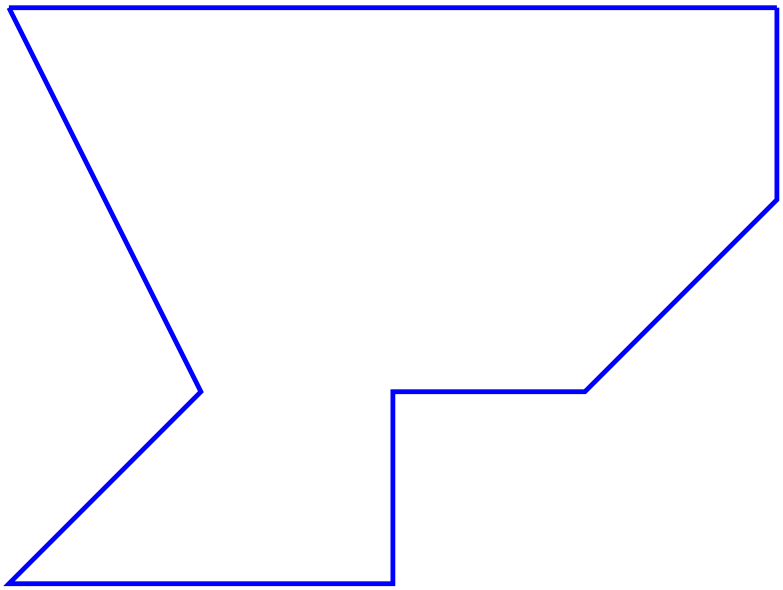
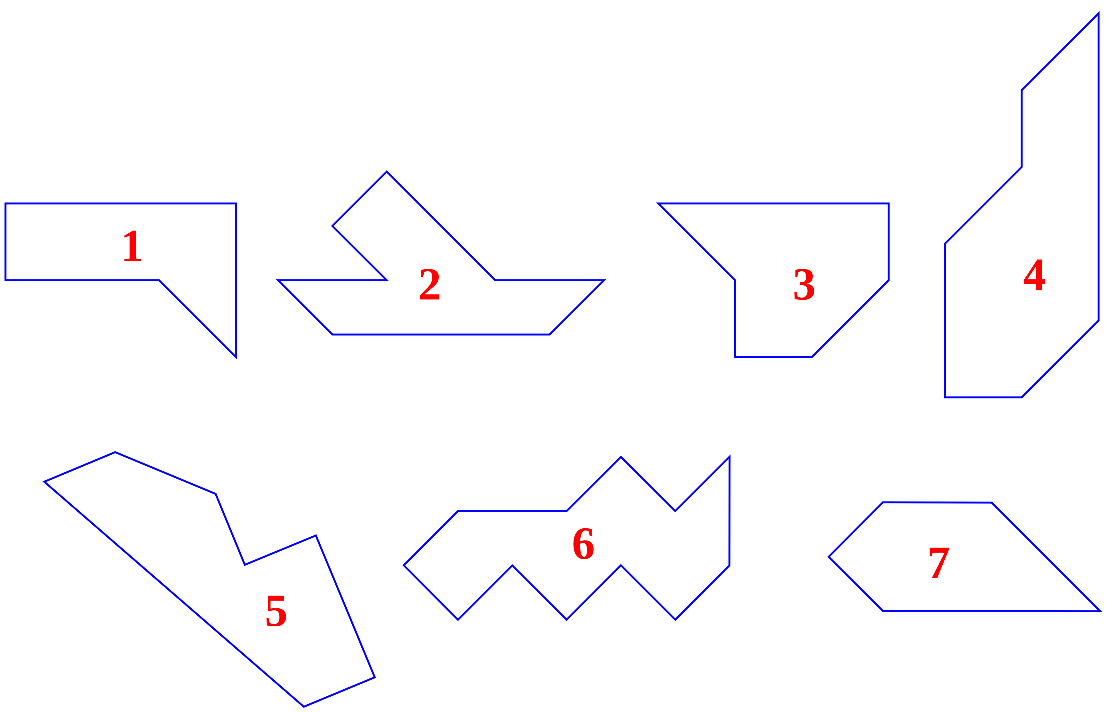
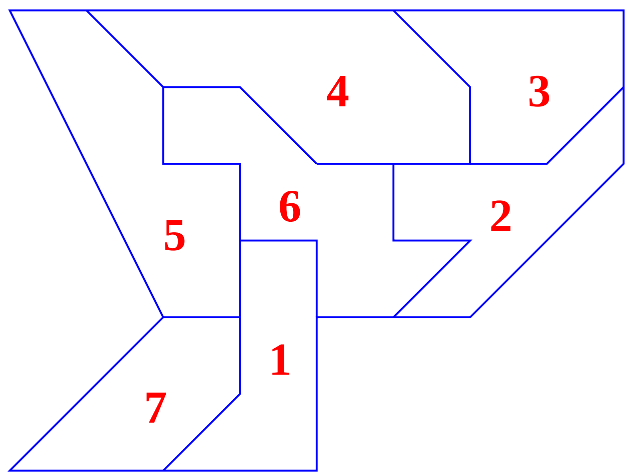
region:

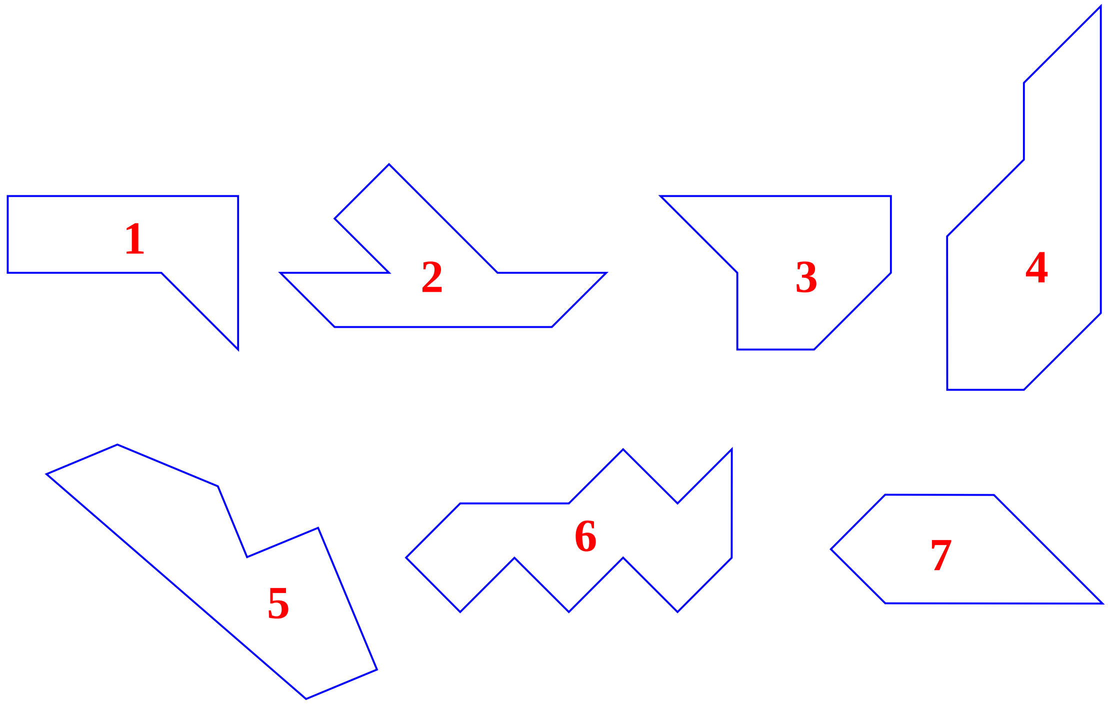


tiles:



tiling:



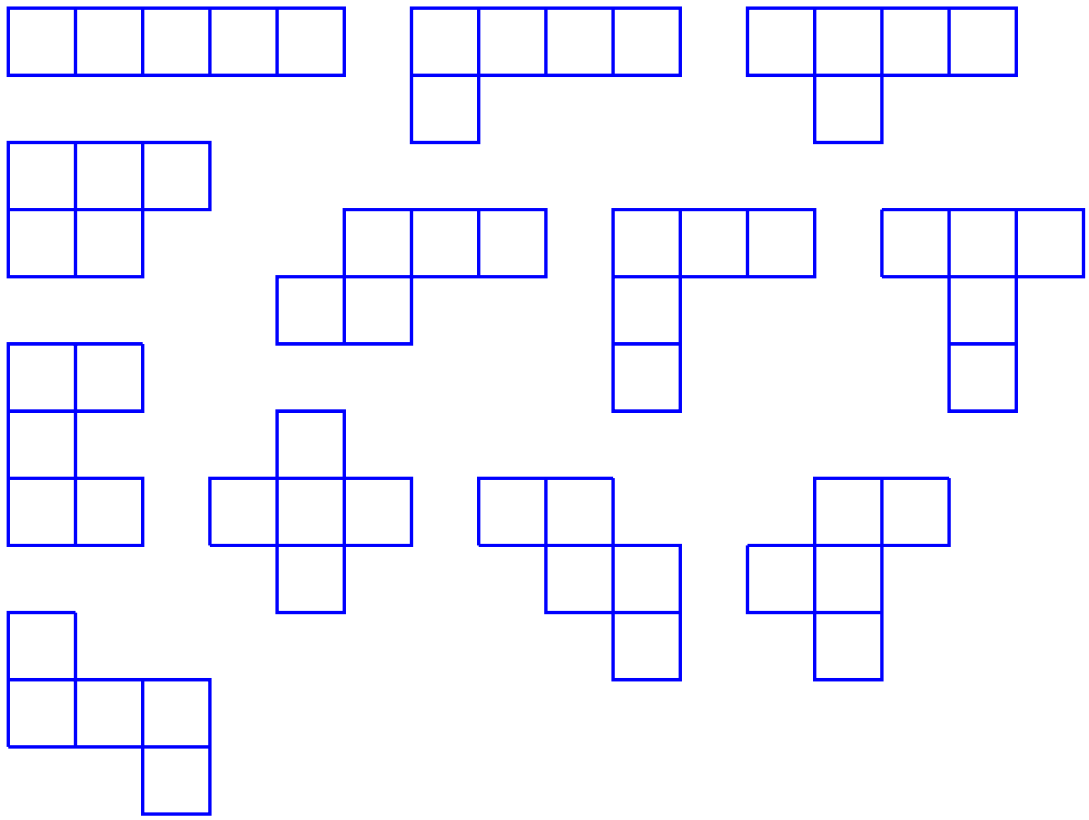


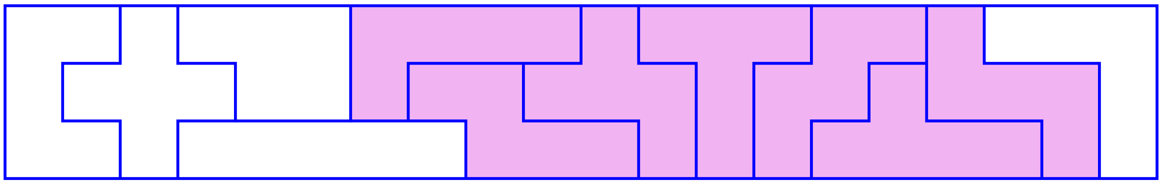
- Is there a tiling?
- How many?
- About how many?
- Is a tiling easy to find?
- Is it easy to prove a tiling doesn't exist?
- Is it easy to convince someone that a tiling doesn't exist?
- What is a "typical" tiling?
- Relations among different tilings
- Special properties, such as symmetry
- Infinite tilings

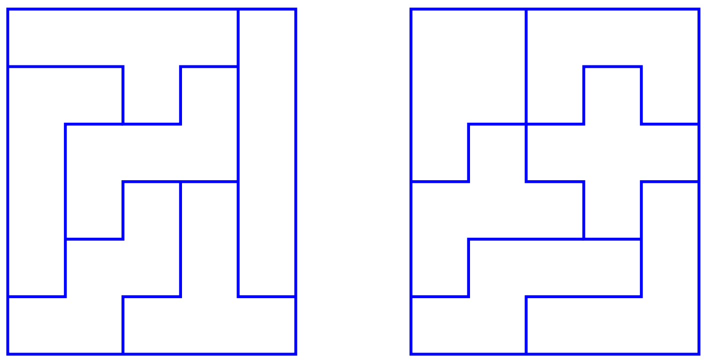
Is there a tiling?

Tiles should be "mathematically interesting."

12 pentominos:

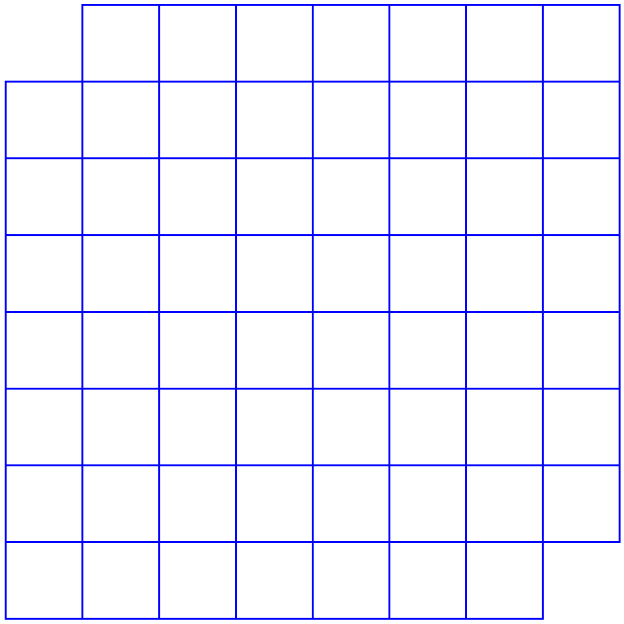




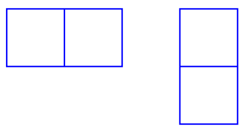


Number of tilings of a 6×10 rectangle: 2339

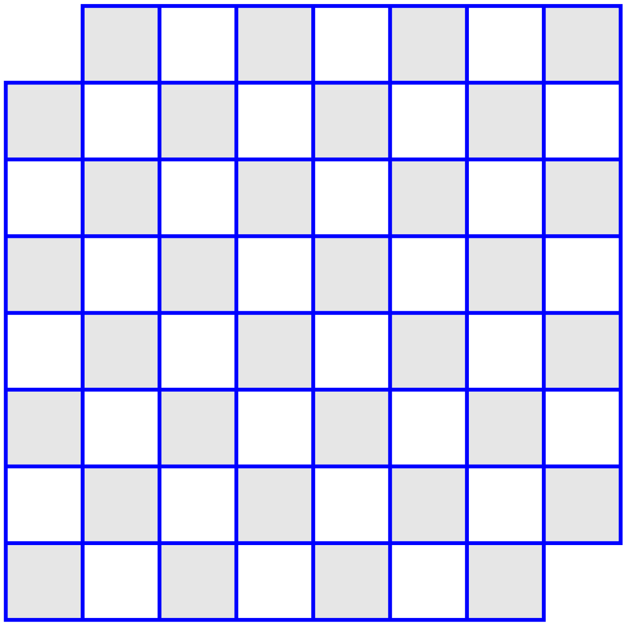
Found by "brute force" computer search (uninteresting)



Is there a tiling with 31 dominos (or dimers)?



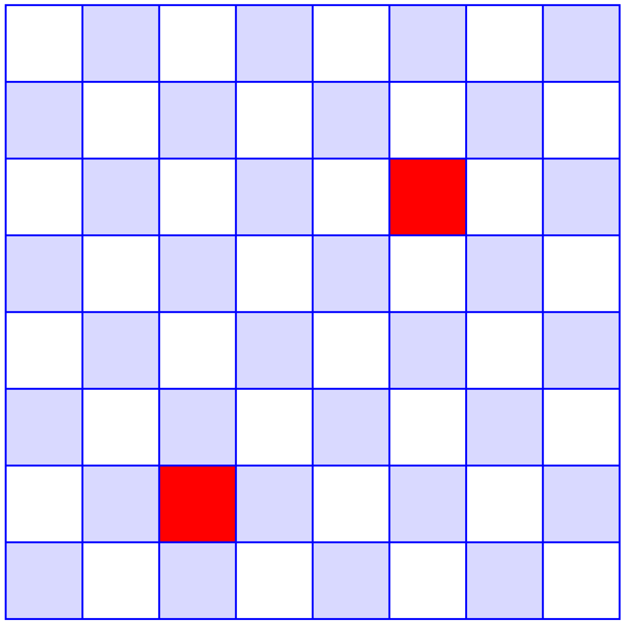
color the chessboard:

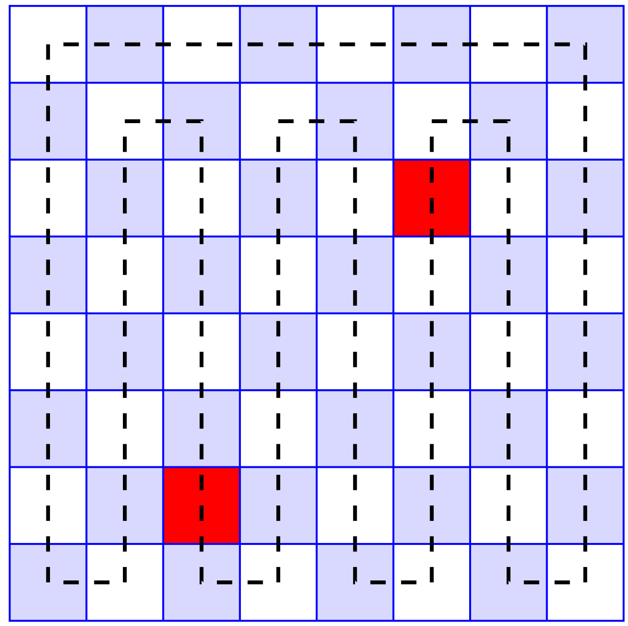


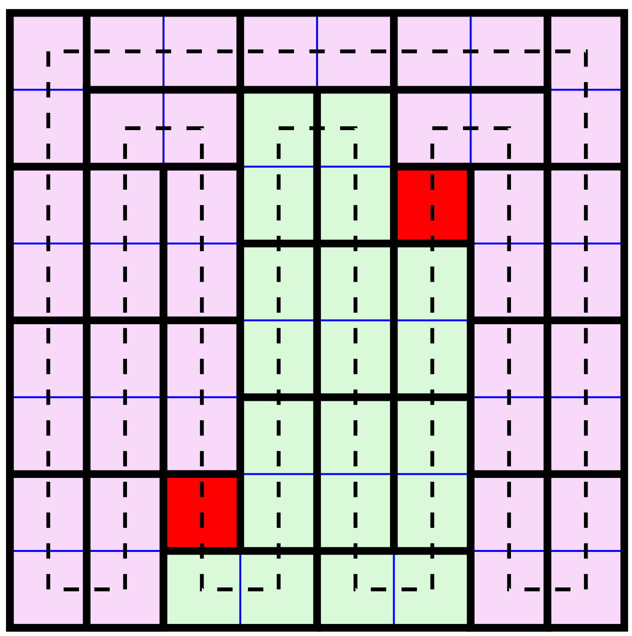
Each domino covers one black and one white square, so 31 dominos cover 31 white squares and 31 black squares. There are 32 white squares and 30 black squares in all, so a tiling does **not** exist.

Example of a **coloring argument**.

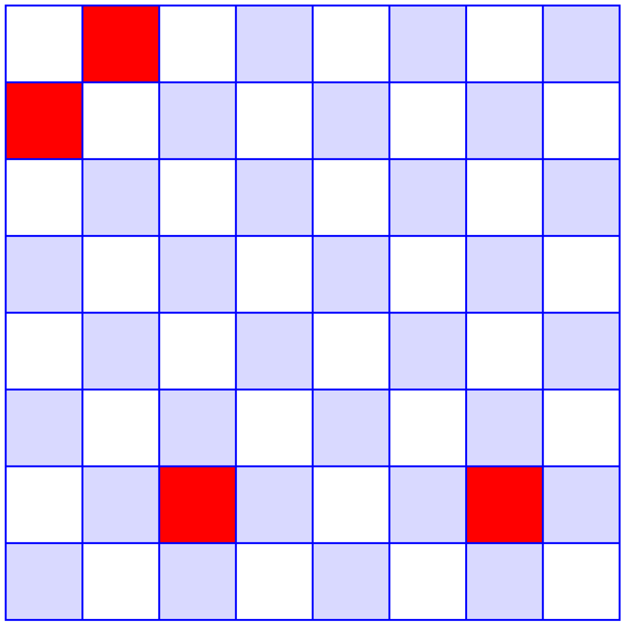
What if we remove one black square and one white square?



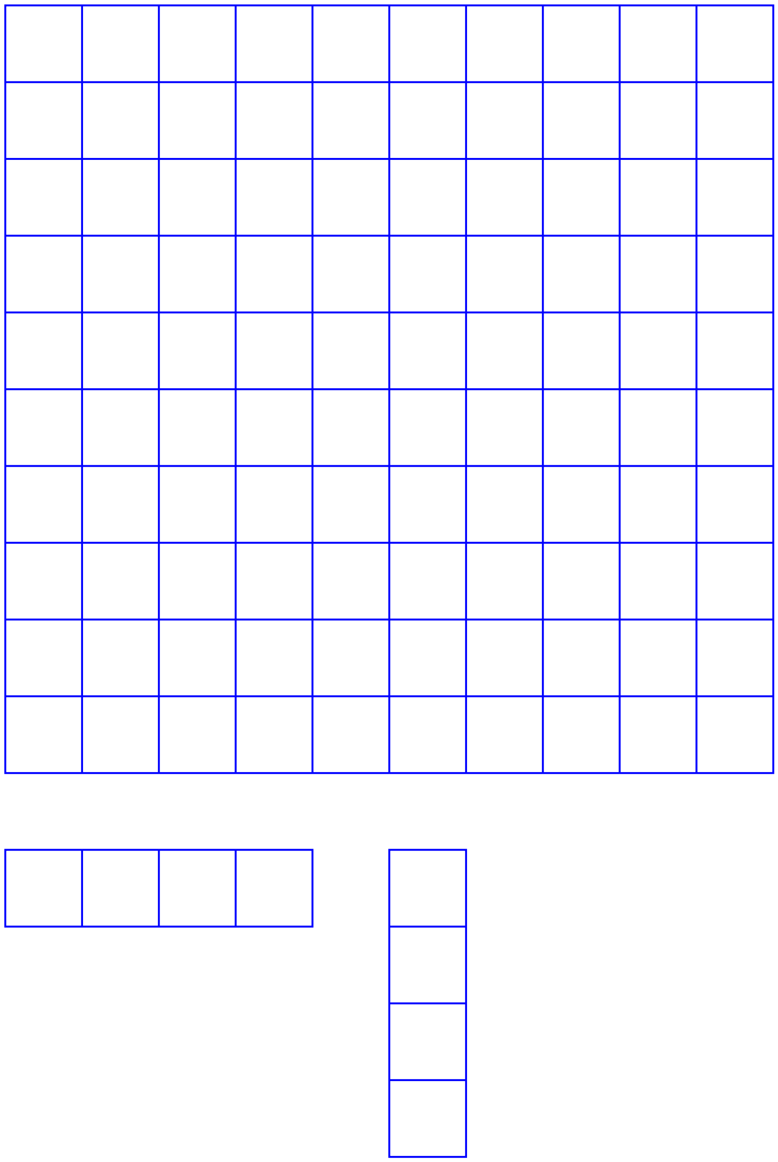


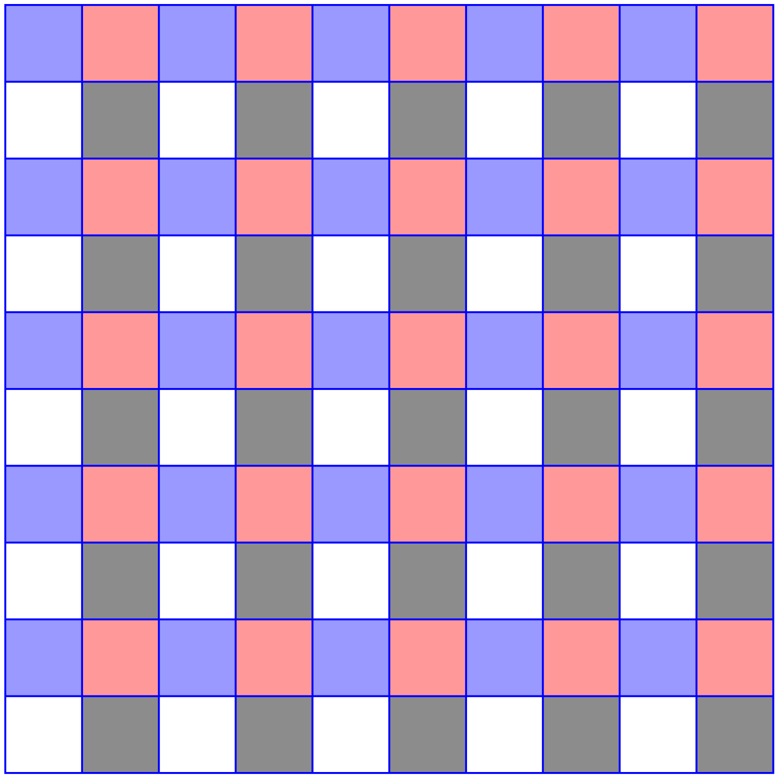


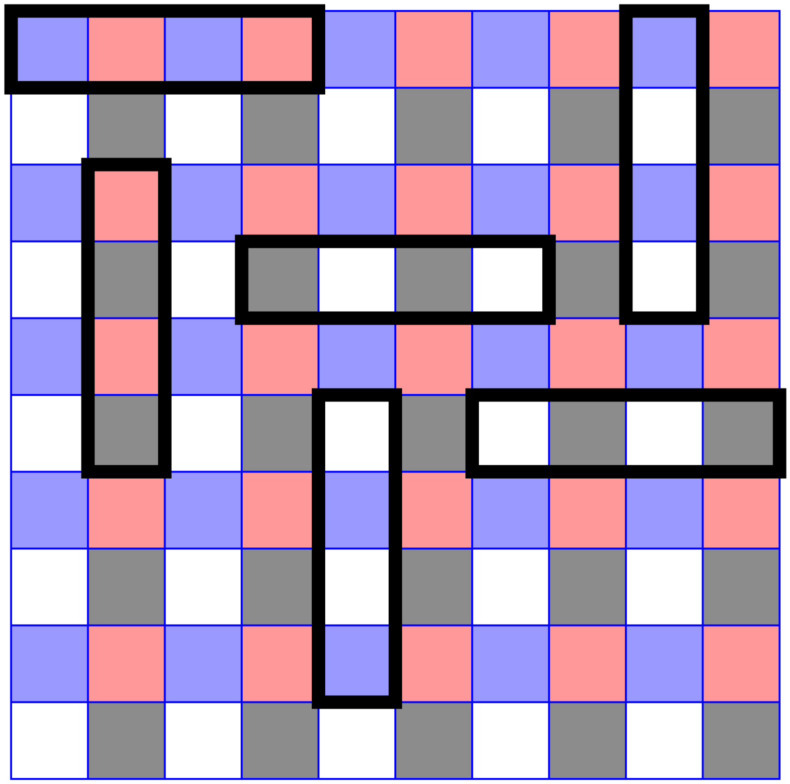
What if we remove **two** black squares and **two** white squares?



Another coloring argument: can a 10×10 board be tiled with 1×4 rectangles (in any orientation)?

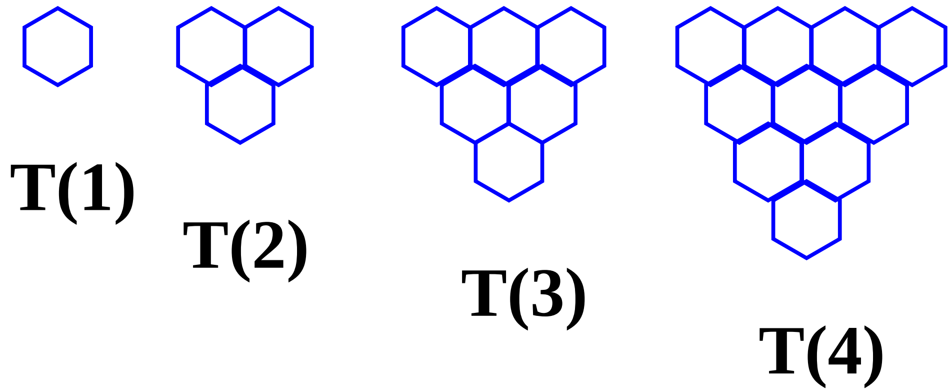






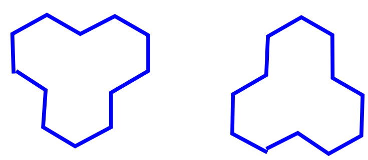
Every tile covers each color an **even** number (including 0) of times. But the board has 25 tiles of each color, so a tiling is impossible.

Coloring doesn't work:

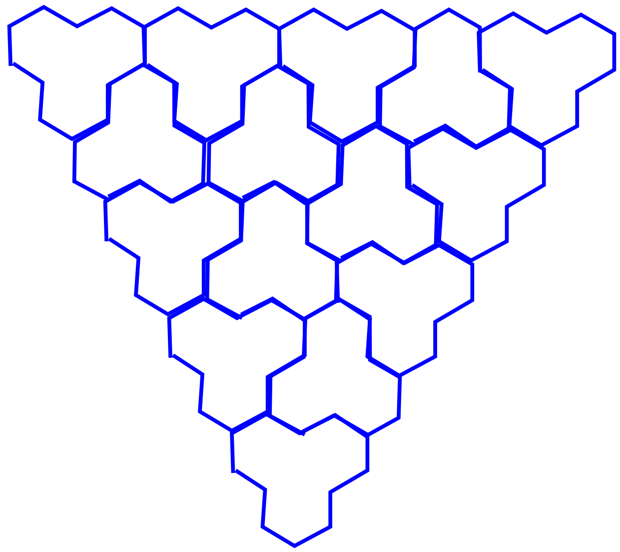


n hexagons on each side ($n(n+1)/2$ hexagons in all)

Can $T(n)$ be covered by "tribones"?



Yes for $T(9)$:



Conway: The triangular array $T(n)$ can be tiled by tribones if and only if $n = 12k, 12k + 2, 12k + 9, 12k + 11$ for some $k \geq 0$.

Smallest values: 0, 2, 9, 11, 12, 14, 21, 23, 24, 26, 33, 35, ...

Cannot be proved by a coloring argument (involves a **nonabelian** group)

How many tilings?

There are 2339 ways (up to symmetry) to tile a 6×10 rectangle with the 12 pentominos. Found by computer search: not so interesting.

First significant result on the enumeration of tilings due to Kasteleyn, Fisher–Temperley (independently, 1961):

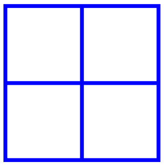
The number of tilings of a $2m \times 2n$ rectangle with $2mn$ dominos is

$$4^{mn} \prod_{j=1}^{m} \prod_{k=1}^{n} \left(\cos^2 \frac{j\pi}{2m+1} + \cos^2 \frac{k\pi}{2n+1} \right).$$

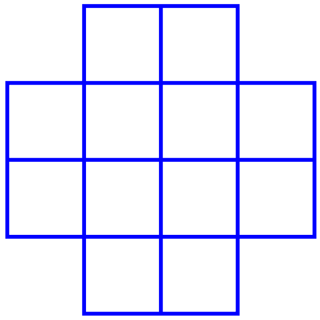
8×8 board: $12988816 = 3604^2$ tilings

Idea of proof: Express number of tilings as the determinant of a matrix. The factors are the eigenvalues.

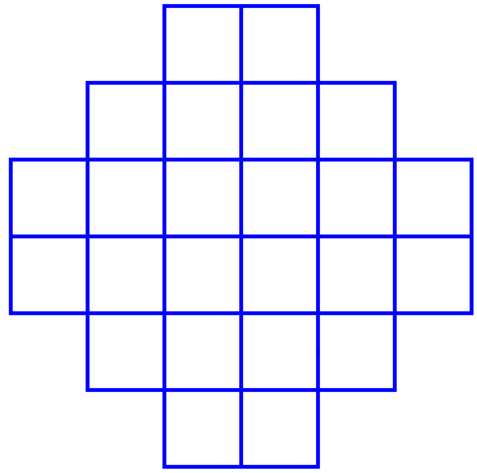
Aztec diamonds:



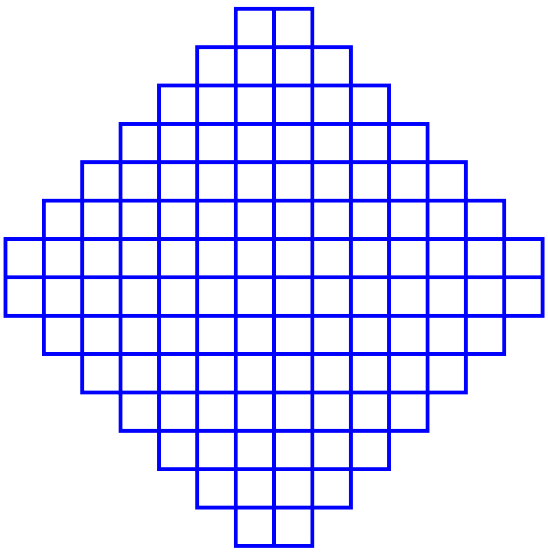
AZ(1)



AZ(2)

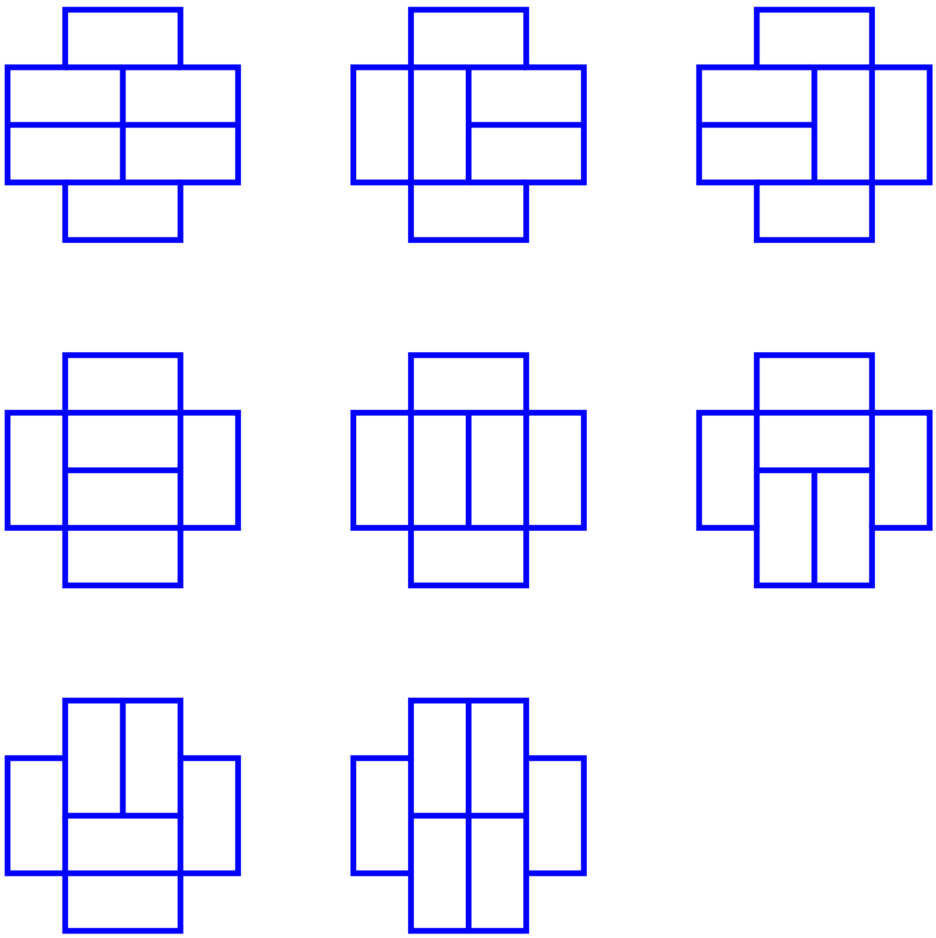


AZ(3)



AZ(7)

Eight domino tilings of AZ(2), the Aztec diamond of order 2:

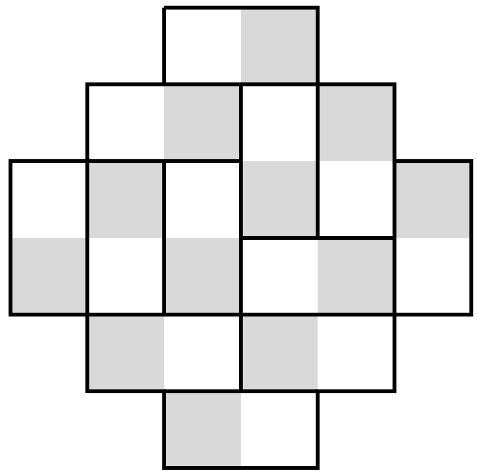


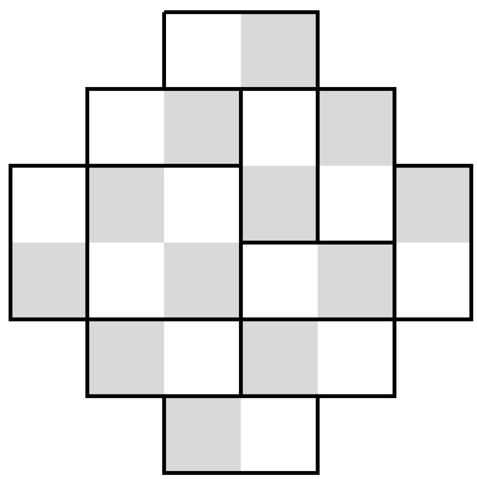
Elkies-Kuperberg-Larsen-Propp (1992): *The number of domino tilings of* $AZ(n)$ *is* $2^{n(n+1)/2}$.

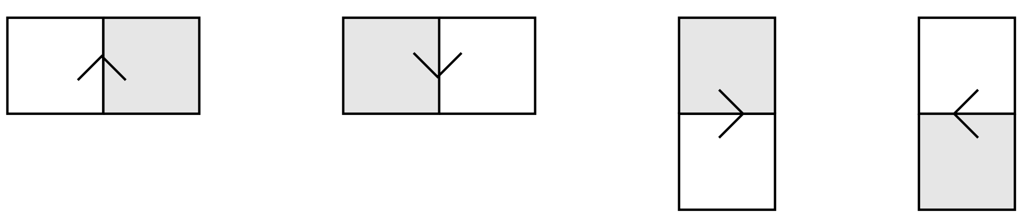
(four proofs originally, now around 12)

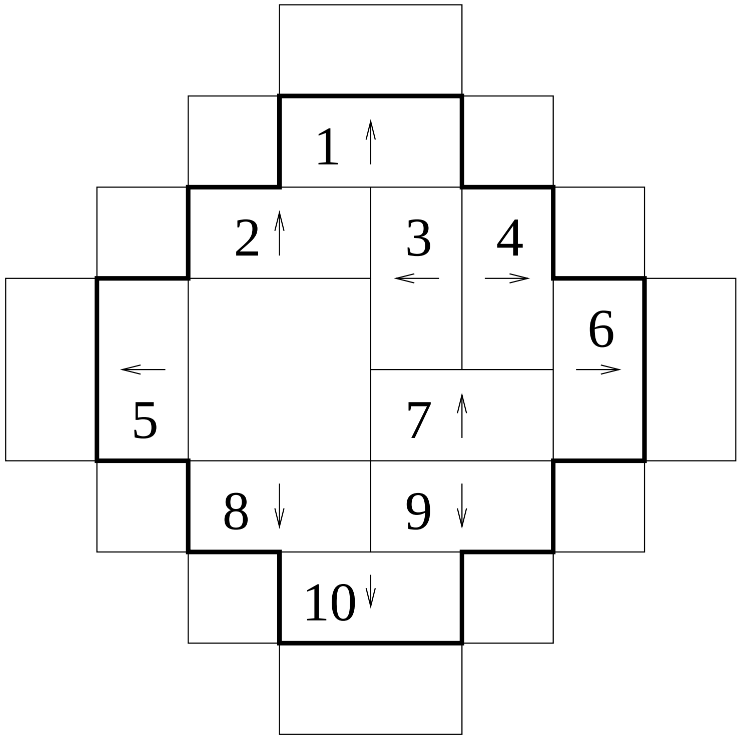
1	2	3	4	5	6	7
2	8	64	1024	32768	2097152	268435456

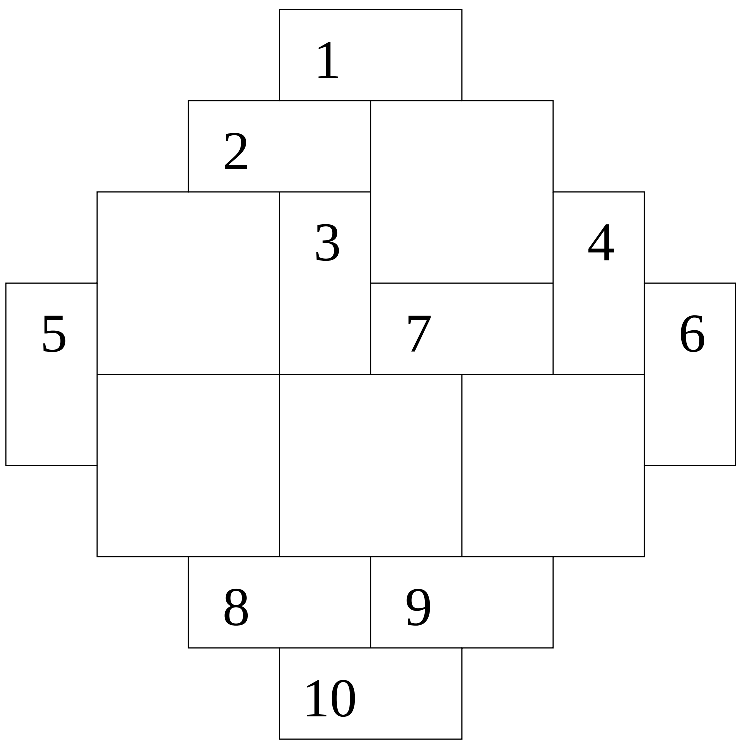
Since $2^{(n+2)(n+1)/2}/2^{(n+1)n/2} = 2^{n+1}$, we would like to associate 2^{n+1} AZ-tilings of order $n+1$ with each AZ-tiling of order n, so that each AZ-tiling of order $n+1$ occurs exactly once. This is done by **domino shuffling**.








1
2
3
4
6
5
7
8
9
10


1
2
3
4
5
7
6
8
9
10

About how many tilings? AZ(n) is a "skewed" $n \times n$ square. How do the number of domino tilings of AZ(n) and an $n \times n$ square (n even) compare?

If a region with N squares has T tilings, then it has (loosely speaking) $\sqrt[N]{T}$ **degrees of freedom per square**.

Number of tilings of AZ(n): $T = 2^{n(n+1)/2}$

Number of squares of AZ(n):

$$N = 2n(n+1)$$

Number of degrees of freedom per square:

$$\sqrt[N]{T} = \sqrt[4]{2} = \mathbf{1.189207115} \cdots$$

Number of tilings of $2n \times 2n$ square:

$$4^{n^2} \prod_{j=1}^{n} \prod_{k=1}^{n} \left(\cos^2 \frac{j\pi}{2n+1} + \cos^2 \frac{k\pi}{2n+1} \right).$$

Theorem (Kasteleyn, et al.). *Let*

$$\begin{aligned} G &= 1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \cdots \\ &= 0.9159655941\cdots \end{aligned}$$

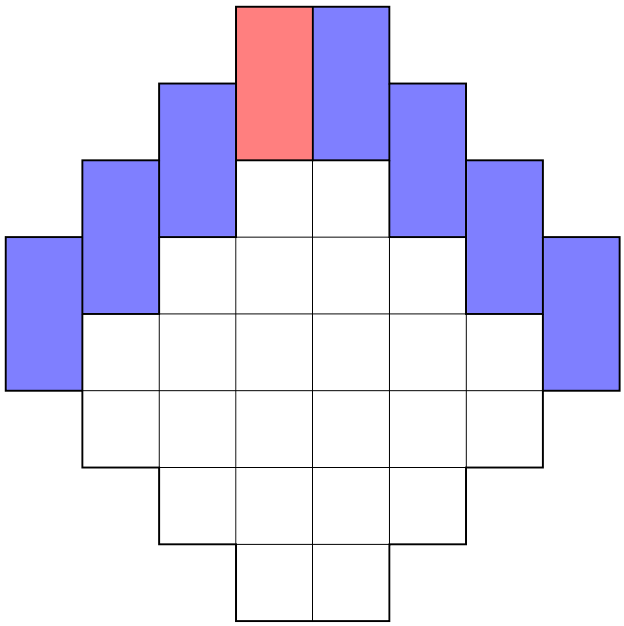
(**Catalan's constant**). *The number of domino tilings of a $2n \times 2n$ square is about C^{4n^2}, where*

$$\begin{aligned} C &= e^{G/\pi} \\ &= \mathbf{1.338515152\cdots}. \end{aligned}$$

More precisely, if $N(n)$ is the number of domino tilings of a $2n \times 2n$ square then

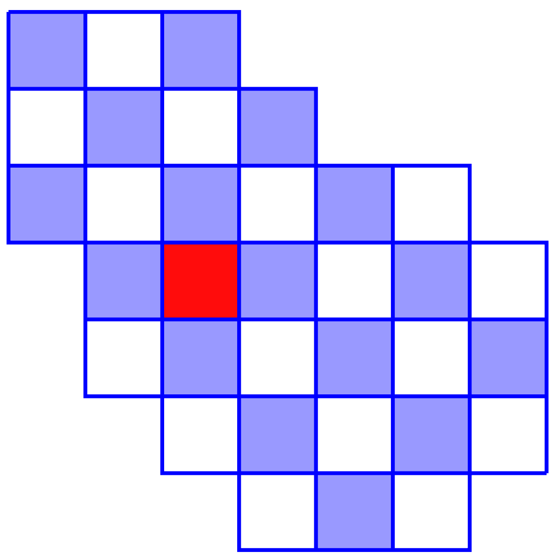
$$\lim_{n \to \infty} N(n)^{1/4n^2} = e^{G/\pi}.$$

Thus the square board is "easier" to tile than the Aztec diamond ($1.3385\cdots$ degrees of freedom per square vs. $1.189207115\cdots$).

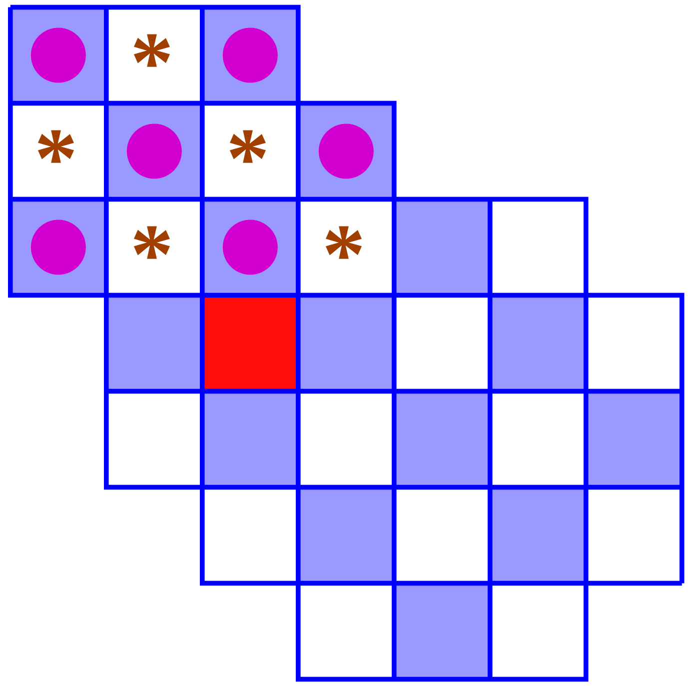


What if a tiling **doesn't** exist? Is it easy to demonstrate that this is the case?

In general, almost certainly **no** (even for 1×3 rectangular tiles, for which the problem is **NP-complete**). But **yes** (!) for domino tilings.



16 white squares and 16 black squares



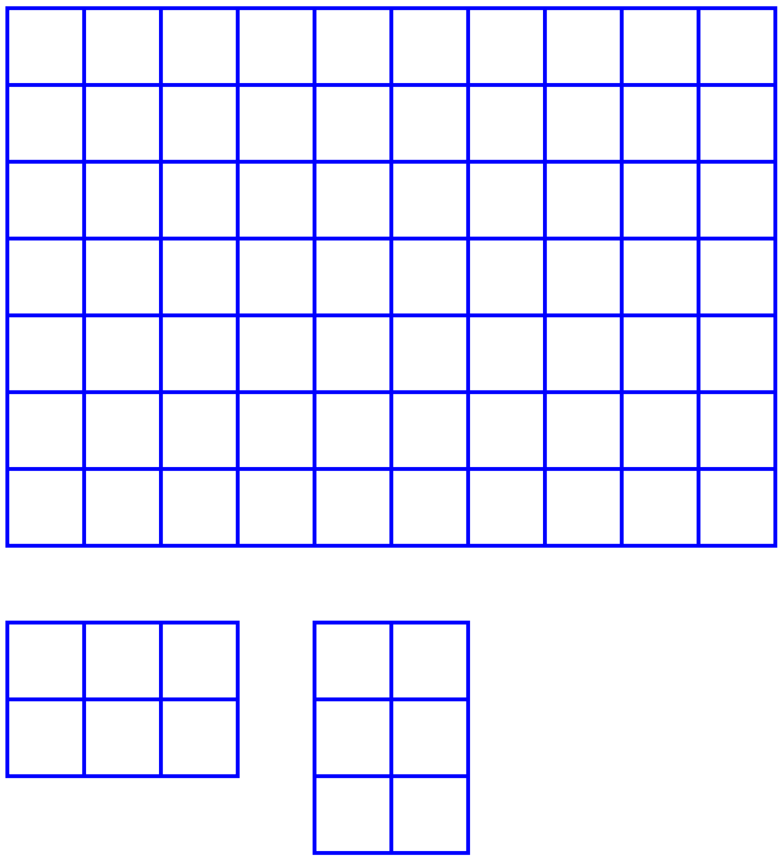
The six black squares with ● are adjacent to a total of five white squares marked ∗. No tiling can cover all six black square marked with ●.

Philip Hall (1935): If a region cannot be tiled with dominos, then one can **always** find such a demonstration of impossibility.

Extends to any bipartite graph: **Marriage Theorem**.

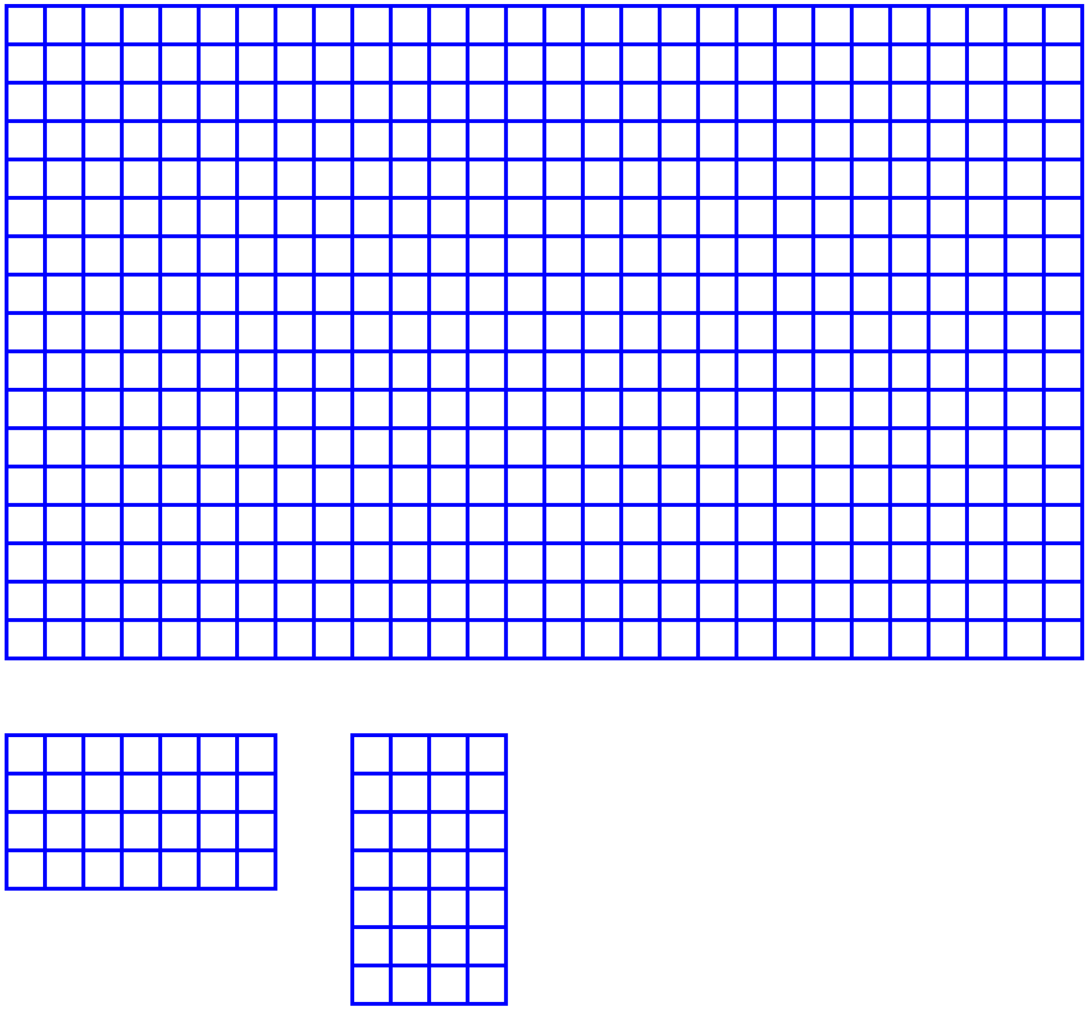
Tilings rectangles with rectangles: two results

Can a 7×10 rectangle be tiled with 2×3 rectangles (in any orientation)?



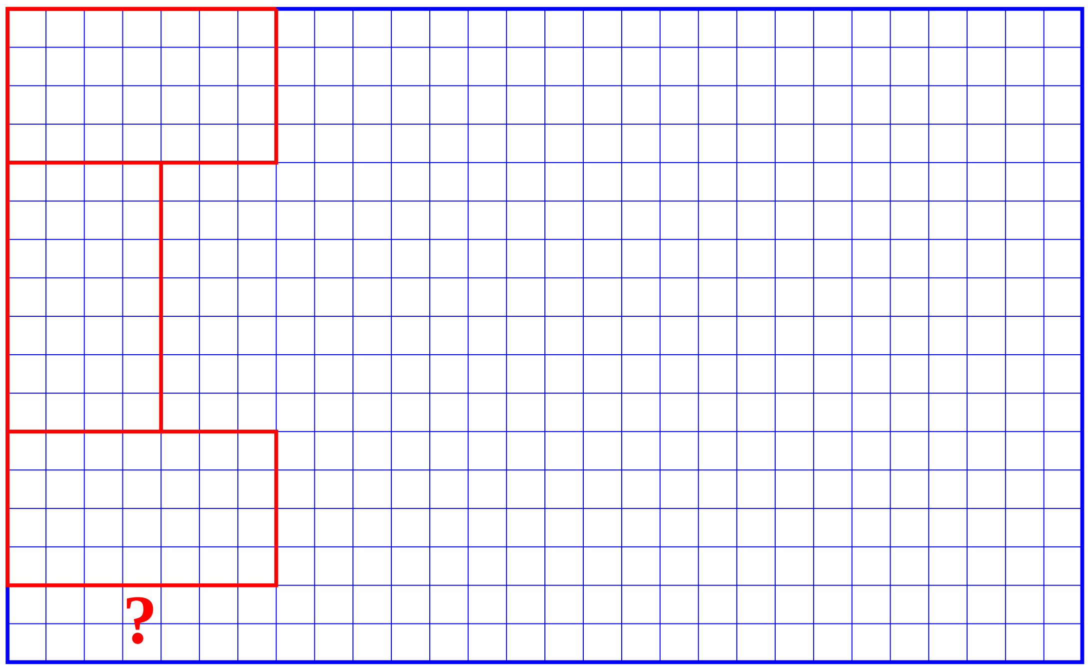
Clearly **no**: a 2×3 rectangle has 6 squares, while a 7×10 rectangle has 70 squares (not divisible by 6).

Can a 17×28 rectangle be tiled with 4×7 rectangles?



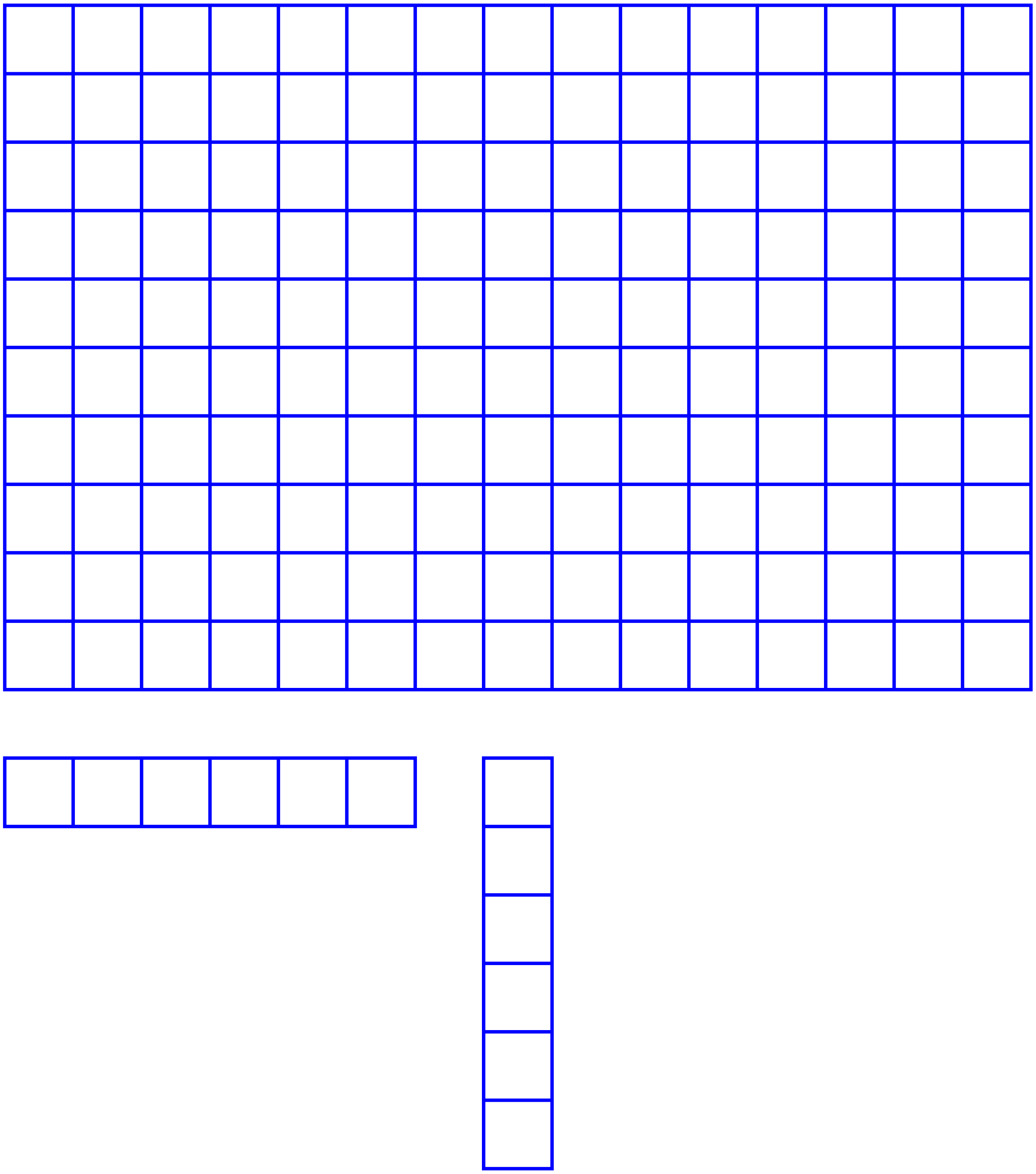
Can a 17×28 rectangle be tiled with 4×7 rectangles?

No: there is no way to cover the first column.



$$17 \neq 4a + 7b$$

Can a 10×15 rectangle be tiled with 1×6 rectangles?

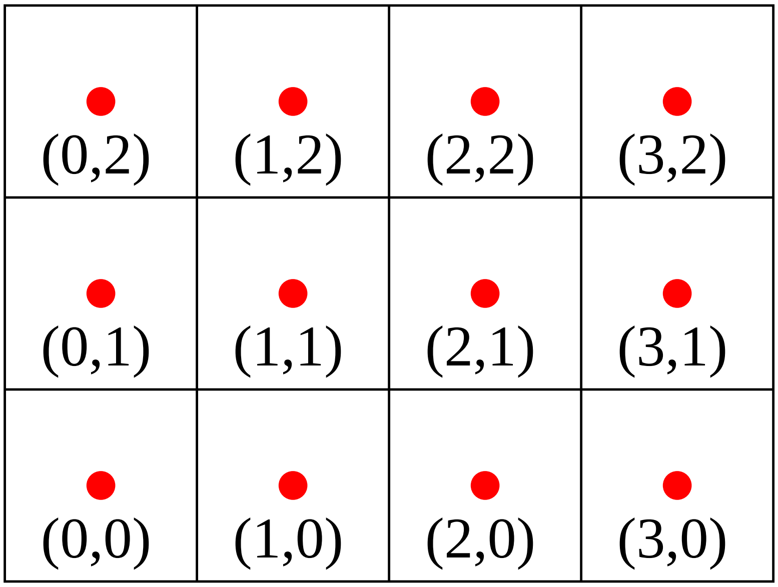


de Bruijn-Klarner: an $m \times n$ rectangle can be tiled with $a \times b$ rectangles if and only if:

- The first row and first column can be covered.
- m or n is divisible by a, and m or n is divisible by b.

Since neither 10 nor 15 are divisible by 6, the 10×15 rectangle **cannot** be tiled with 1×6 rectangles.

Idea of proof.



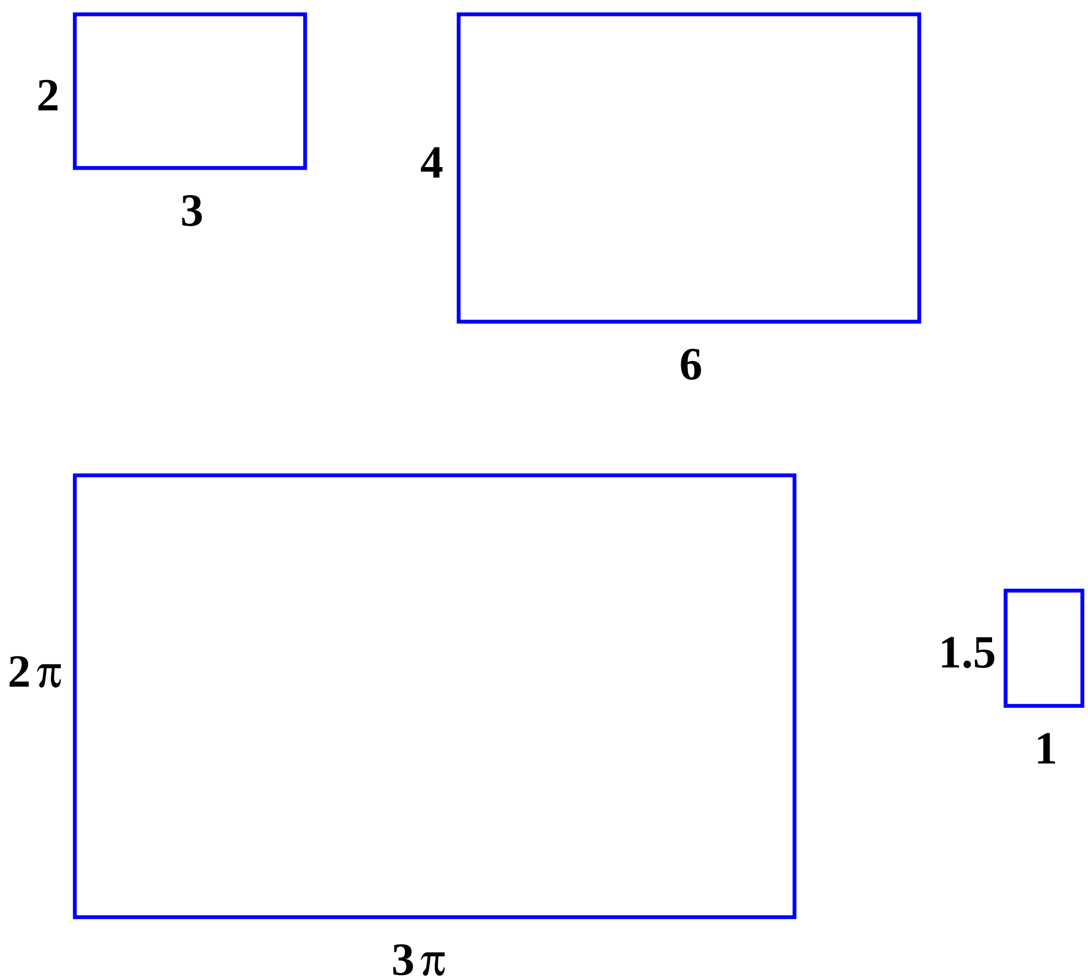
$$(1+x+\cdots+x^{m-1})(1+y+\cdots+y^{n-1}) =$$

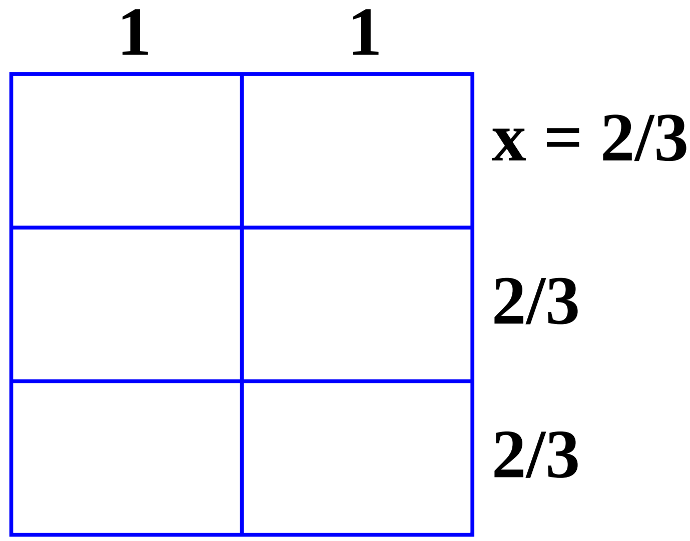
$$\sum x^r y^s (1+x+\cdots+x^{a-1})(1+y+\cdots+y^{b-1})$$

$$+\sum x^r y^s (1+x+\cdots+x^{b-1})(1+y+\cdots+y^{a-1})$$

Set $x = y = e^{2\pi i/a}$, etc.

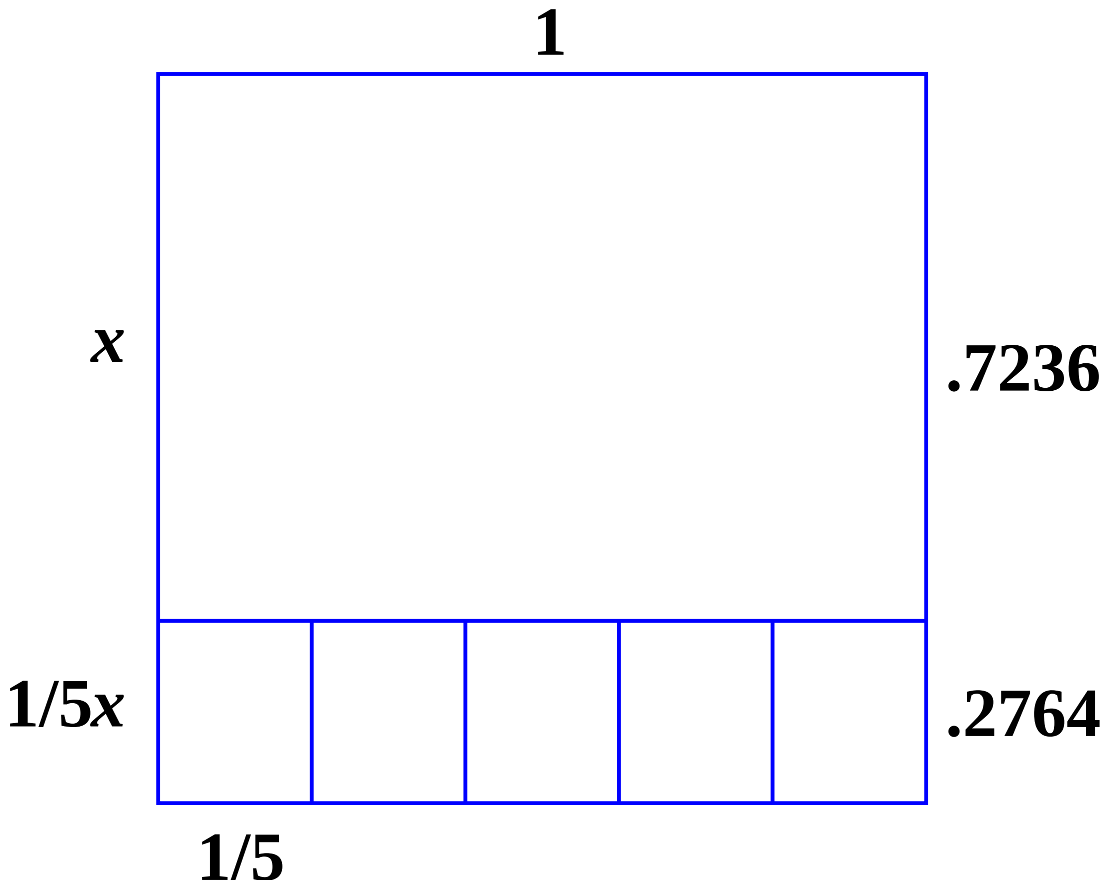
Let $x > 0$, such as $x = \sqrt{2}$. Can a square be tiled with finitely many rectangle **similar** to a $1 \times x$ rectangle (in any orientation)? In other words, can a square be tiled with finitely many rectangles all of the form $a \times ax$ (where a may vary)?





$$x = 2/3$$

$$3x - 2 = 0$$

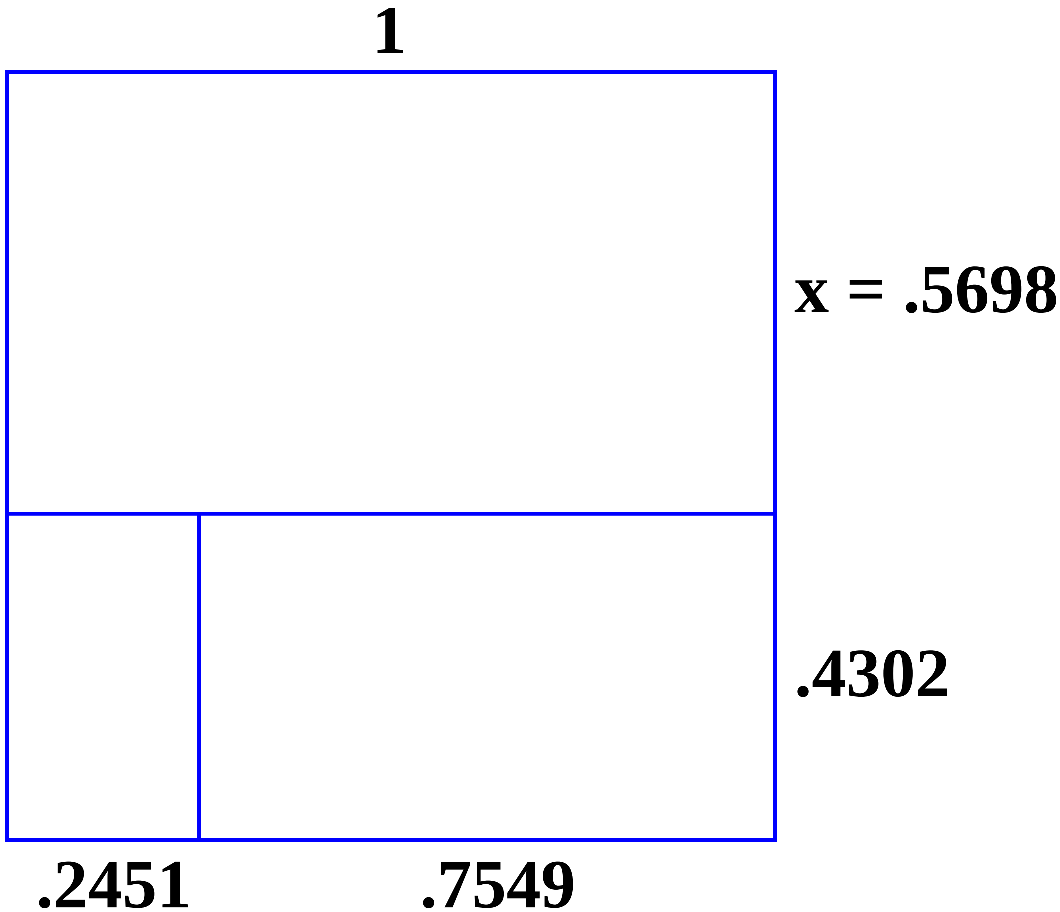


$$x + \frac{1}{5x} = 1$$

$$5x^2 - 5x + 1 = 0$$

$$x = \frac{5 + \sqrt{5}}{10} = 0.7236067977\cdots$$

$$\text{Other root: } \frac{5 - \sqrt{5}}{10} = 0.2763932023\cdots$$



$$x = 0.5698402910\cdots$$
$$x^3 - x^2 + 2x - 1 = 0$$

Other roots:

$$0.215 + 1.307\sqrt{-1}$$
$$0.215 - 1.307\sqrt{-1}$$

Freiling-Rinne (1994), **Laczkovich-Szekeres** (1995): A square can be tiled with finitely many rectangles similar to a $1 \times x$ rectangle if and only if:

- x is the root of a polynomial with integer coefficients.
- If $a + b\sqrt{-1}$ is another root of the polynomial of least degree satisfied by x, then $a > 0$.

Idea of proof. A tiling of a rectangle with similar rectangles can be encoded by a continued fraction.

H. S. Wall (1945): Let

$$\mathbf{F(x)} = x^n + a_1x^{n-1} + \cdots + a_n$$

have real coefficients. Let

$$\mathbf{Q(x)} = a_1x^{n-1} + a_3x^{n-3} + \cdots.$$

Write (uniquely)

$$\frac{Q(x)}{F(x)} = \cfrac{1}{c_1x + 1 + \cfrac{1}{c_2x + \cfrac{1}{\ddots + \cfrac{1}{c_nx}}}}.$$

Then all zeros of $F(x)$ have negative real parts if and only if all $c_i > 0$.

Examples. $x = \boldsymbol{\sqrt{2}}$. Then $x^2 - 2 = 0$. Other root is $-\sqrt{2} < 0$. Thus a square **cannot** be tiled with finitely many rectangles similar to a $1 \times \sqrt{2}$ rectangle.

$x = \boldsymbol{\sqrt{2} + \frac{17}{12}}$. Then

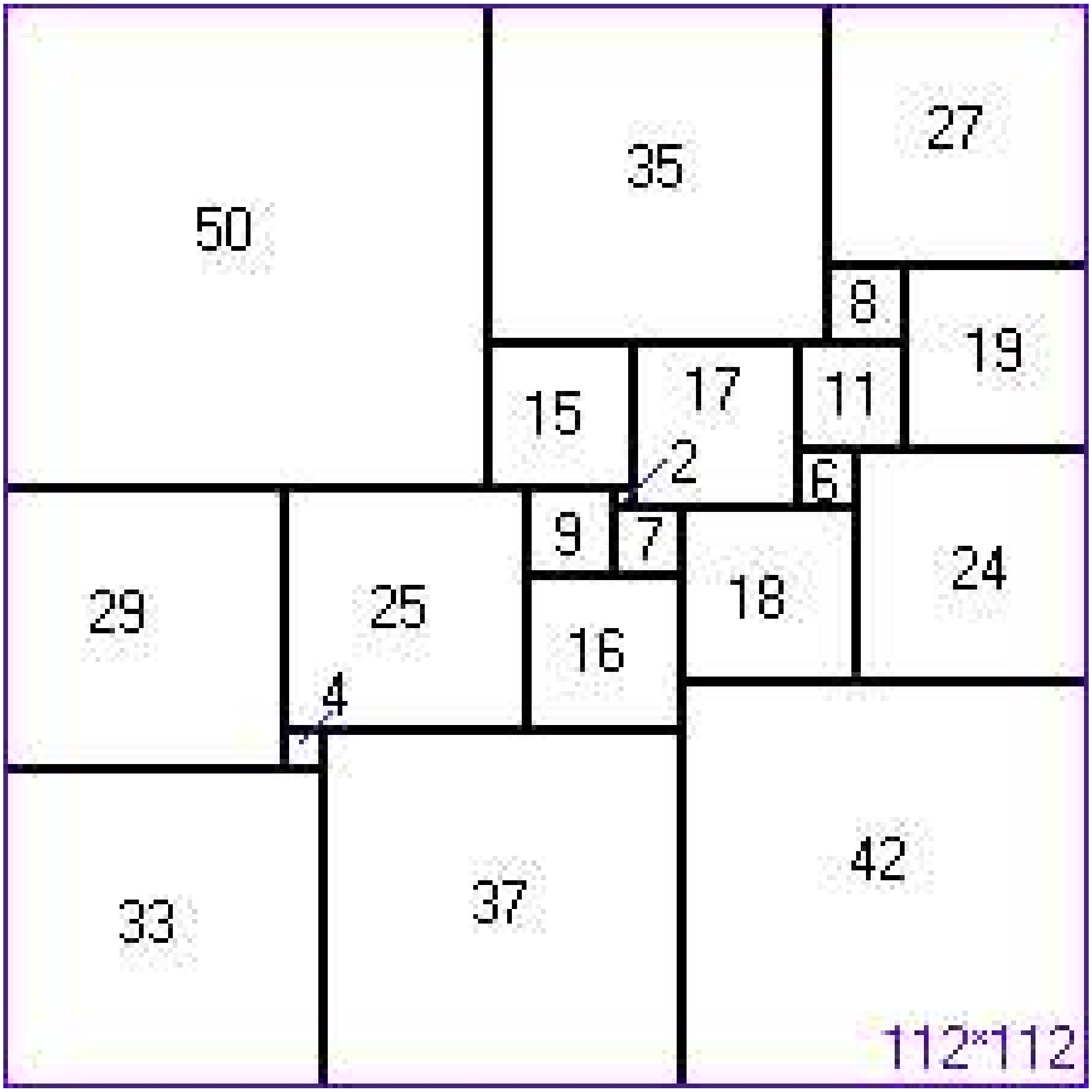
$$144x^2 - 408x + 1 = 0.$$

Other root is

$$-\sqrt{2} + \frac{17}{12} = 0.002453 \cdots > 0,$$

so a square **can** be tiled with finitely many rectangles similar to a $1 \times (\sqrt{2} + \frac{17}{12})$ rectangle.

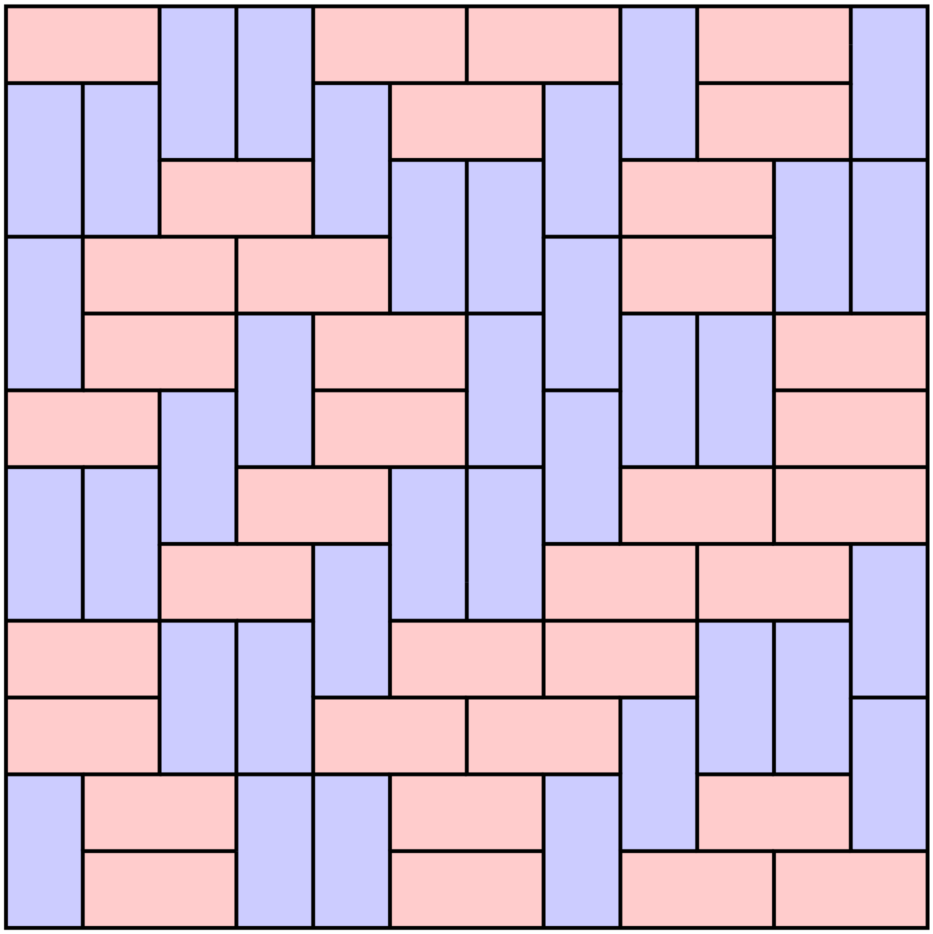
Squaring the square:



(from Wikipedia)

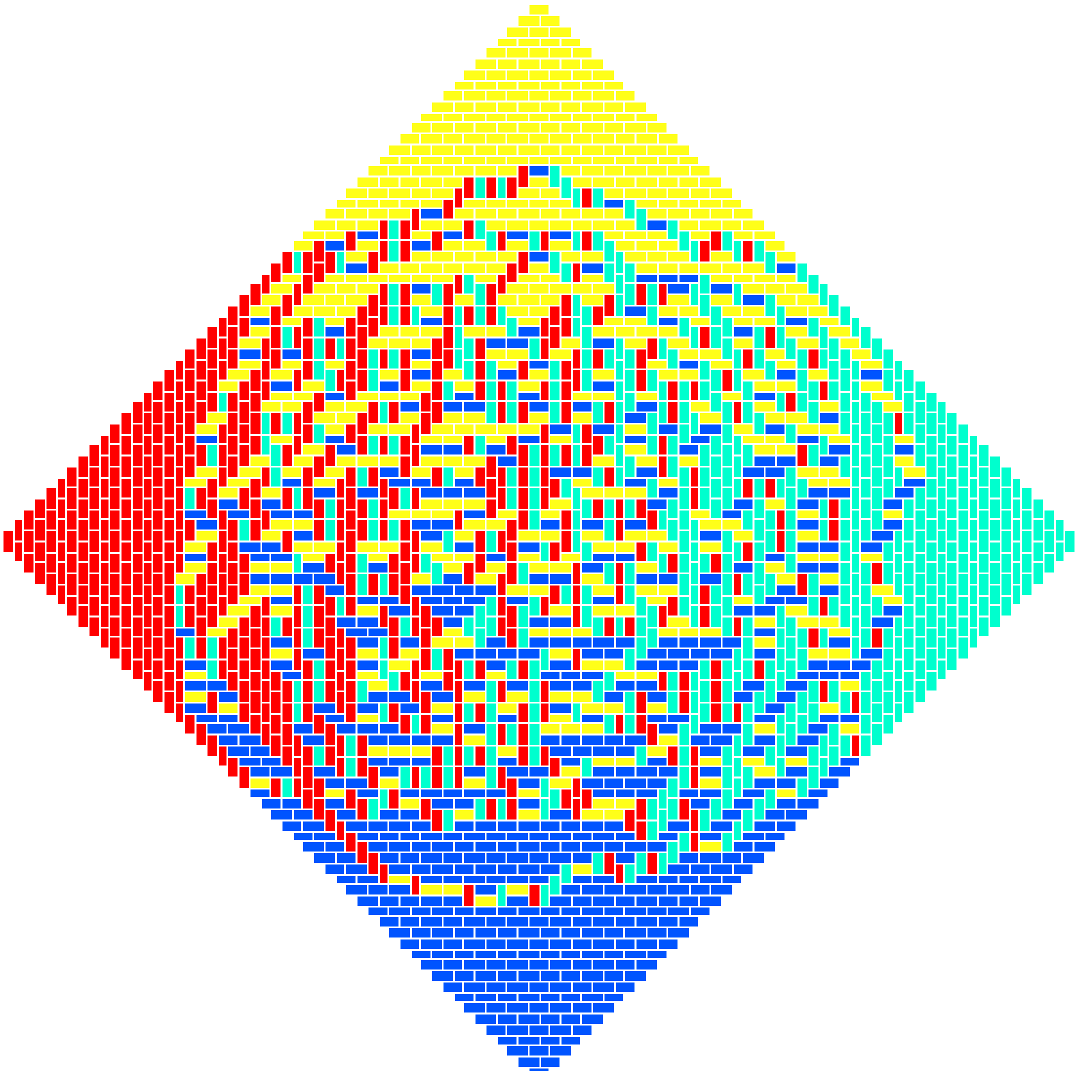
What is a “typical” tiling?

A random domino tiling of a 12×12 square:



No obvious structure.

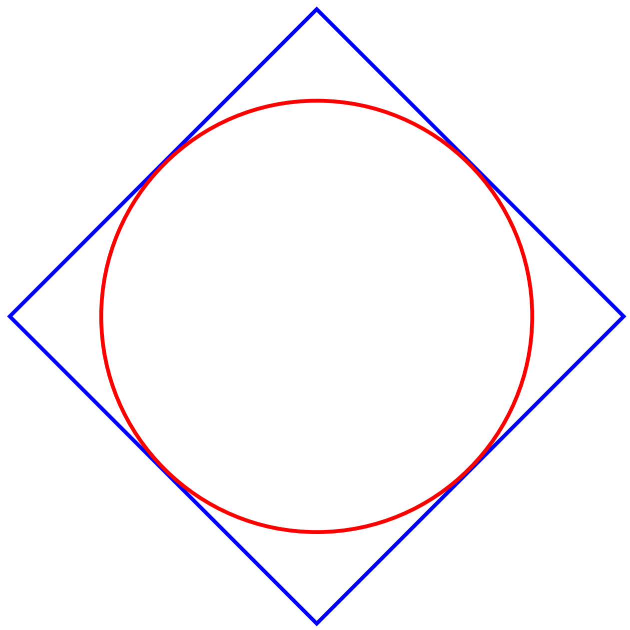
A random tiling of the Aztec diamond of order 50:



“Regular” at the corners, chaotic in the middle.

What is the **region of regularity**?

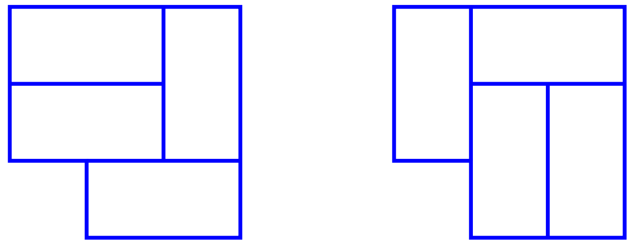
Arctic Circle Theorem (Jockusch-Propp-Shor, 1995). *For very large* n*, and for "most" domino tilings of the Aztec diamond* $\mathrm{AZ}(n)$*, the region of regularity "approaches" the outside of a circle tangent to the four limiting sides of* AZ_n.



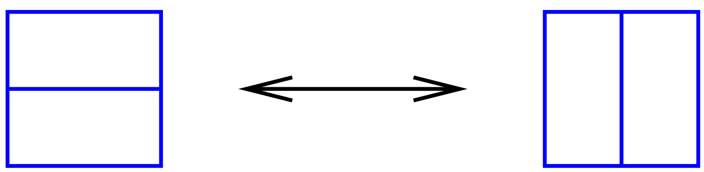
The tangent circle is the **Arctic circle**. Outside this circle the tiling is "frozen."

Relations among tilings

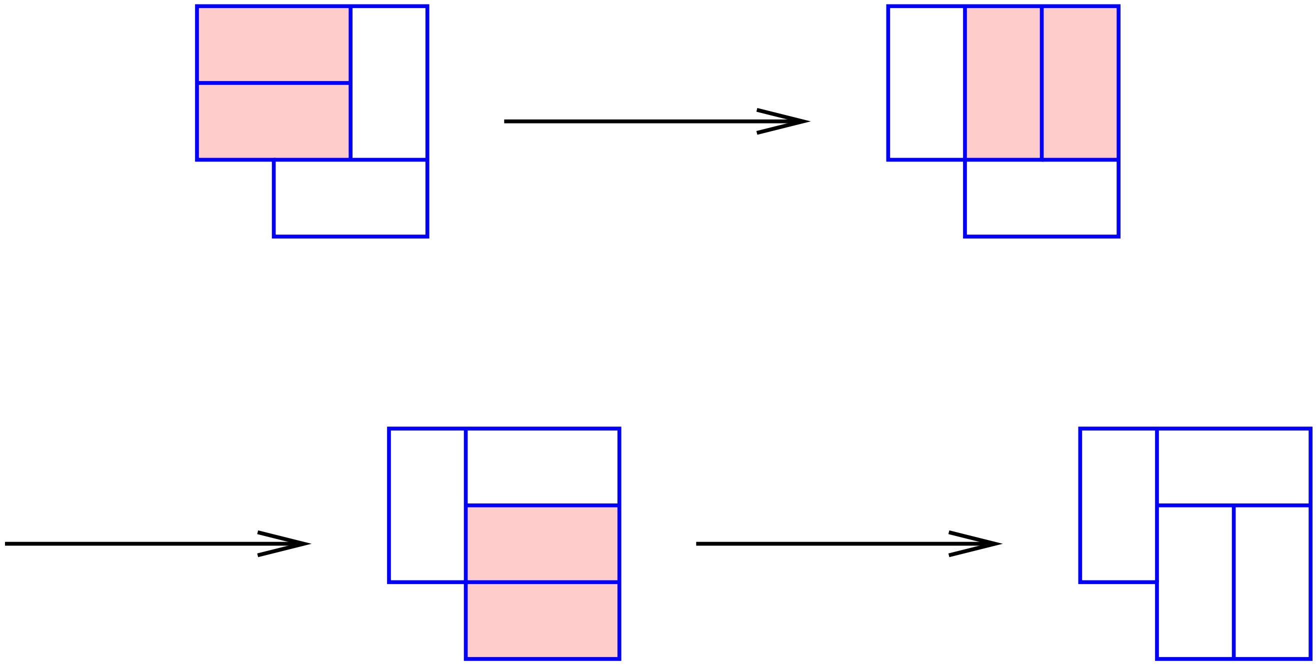
Two domino tilings of a region in the plane:



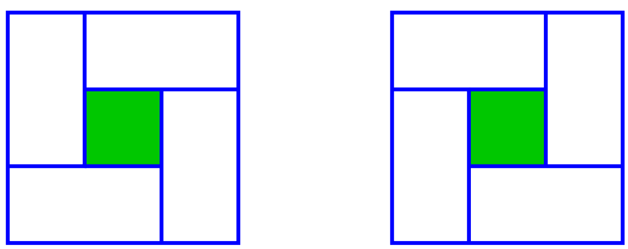
A **flip** consists of reversing the orientation of two dominos forming a 2×2 square.



Domino flipping theorem (Thurston, et al.). *If R has no holes (**simply-connected**), then any domino tiling of R can be reached from any other by a sequence of flips.*



Flipping theorem is **false** if holes are allowed.



Corollary. *Let R be simply-connected. In a tiling by dominos, let*

$$\boldsymbol{A} = \#(\text{horizontal dominos})$$
$$\boldsymbol{B} = \#(\text{vertical dominos})$$

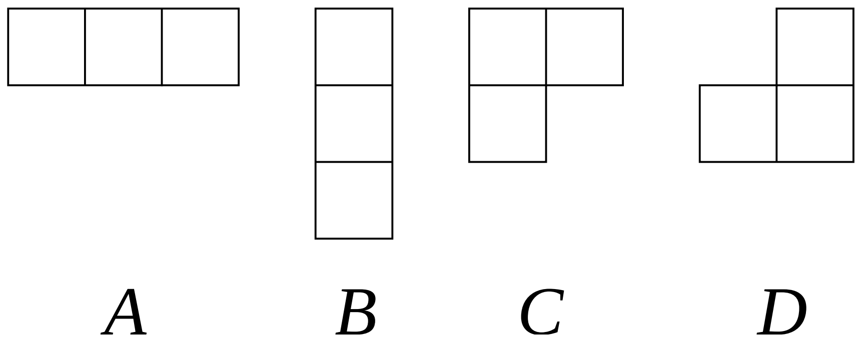
Then

$$A + B \text{ depends only on } R$$

$$A \pmod 2 \text{ depends only on } R.$$

No other (independent) such conditions: **tiling group** is $\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})$.

Let R be simply-connected and tilable using the given number of each of the four tiles (translations only):



Then

$$A + B + C + D \text{ depends only on } R$$

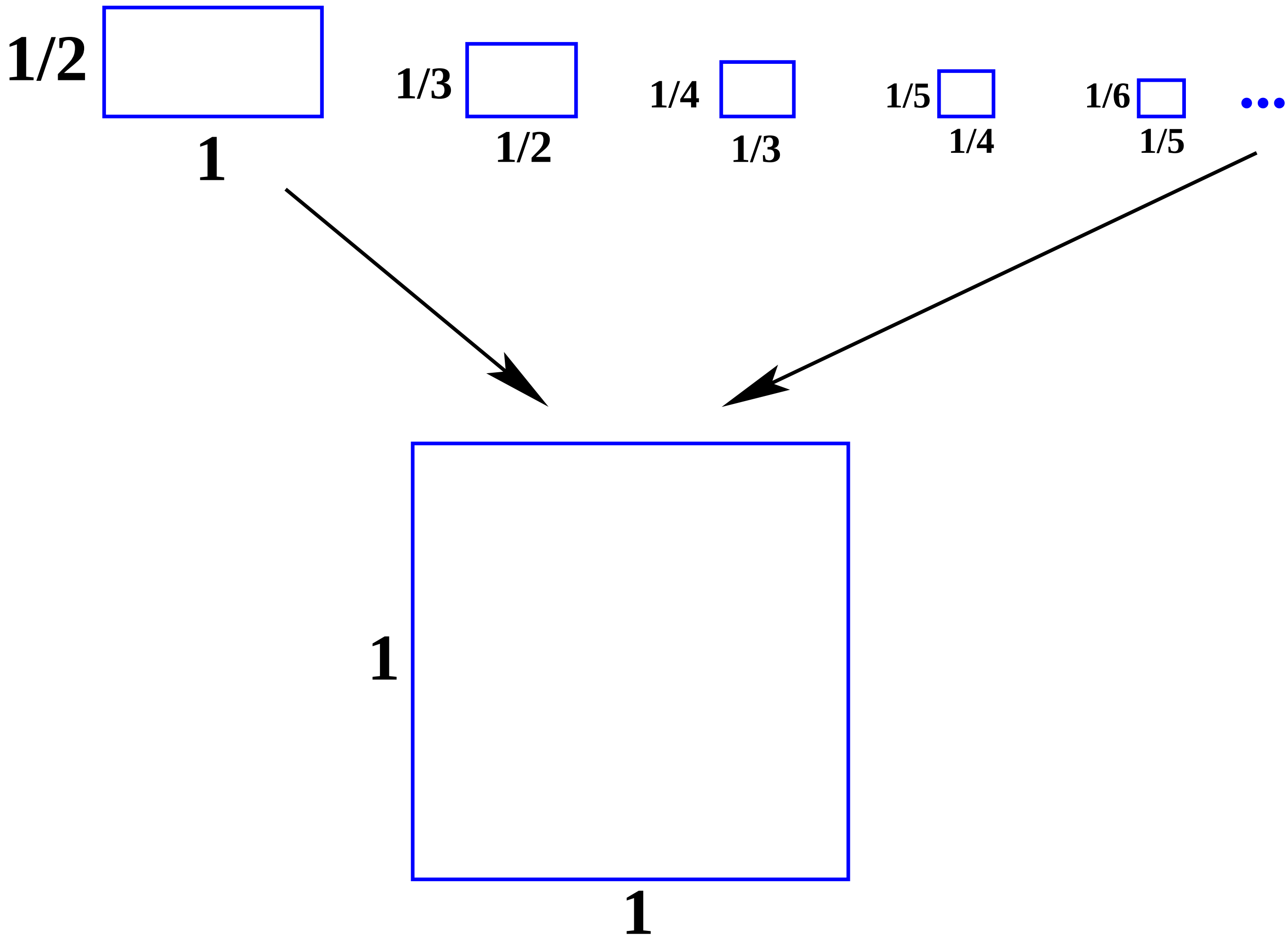
$$C - D \text{ depends only on } R.$$

No other (independent) such conditions: **tiling group** is $\mathbb{Z} \times \mathbb{Z}$.

General theory due to **Igor Pak**.

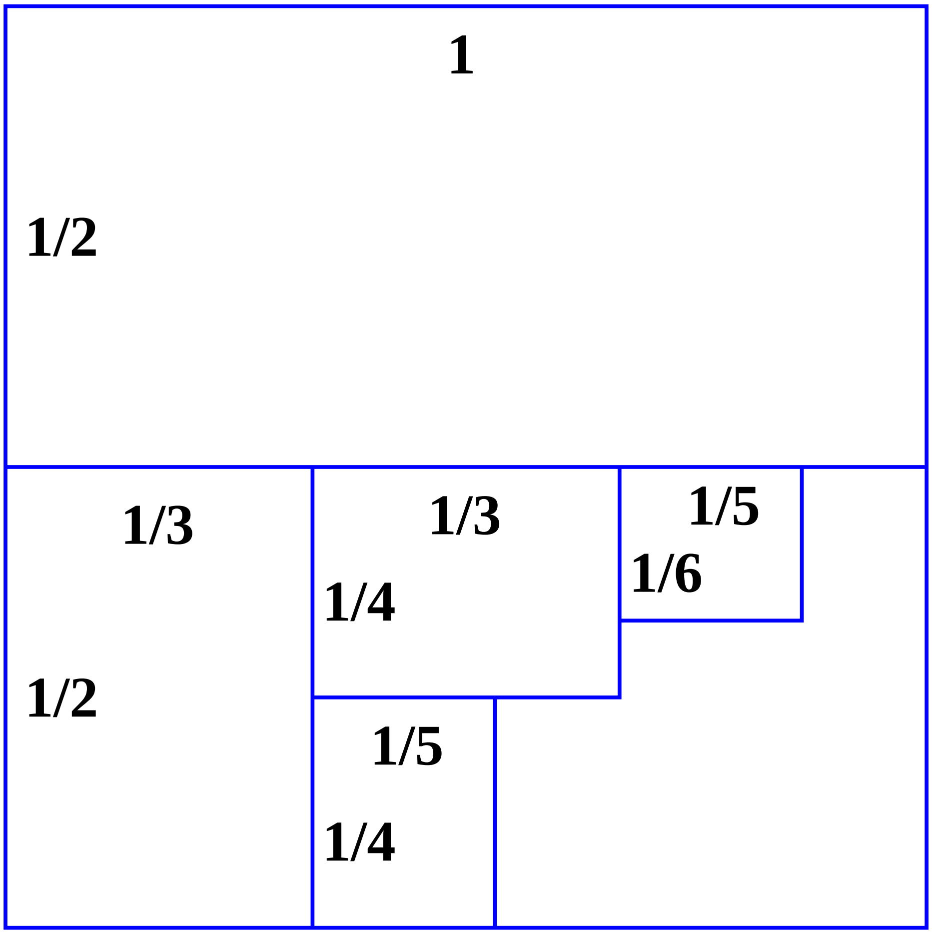
Confronting infinity:

(1) A finite (bounded) region, infinitely many tiles.



Total area: $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots = 1$

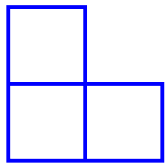
Can a square of side 1 be tiled with squares of sides $1/2, 1/6, 1/12, \ldots$ (once each)?

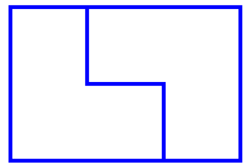


Unsolved, but (Paulhus, 1998) the tiles will fit into a square of side $1 + 10^{-9}$ (not a tiling, since there is leftover space).

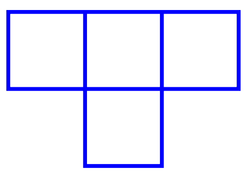
Finitely many tiles, but an indeterminately large region.

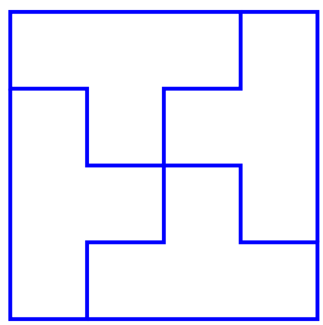
Which polyominos can tile rectangles?





order 2

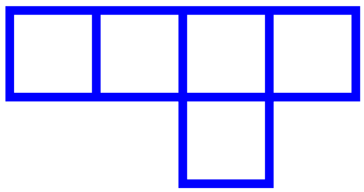




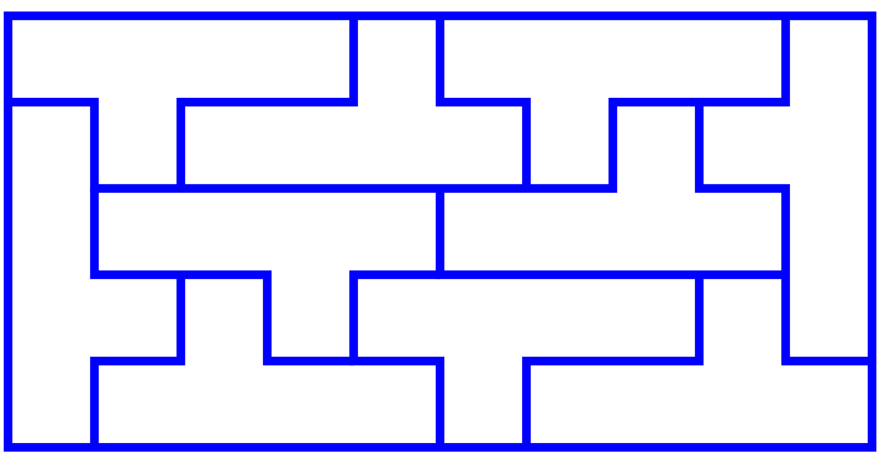
order 4

The **order** of a polyomino is the least number of copies of it needed to tile some rectangle.

No polyomino has order 3.

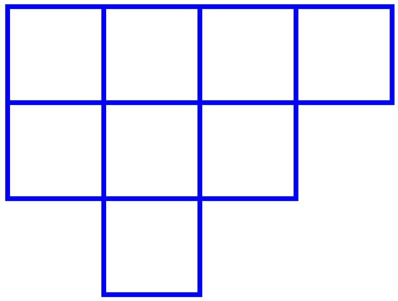


order 10

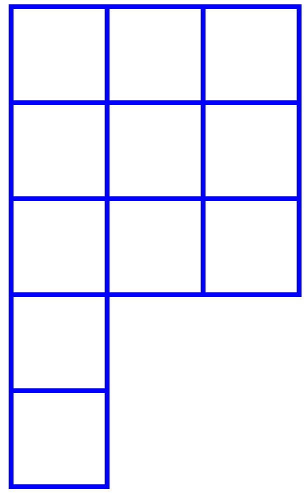


Known orders: $4, 8, 12, 16, \ldots, 4n, \ldots$

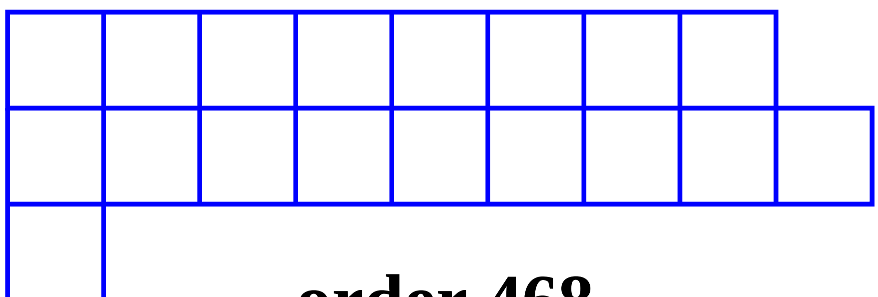
$$1, 2, 10, 18, 50, 138, 246, 270$$



order 246

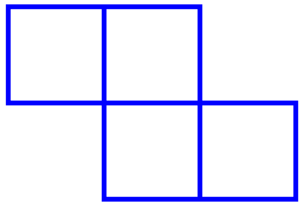


order 270



order 468

Unknown: order 6? odd order?



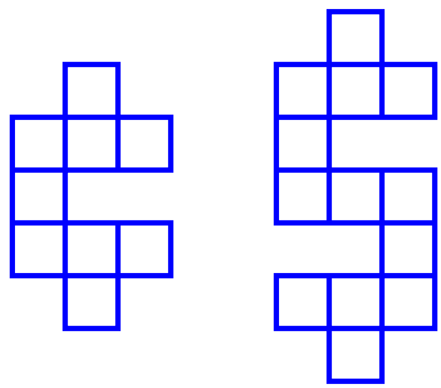
no order

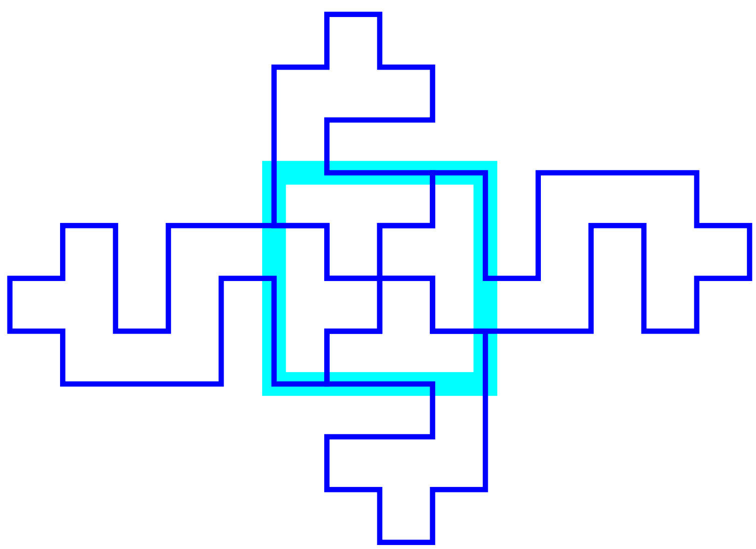
Cannot tile a rectangle (order does not exist).

Deep result from mathematical logic: there does not exist an algorithm (computer program) to decide whether a finite set of polyominos tiles some region **containing** a given square.

Conjecture. There does not exist an algorithm to decide whether a polyomino P tiles **some** rectangle.

Consequence. Let LS(n) be the largest size square that can be **covered** by disjoint copies of some set S of polynominos with a total of n squares, such that LS(n) is finite.





If $f(n)$ is any function that can be computed on a computer (with infinite memory), such as

$$f(n) = n^n, \quad f(n) = n^{n^n}$$

$$f(n) = n^{n^{\cdot^{\cdot^{\cdot^{n}}}}} \ (n\ n\text{'s}), \quad \ldots,$$

then $\mathrm{LS}(n) > f(n)$ for large n. (Otherwise a computer could simply check all possible tilings such that each polyomino in the tiling overlaps a given square of side length k, for all $k \le f(n)$.)

In other words, $\mathrm{LS}(n)$ grows faster than any recursive function.

Confronting infinity: (3) Tiling the plane