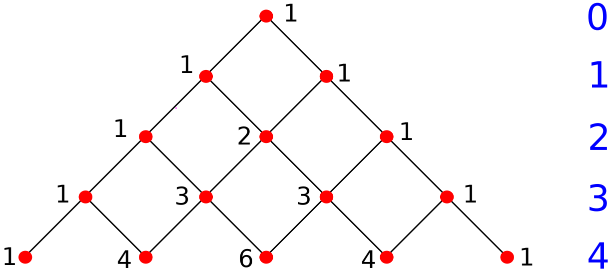

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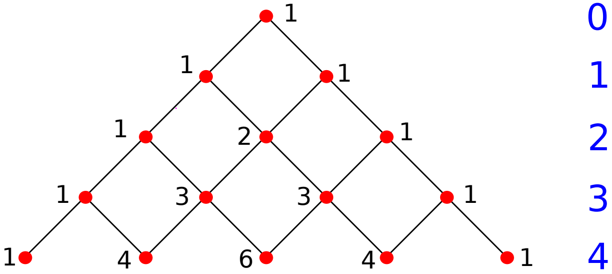

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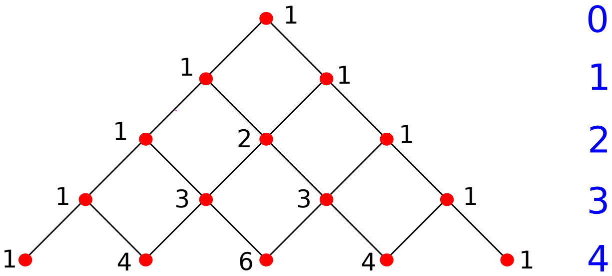
These properties **characterize** the diagram.

Two further properties



- Each label is the sum of those on the level above connected by an edge
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\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & 1 & & & & 1 & & \\
 & 1 & & & & & & 1 & \\
1 & & & & & & & & 1 \\
 & & & & \vdots & & & &
\end{array}
$$

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$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & {\color{blue}1} & 1 & & & \\
 & & 1 & & & & 1 & & \\
 & 1 & & & & & & 1 & \\
 & & & & \vdots & & & &
\end{array}
$$

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\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & 1 & & 1 & & 1 & & \\
 & 1 & & {\color{red}2} & & {\color{red}2} & & 1 & \\
1 & & & & & & & & 1 \\
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 & & & & 1 & & & 1 & & & 1 & & & & \\
 & & 1 & & 1 & & 2 & 1 & 2 & & 1 & & 1 & & \\
1 & & \color{red}{2} & & \color{red}{3} & & \color{red}{3} & & \color{red}{3} & & \color{red}{3} & & \color{red}{2} & & 1 \\
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 & 1 & & 1 & & 2 & & 1 & & 2 & & 1 & & 1 & \\
1 & {\color{blue}1} & 2 & {\color{blue}1} & 3 & {\color{blue}2} & 3 & {\color{blue}1} & 3 & {\color{blue}2} & 3 & {\color{blue}1} & 2 & {\color{blue}1} & 1 \\
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$$

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 & & & 1 & & & & 1 & & & & 1 & & & \\
 & 1 & & 1 & & 2 & & 1 & & 2 & & 1 & & 1 & \\
1 & 1 & 2 & 1 & 3 & 2 & 3 & 1 & 3 & 2 & 3 & 1 & 2 & 1 & 1 \\
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Some properties

- Number of entries in row n (beginning with row 0): $2^{n+1} - 1$

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Some properties

- Number of entries in row n (beginning with row 0): $2^{n+1}-1$
- Sum of entries in row n: 3^n
- Largest entry in row n: F_{n+1} (Fibonacci number)
- Let $\left\langle {n \atop k} \right\rangle$ be the kth entry (beginning with $k=0$) in row n. Write

$$P_n(x) = \sum_{k\geq 0} \left\langle {n \atop k} \right\rangle x^k.$$

Then $\boxed{P_{n+1}(x) = (1+x+x^2)P_n(x^2)}$, since $x\,P_n(x^2)$ corresponds to bringing down the previous row, and $(1+x^2)P_n(x^2)$ to summing two consecutive entries.

Stern analogue of binomial theorem

Corollary. $P_n(x) = \prod_{i=0}^{n-1} \left(1 + x^{2^i} + x^{2\cdot 2^i}\right)$

Historical note

An essentially equivalent array is due to **Moritz Abraham Stern** around 1858 and is known as **Stern's diatomic array**:

$$\begin{array}{ccccccccccccccccc}
1 & & & & & & & & & & & & & & & & 1 \\
1 & & & & & & & & 2 & & & & & & & & 1 \\
1 & & & & 3 & & & & 2 & & & & 3 & & & & 1 \\
1 & & 4 & & 3 & & 5 & & 2 & & 5 & & 3 & & 4 & & 1 \\
1 & 5 & 4 & 7 & 3 & 8 & 5 & 7 & 2 & 7 & 5 & 8 & 3 & 7 & 4 & 5 & 1 \\
 & & & & & & & & \vdots & & & & & & & &
\end{array}$$

Sums of squares

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & 1 & & & & 1 & & & & 1 & & & \\
 & 1 & & 1 & & 2 & & 1 & & 2 & & 1 & & 1 & \\
1 & 1 & 2 & 1 & 3 & 2 & 3 & 1 & 3 & 2 & 3 & 1 & 2 & 1 & 1 \\
 & & & & & & & \vdots & & & & & & &
\end{array}
$$

$$
\boldsymbol{u_2(n)} \coloneqq \sum_k \left\langle {n \atop k} \right\rangle^2 = 1,\ 3,\ 13,\ 59,\ 269,\ 1227,\ \ldots
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$$\begin{array}{ccccccccccccccc}
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\end{array}$$

$$\boldsymbol{u_2(n)} := \sum_k \left\langle {n \atop k} \right\rangle^2 = 1,\ 3,\ 13,\ 59,\ 269,\ 1227,\ \ldots$$

$$u_2(n+1) = 5u_2(n) - 2u_2(n-1), \quad n \geq 1$$

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$$\boldsymbol{u_2(n)} := \sum_k \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle^2 = 1,\ 3,\ 13,\ 59,\ 269,\ 1227,\ \ldots$$

$$u_2(n+1) = 5u_2(n) - 2u_2(n-1), \quad n \geq 1$$

$$\sum_{n\geq 0} u_2(n) x^n = \frac{1-2x}{1-5x+2x^2}$$

Proof

$$\begin{aligned} u_2(n+1) &= \cdots + \left\langle {n \atop k} \right\rangle^2 + \left(\left\langle {n \atop k} \right\rangle + \left\langle {n \atop k+1} \right\rangle \right)^2 + \left\langle {n \atop k+1} \right\rangle^2 + \cdots \\ &= 3u_2(n) + 2\sum_k \left\langle {n \atop k} \right\rangle \left\langle {n \atop k+1} \right\rangle . \end{aligned}$$

Proof

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\end{aligned}
$$

Thus define $\boldsymbol{u_{1,1}(n)} \coloneqq \sum_k \left\langle {n \atop k} \right\rangle \left\langle {n \atop k+1} \right\rangle$, so

$$
u_2(n+1) = 3u_2(n) + 2u_{1,1}(n).
$$

What about $u_{1,1}(n)$?

$$\begin{aligned} u_{1,1}(n+1) &= \cdots + \left(\left\langle {n \atop k-1} \right\rangle + \left\langle {n \atop k} \right\rangle\right)\left\langle {n \atop k} \right\rangle + \left\langle {n \atop k} \right\rangle\left(\left\langle {n \atop k} \right\rangle + \left\langle {n \atop k+1} \right\rangle\right) \\ &\quad + \left(\left\langle {n \atop k} \right\rangle + \left\langle {n \atop k+1} \right\rangle\right)\left\langle {n \atop k+1} \right\rangle + \cdots \\ &= 2u_2(n) + 2u_{1,1}(n) \end{aligned}$$

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Recall also $u_2(n+1) = 3u_2(n) + 2u_{1,1}(n)$.

Two recurrences in two unknowns

Let $\boldsymbol{A} := \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}$. Then

$$A \begin{bmatrix} u_2(n) \\ u_{1,1}(n) \end{bmatrix} = \begin{bmatrix} u_2(n+1) \\ u_{1,1}(n+1) \end{bmatrix}.$$

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$$\begin{aligned} (A^2 - 5A + 2I)A^{n-1} &= 0_{2\times 2} \\ \Rightarrow u_2(n+1) &= 5u_2(n) - 2u_2(n-1) \end{aligned}$$

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Also $u_{1,1}(n+1) = 5u_{1,1}(n) - 2u_{1,1}(n-1)$.

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Equivalently, if $\prod_{i=0}^{n-1} \left(1 + x^{2^i} + x^{2 \cdot 2^i}\right) = \sum a_j x^j$, then

$$\sum a_j^3 = 3 \cdot 7^{n-1}.$$

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Same method gives the matrix $\begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}$.

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Thus $u_3(n+1) = 7u_3(n)$ and

$$u_{2,1}(n+1) = 7u_{2,1}(n) \coloneqq 7\sum_k \left\langle {n \atop k} \right\rangle^2 \left\langle {n \atop k+1} \right\rangle \; (n \geq 1).$$

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In fact,

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Much nicer than $\sum_k \binom{n}{k}^3$

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By the same technique, can show that

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Much more can be said!

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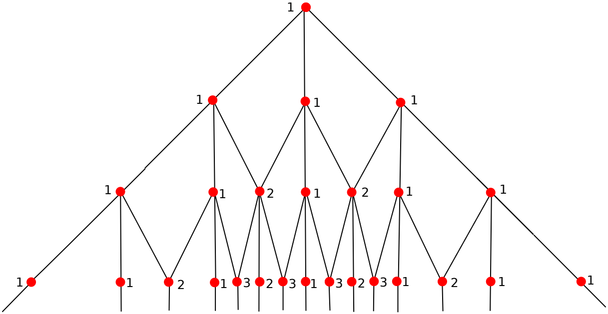

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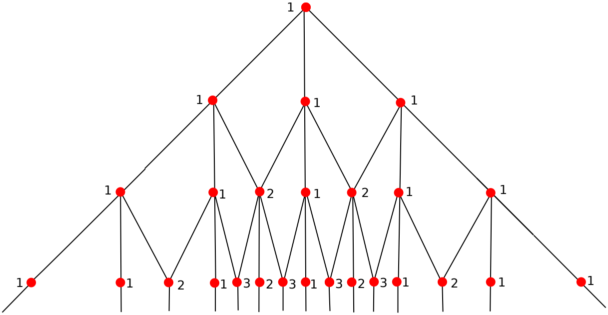

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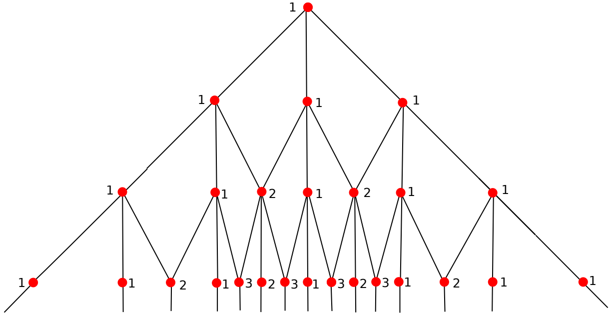

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- The diagram is planar.

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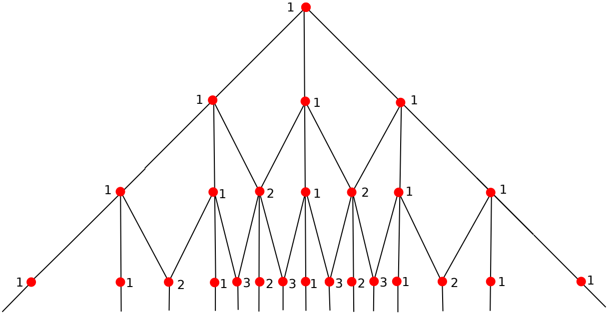

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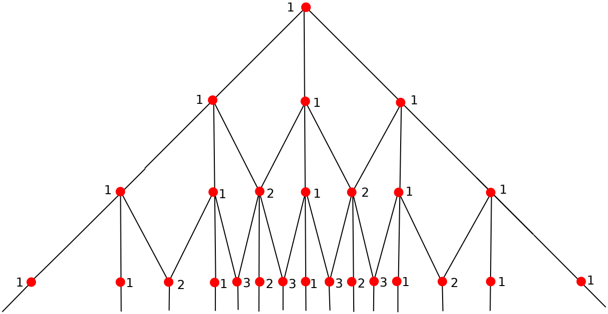

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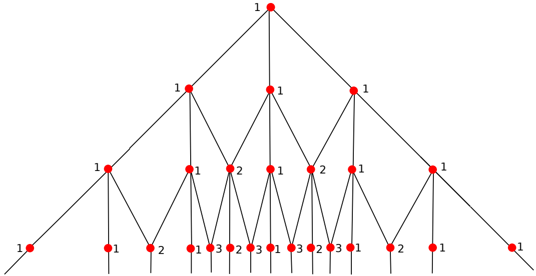

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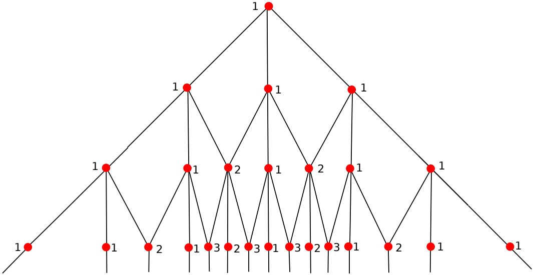

- Each label is the sum of those on the level above connected by an edge
- Each label is the number of paths from that label to the top.

The kth label (beginning with $k = 0$) at rank n is $\left\langle {n \atop k} \right\rangle$:

$$\sum_k \left\langle {n \atop k} \right\rangle x^k = \prod_{i=0}^{n-1} \left(1 + x^{2^i} + x^{2 \cdot 2^i}\right).$$

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Fibonacci numbers: $F_1 = F_2 = 1,\ \ F_n = F_{n-1} + F_{n-2}\ \ (n \geq 3)$

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$v_2(n)$: sum of squares of coefficients of $I_n(x)$

Goal: $\displaystyle\sum_{n\geq 0} v_2(n)x^n = \frac{1-2x^2}{1-2x-2x^2+2x^3}$

The Fibonacci triangle $\mathcal{F}$

														1														
											1						1											
					1						1			$\bullet$			1						1					
			1		1			$\bullet$			1			2			1			$\bullet$			1		1			
	1		1	$\bullet$	1			2			1	$\bullet$	2		2	$\bullet$	1			2			1	$\bullet$	1		1	
1	1	$\bullet$	1	2	1	$\bullet$	2		2	$\bullet$	1	3	2	$\bullet$	2	3	1	$\bullet$	2		2	$\bullet$	1	2	1	$\bullet$	1	1

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					1						1			•			1						1					
			1		1			•			1			2			1			•			1		1			
	1		1	•	1			2			1	•	2		2	•	1			2			1	•	1		1	
1	1	•	1	2	1	•	2		2	•	1	3	2	•	2	3	1	•	2		2	•	1	2	1	•	1	1

- Copy each entry of row $n \geq 1$ to the next row.
- Add two entries if separated by at bullet (and form group of 3)
- Copy once more the middle entry of a group of 3 (group of 2)
- Adjoin 1 at beginning and end of each row after row 0.

“Binomial theorem” for $\mathcal{F}$

$\left[{n \atop k} \right]$: kth entry (beginning with $k = 0$) in row n (beginning with $n = 0$) in $\mathcal{F}$

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Proof omitted.

$\sum_k \left[{n \atop k}\right]^2$

Now can obtain a system of recurrences analogous to

$$\begin{aligned} u_2(n+1) &= 3u_2(n) + 2u_{1,1}(n) \\ u_{1,1}(n+1) &= 2u_2(n) + 2u_{1,1}(n) \end{aligned}$$

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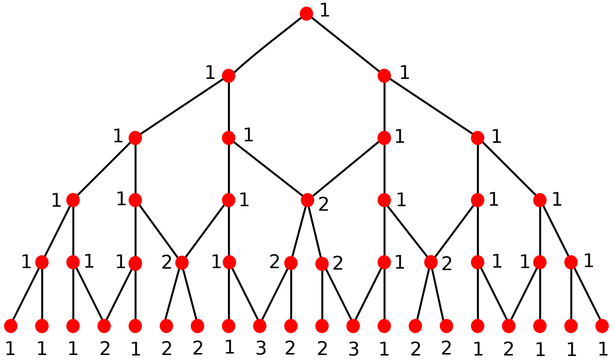
Probably a simpler argument using this method.

A diagram (poset) associated with $\mathfrak{F}$



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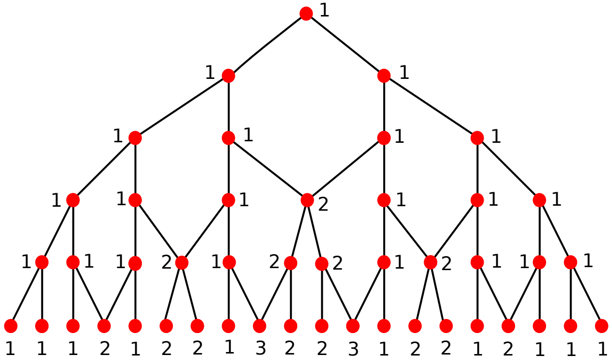

- Each point lies directly above **two** points.

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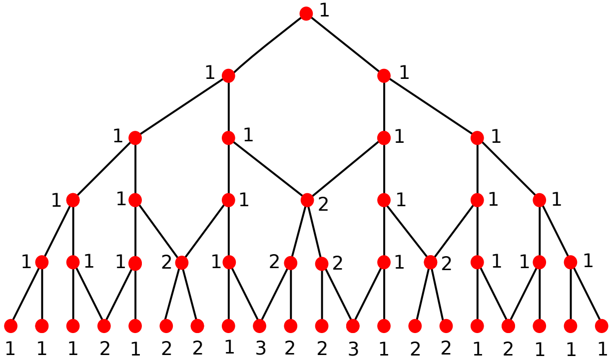

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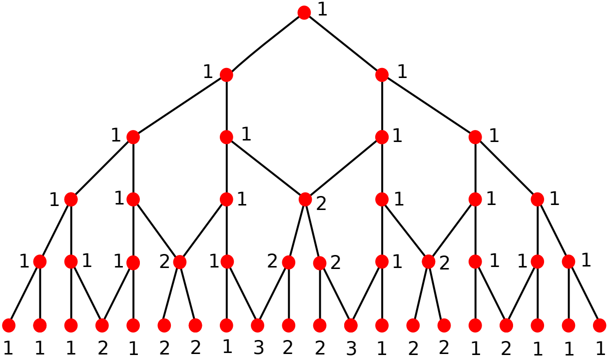

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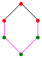
- Every  extends to 

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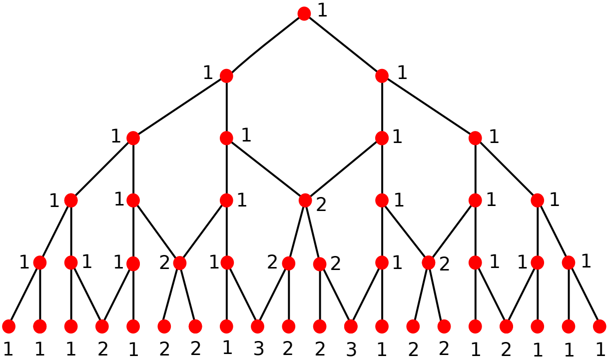

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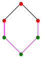
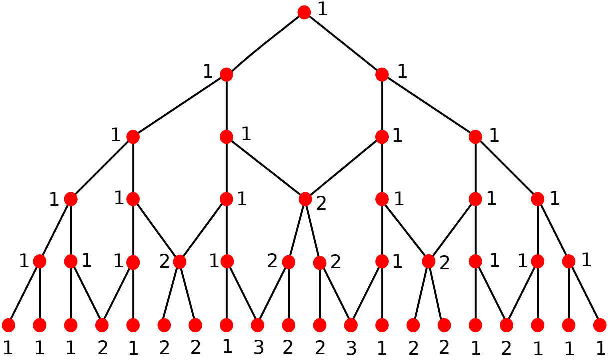
- Every  extends to 

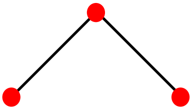
These properties **characterize** the diagram.

Two further properties



- Each label is the sum of those on the level above connected by an edge
- Each label is the number of paths from that label to the top.

Constructing $\mathfrak{F}$



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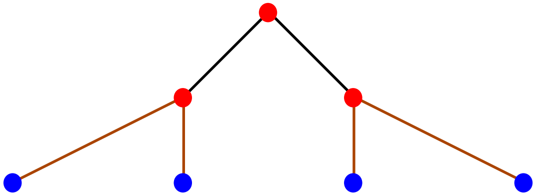

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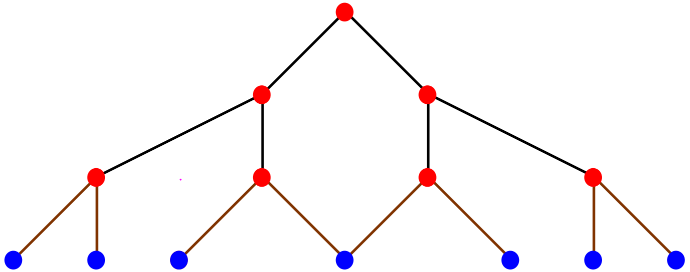

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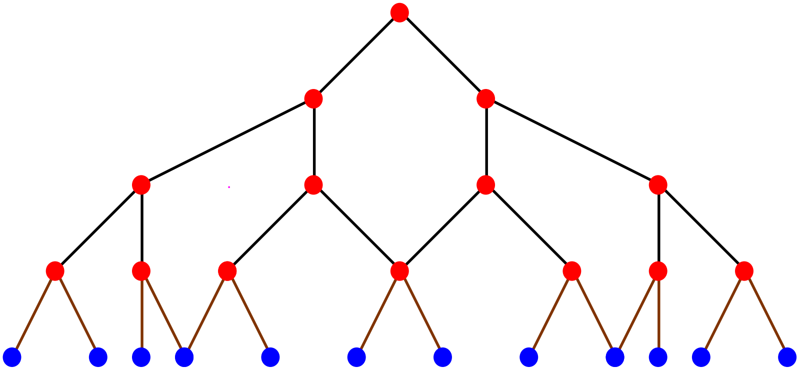

Number of elements at level n

p_n: number of elements of $\mathfrak{F}$ at level n

$(p_0, p_1, \dots) = (1, 2, 4, 7, 12, 20, \dots)$

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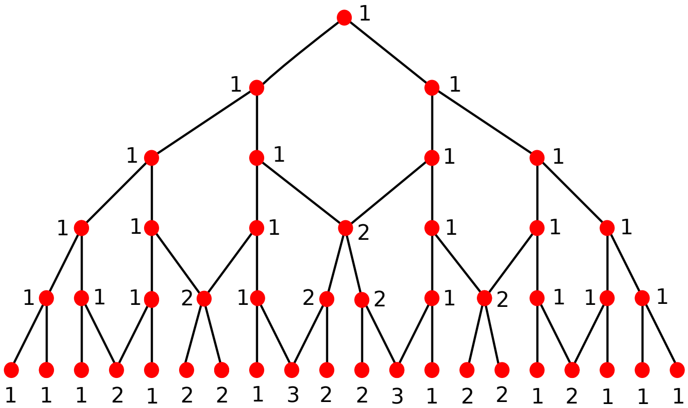
Solution with $p_0 = 1, p_1 = 2$ is $\boxed{p_n = F_{n+3} - 1}$

The groups of size two and three



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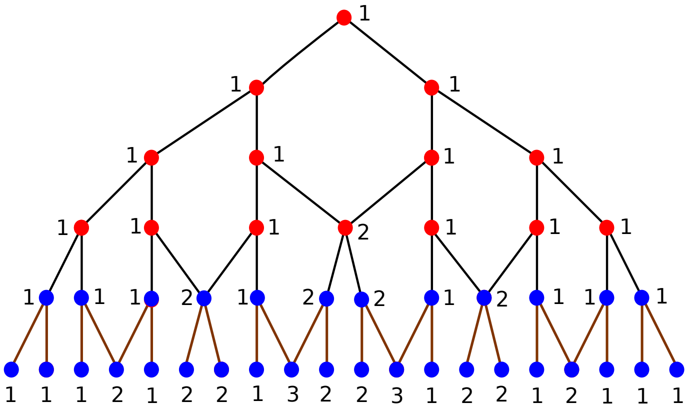

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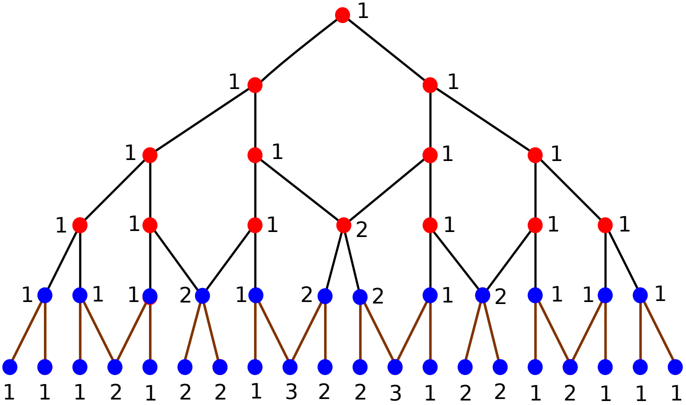

What is the sequence of group sizes on each level? E.g., on level 5, the sequence $2, 3, 2, 3, 3, 2, 3, 2$.

The limiting sequence

As $n \to \infty$, we get a "limiting sequence"

$$2, 3, 2, 3, 3, 2, 3, 2, 3, 3, 2, 3, 3, 2, 3, 2, 3, 3, 2, 3, \ldots.$$

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Let $\phi = (1 + \sqrt{5})/2$, the golden mean.

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Let $\phi = (1 + \sqrt{5})/2$, the golden mean.

Theorem. *The limiting sequence $(c_1, c_2, \dots)$ is given by*

$$c_n = 1 + \lfloor n\phi \rfloor - \lfloor (n-1)\phi \rfloor.$$

Properties of $c_n = 1 + \lfloor n\phi \rfloor - \lfloor (n-1)\phi \rfloor$

$$2, 3, 2, 3, 3, 2, 3, 2, 3, 3, 2, 3, 3, 2, 3, 2, 3, 3, 2, 3, \ldots .$$

- $\gamma = (c_2, c_3, \ldots)$ characterized by invariance under $2 \to 3$, $3 \to 32$ (**Fibonacci word** in the letters 2,3).

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- $\gamma = z_1 z_2 \ldots$ (concatenation), where $z_1 = 3$, $z_2 = 23$, $z_k = z_{k-2} z_{k-1}$

$$3 \cdot 23 \cdot 323 \cdot 23323 \cdot 32323323 \cdots$$

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- Sequence of number of 3's between consecutive 2's is the original sequence with 1 subtracted from each term.

$$2\underbrace{3}_{1}2\underbrace{33}_{2}2\underbrace{3}_{1}2\underbrace{33}_{2}2\underbrace{33}_{2}2\underbrace{3}_{1}2\underbrace{33}_{2}2\ldots .$$

Coefficients of $I_n(x)$

$$I_n(x) = \prod_{i=1}^{n} \left(1 + x^{F_{i+1}}\right)$$

Coefficient of x^m: number of ways to write m as a sum of distinct Fibonacci numbers from $\{F_2, F_3, \ldots, F_{n+1}\}$.

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$$8 = 5 + 3 = 5 + 2 + 1.$$

Can we see these sums from $\mathfrak{F}$? Each path from the top to a point $t \in \mathfrak{F}$ should correspond to a sum.

An edge labeling of $\mathfrak{F}$

The edges between ranks $2k$ and $2k+1$ are labelled alternately $0, F_{2k+2}, 0, F_{2k+2}, \ldots$ from left to right.

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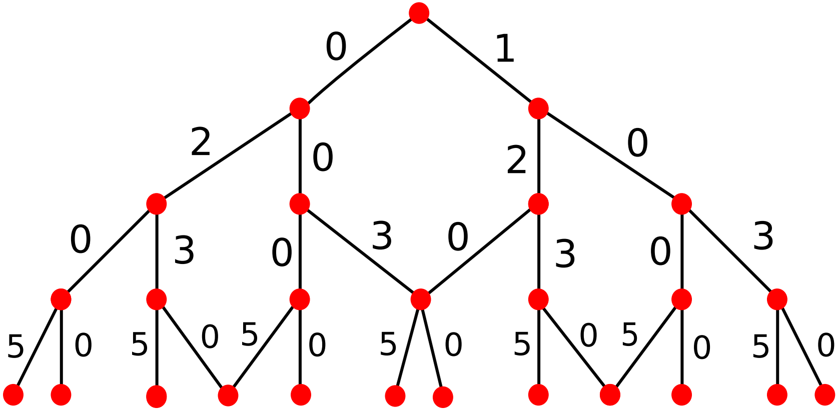
The edges between ranks $2k-1$ and $2k$ are labelled alternately $F_{2k+1}, 0, F_{2k+1}, 0, \ldots$ from left to right.

Diagram of the edge labeling



Connection with sums of Fibonacci numbers

Let $\boldsymbol{t} \in \mathfrak{F}$. All paths (saturated chains) from the top to t have the same sum of their elements $\boldsymbol{\sigma(t)}$.

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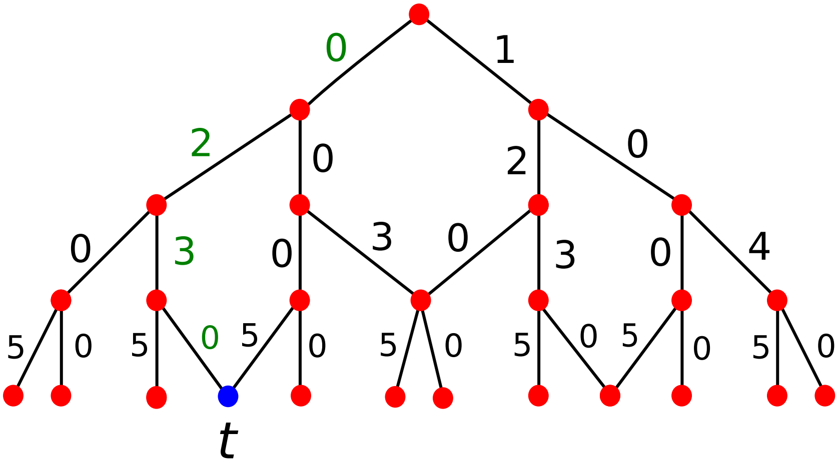
If $\mathrm{rank}(t) = n$, this gives all ways to write $\sigma(t)$ as a sum of distinct Fibonacci numbers from $\{F_2, F_3, \ldots, F_{n+1}\}$.

An example



$$2 + 3 = F_3 + F_4$$

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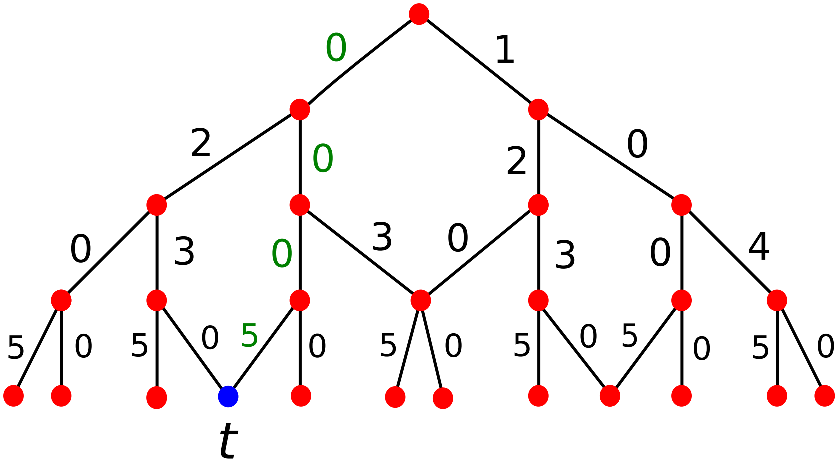
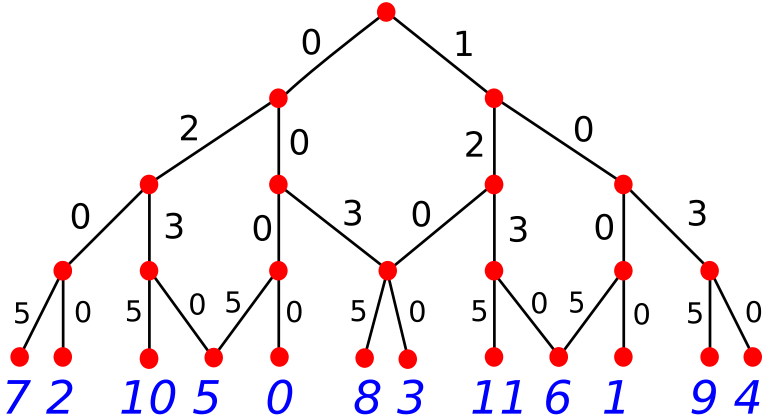


$5 = F_5$

An ordering of $\mathbb{N}$



In the limit as rank $\to \infty$, gives an interesting linear ordering of $\mathbb{N}$.

Second proof: factorization in a free monoid

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I_n(x) &:= \prod_{i=1}^{n} \left(1 + x^{F_{i+1}}\right) \\
&= \sum_k \begin{bmatrix} n \\ k \end{bmatrix} x^k
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$$\begin{bmatrix} n \\ k \end{bmatrix} = \#\left\{(a_1, \ldots, a_n) \in \{0,1\}^n \ : \ \sum_i a_i F_{i+1} = k\right\}$$

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v_2(n) &:= \sum_k \begin{bmatrix} n \\ k \end{bmatrix}^2 \\
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\end{aligned}
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A concatenation product

$$\mathcal{M}_n := \left\{ \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \end{pmatrix} : \sum a_i F_{i+1} = \sum b_i F_{i+1} \right\}$$

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Let

$$\alpha = \begin{pmatrix} a_1 & \cdots & a_n \\ b_1 & \cdots & b_n \end{pmatrix} \in \mathcal{M}_n, \quad \beta = \begin{pmatrix} c_1 & \cdots & c_m \\ d_1 & \cdots & d_m \end{pmatrix} \in \mathcal{M}_m.$$

Define

$$\alpha\beta = \begin{pmatrix} a_1 & \cdots & a_n & c_1 & \cdots & c_m \\ b_1 & \cdots & b_n & d_1 & \cdots & d_m \end{pmatrix},$$

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Easy to check: $\alpha\beta \in \mathcal{M}_{n+m}$

The monoid $\mathcal{M}$

$$\boldsymbol{\mathcal{M}} := \mathcal{M}_0 \cup \mathcal{M}_1 \cup \mathcal{M}_2 \cup \cdots,$$

a **monoid** (semigroup with identity) under concatenation. The identity element is $\varnothing \in \mathcal{M}_0$.

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Definition. A subset $\mathcal{G} \subset \mathcal{M}$ **freely generates** $\mathcal{M}$ if every $\alpha \in \mathcal{M}$ can be written uniquely as a product of elements of $\mathcal{G}$. (We then call $\mathcal{M}$ a **free** monoid.)

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Suppose $\mathcal{G}$ freely generates $\mathcal{M}$, and let $\boldsymbol{G(x)} = \sum_{n\geq 1} \#(\mathcal{M}_n \cap \mathcal{G})x^n$. Then

$$\begin{aligned} \sum_n v_2(n)x^n &= \sum_n \#\mathcal{M}_n \cdot x^n \\ &= 1 + G(x) + G(x)^2 + \cdots \\ &= \frac{1}{1 - G(x)}. \end{aligned}$$

Free generators of $\mathcal{M}$

Theorem. *$\mathcal{M}$ is freely generated by the following elements:*

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 11 & * & 1 & * & 1 & * & 1 & * & \cdots & * & 1 & 0 \\ 00 & * & 0 & * & 0 & * & 0 & * & \cdots & * & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 00 & * & 0 & * & 0 & * & 0 & * & \cdots & * & 0 & 1 \\ 11 & * & 1 & * & 1 & * & 1 & * & \cdots & * & 1 & 0 \end{pmatrix},$$

where each $*$ can be 0 or 1, but two $*$'s in the same column must be equal.

Free generators of $\mathcal{M}$

Theorem. *$\mathcal{M}$ is freely generated by the following elements:*

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 11 & * & 1 & * & 1 & * & 1 & * & \cdots & * & 1 & 0 \\ 00 & * & 0 & * & 0 & * & 0 & * & \cdots & * & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 00 & * & 0 & * & 0 & * & 0 & * & \cdots & * & 0 & 1 \\ 11 & * & 1 & * & 1 & * & 1 & * & \cdots & * & 1 & 0 \end{pmatrix},$$

where each $*$ can be 0 or 1, but two $*$'s in the same column must be equal.

Example. $\begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}$: $1 + 2 + 3 + 5 = 3 + 8$

$G(x)$

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Two elements of length one: $G(x) = 2x + \cdots$

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Two elements of length one: $G(x) = 2x + \cdots$

Let $\boldsymbol{k}$ be the number of columns of $*$'s. Length is $2k+3$. Thus

$$\begin{aligned} G(x) &= 2x + 2\sum_{k\geq 0} 2^k x^{2k+3} \\ &= 2x + \frac{2x^3}{1-2x^2}. \end{aligned}$$

Completion of proof

$$\begin{aligned}
\sum_n v_2(n)x^n &= \frac{1}{1-G(x)} \\
&= \frac{1}{1-\left(2x+\frac{2x^3}{1-2x^2}\right)} \\
&= \frac{1-2x^2}{1-2x-2x^2+2x^3} \qquad \square
\end{aligned}$$

Further vistas?

Let $i, j \geq 1$. Define the diagram (poset) $\boldsymbol{P_{ij}}$ by

- Each point lies directly above i points.

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P_{21}: diagram for Stern's triangle
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What can be said about P_{ij}?

References

These slides:
www-math.mit.edu/~rstan/transparencies/uncg.pdf

The Stern triangle: *Amer. Math. Monthly* **127** (2020), 99–111;
arXiv:1901.04647

The Fibonacci triangle (and much more): **arXiv:2101.02131**

Fibonacci word: **Wikipedia**

Factorization in free monoids: **EC1, second ed., §4.7.4**

The final slide

The final slide