

Increasing and decreasing sub-sequences

3 1 8 **4** 9 **6** **7** 2 5 (i.s.)

3 1 **8** **4** 9 6 7 **2** 5 (d.s.)

$$\mathbf{is}(w) = |\text{longest i.s.}| = 4$$

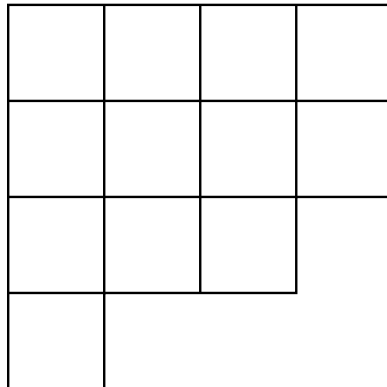
$$\mathbf{ds}(w) = |\text{longest d.s.}| = 3$$

partition $\lambda \vdash n: \lambda = (\lambda_1, \lambda_2, \dots)$

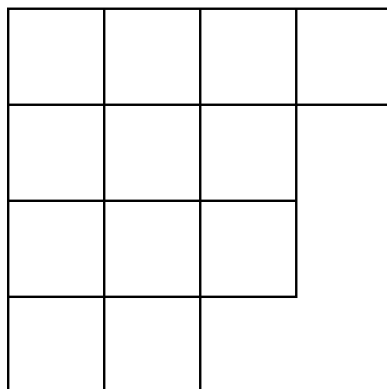
$$\lambda_1 \geq \lambda_2 \geq \dots \geq 0$$

$$\sum \lambda_i = n$$

(Young) diagram of $\lambda = (4, 4, 3, 1)$:



Young diagram of the **conjugate** partition $\lambda' = (4, 3, 3, 2)$:



standard Young tableau (SYT) of shape $\lambda \vdash n$, e.g., $\lambda = (4, 4, 3, 1)$:

$$\begin{array}{c} < \\ \wedge \\ \begin{array}{|c|c|c|c|} \hline 1 & 2 & 7 & 10 \\ \hline 3 & 5 & 8 & 12 \\ \hline 4 & 6 & 11 & \\ \hline 9 & & & \\ \hline \end{array} \end{array}$$

$f^\lambda = \#$ of SYT of shape λ

E.g., $f^{(3,2)} = 5$:

1 2 3	1 2 4	1 2 5	1 3 4	1 3 5
4 5	3 5	3 4	2 5	2 4

\exists simple formula for f^λ (Frame-Robinson-Thrall **hook-length formula**)

Note. $f^\lambda = \dim(\text{irrep. of } \mathfrak{S}_n)$, where \mathfrak{S}_n is the **symmetric group** of all permutations of $1, 2, \dots, n$.

RSK algorithm: a bijection

$$w \xrightarrow{\text{rsk}} (P, Q),$$

where $w \in \mathfrak{S}_n$ and P, Q are SYT of the same shape $\lambda \vdash n$.

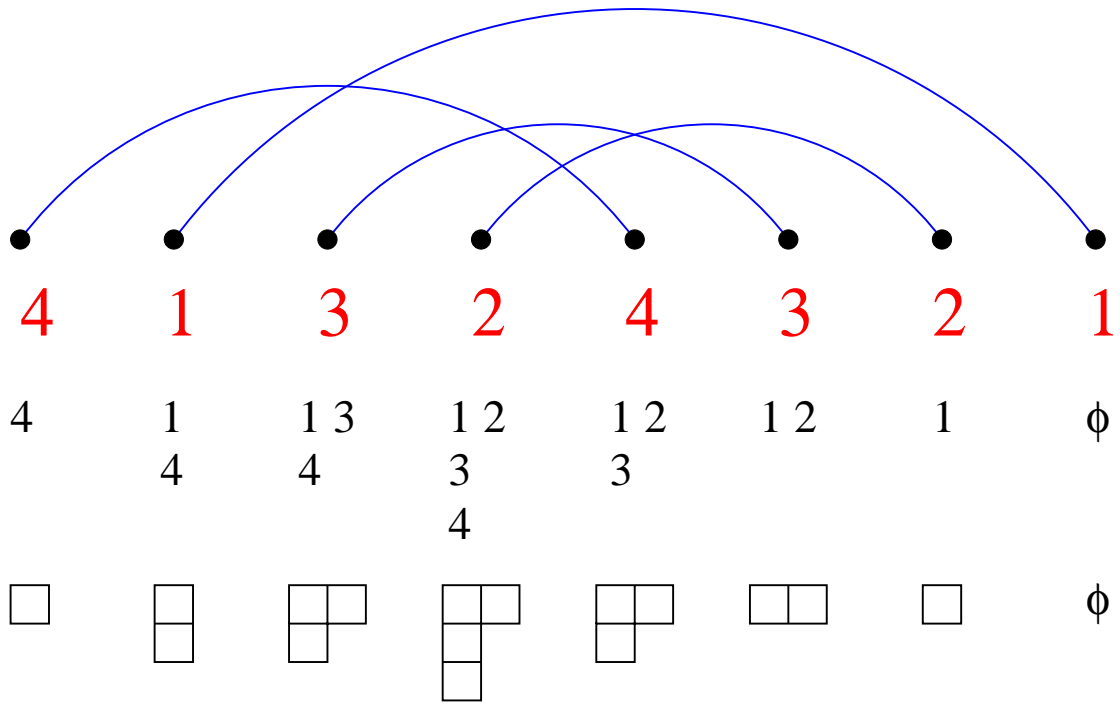
Write $\lambda = \mathbf{sh}(w)$, the **shape** of w .

R = Gilbert de Beauregard Robinson

S = Craige Schensted (= Ea Ea)

K = Donald Ervin Knuth

$w = 4132:$



$$(P, Q) = \begin{pmatrix} 12 & 13 \\ 3 & 2 \\ 4 & 4 \end{pmatrix}$$

Schensted's theorem: Let $w \xrightarrow{\text{rsk}} (P, Q)$, where $\text{sh}(P) = \text{sh}(Q) = \lambda$. Then

$$\text{is}(w) = \text{longest row length} = \lambda_1$$

$$\text{ds}(w) = \text{longest column length} = \lambda'_1.$$

Corollary (Erdős-Szekeres, Seidenberg). Let $w \in \mathfrak{S}_{pq+1}$. Then either $\text{is}(w) > p$ or $\text{ds}(w) > q$.

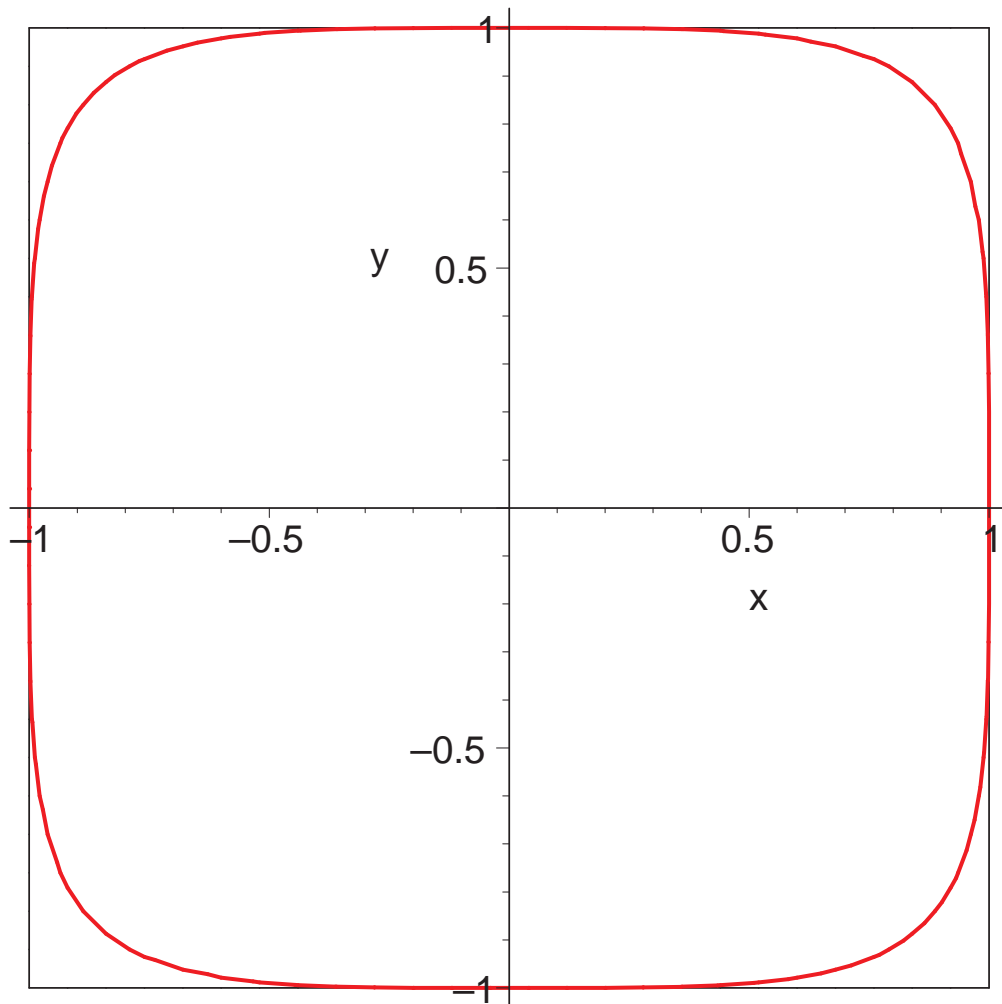
Proof. Let $\lambda = \text{sh}(w)$. If $\text{is}(w) \leq p$ and $\text{ds}(w) \leq q$ then $\lambda_1 \leq p$ and $\lambda'_1 \leq q$, so $\sum \lambda_i \leq pq$. \square

Corollary. *Say $p \leq q$. Then*

$$\begin{aligned} \#\{w \in \mathfrak{S}_{pq} : \text{is}(w) = p, \text{ds}(w) = q\} \\ = \left(f^{(p^q)}\right)^2 \end{aligned}$$

By hook-length formula, this is

$$\left(\frac{(pq)!}{1^1 2^2 \cdots p^p (p+1)^p \cdots q^p (q+1)^{p-1} \cdots (p+q-1)^1} \right)^2.$$



$$(x^2 - y^2)^2 + 2(x^2 + y^2) = 3$$

Area enclosed by curve:

$$\begin{aligned}\alpha &= 2 \int_0^1 \frac{1}{\sqrt{(1-t^2)(1-(t/3)^2)}} dt \\ &\quad - \frac{3}{2} \int_0^1 \sqrt{\frac{1-(t/3)^2}{1-t^2}} dt \\ &= 4(0.94545962 \dots)\end{aligned}$$

Distribution of $\text{is}(w)$

$$\begin{aligned} \mathbf{E}(n) &= \text{expectation of } \text{is}(w), \quad w \in \mathfrak{S}_n \\ &= \frac{1}{n!} \sum_{\lambda \vdash n} \lambda_1 \left(f^\lambda \right)^2 \end{aligned}$$

Ulam: what is distribution of $\text{is}(w)$?
rate of growth of $E(n)$?

Hammersley (1972):

$$\exists c = \lim_{n \rightarrow \infty} n^{-1/2} E(n),$$

and

$$\frac{\pi}{2} \leq c \leq e.$$

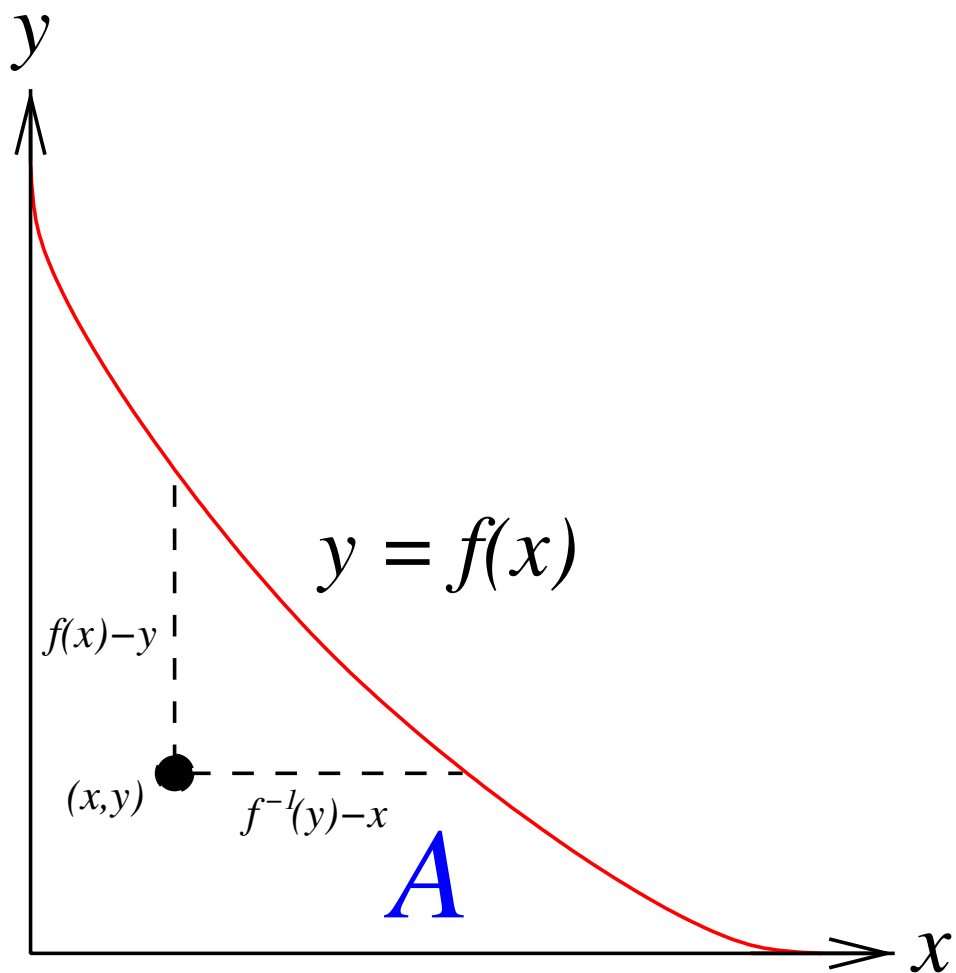
Conjectured $c = 2$.

Logan-Shepp, Vershik-Kerov (1977):
 $c = 2$

Idea of proof.

$$\begin{aligned} E(n) &= \frac{1}{n!} \sum_{\lambda \vdash n} \lambda_1 \left(f^\lambda \right)^2 \\ &\approx \frac{1}{n!} \max_{\lambda \vdash n} \lambda_1 \left(f^\lambda \right)^2. \end{aligned}$$

Find “limiting shape” of $\lambda \vdash n$ maximizing λ as $n \rightarrow \infty$ using hook-length formula.



$$\min \iint_A \log(f(x) + f^{-1}(y) - x - y) dx dy,$$

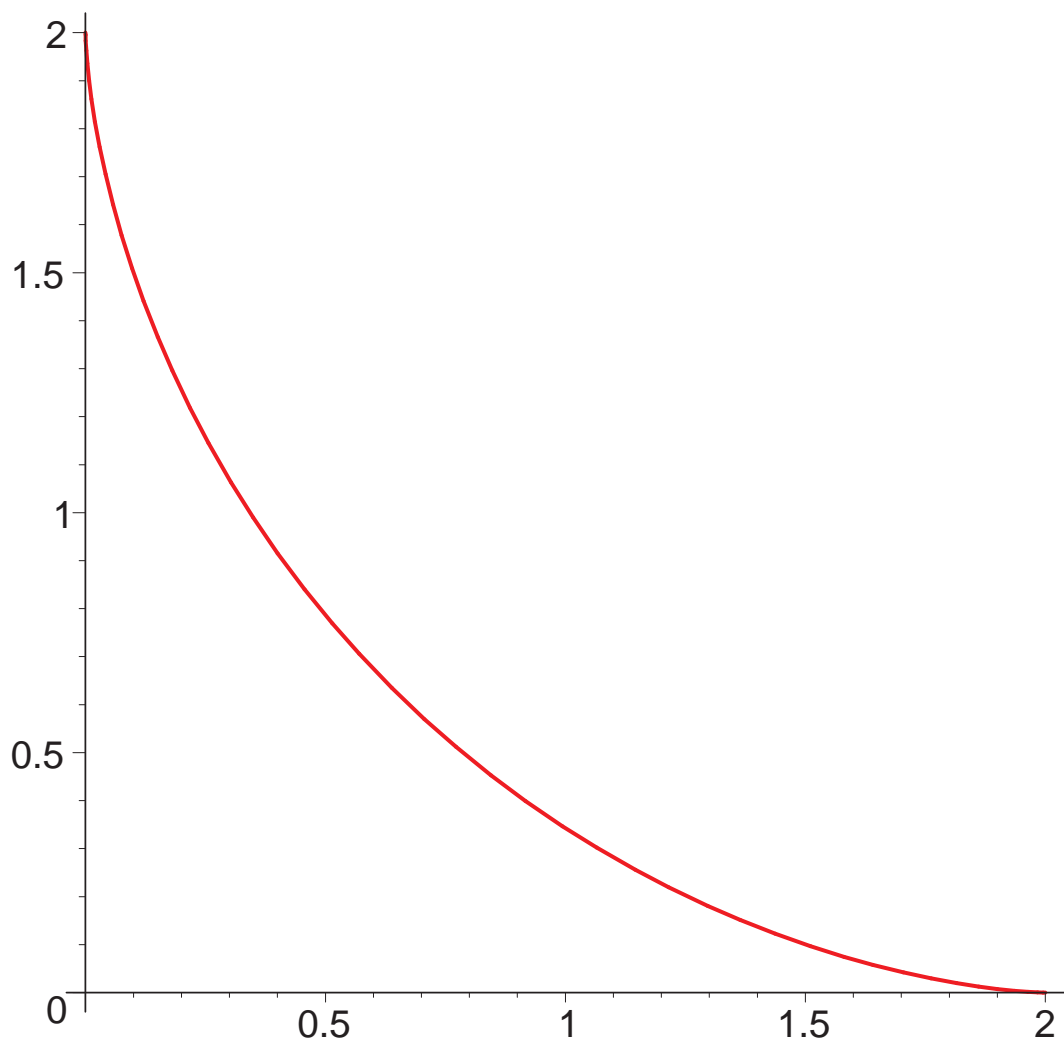
subject to

$$\iint_A dx dy = 1.$$

$$x = y + 2 \cos \theta$$

$$y = \frac{2}{\pi}(\sin \theta - \theta \cos \theta)$$

$$0 \leq \theta \leq \pi$$



$$u_k(n) := \#\{w \in \mathfrak{S}_n : \text{is}_n(w) \leq k\}.$$

J. M. Hammersley (1972):

$$u_2(n) = C_n = \frac{1}{n+1} \binom{2n}{n},$$

a **Catalan number**.

For ≥ 130 combinatorial interpretations of C_n , see

`www-math.mit.edu/~rstan/ec`

I. Gessel (1990):

$$\sum_{n \geq 0} u_k(n) \frac{x^{2n}}{n!^2} = \det \left[I_{|i-j|}(2x) \right]_{i,j=1}^k,$$

where

$$I_m(2x) = \sum_{j \geq 0} \frac{x^{m+2j}}{j!(m+j)!},$$

a **hyperbolic Bessel function** of the first kind of order m .

E.g.,

$$\begin{aligned} \sum_{n \geq 0} u_2(n) \frac{x^{2n}}{n!^2} &= U_0(2x)^2 - U_1(2x)^2 \\ &= \sum_{n \geq 0} C_n \frac{x^{2n}}{n!^2}. \end{aligned}$$

Corollary. For fixed k , $u_k(n)$ is **P-recursive**, e.g.,

$$\begin{aligned} & (n+4)(n+3)^2 u_4(n) \\ = & (20n^3 + 62n^2 + 22n - 24)u_4(n-1) \\ & - 64n(n-1)^2 u_4(n-2) \end{aligned}$$

$$\begin{aligned} & (n+6)^2(n+4)^2 u_5(n) \\ = & (375 - 400n - 843n^2 - 322n^3 - 35n^4)u_5(n-1) \\ & + (259n^2 + 622n + 45)(n-1)^2 u_5(n-2) \\ & - 225(n-1)^2(n-2)^2 u_5(n-3). \end{aligned}$$

Conjectures on form of recurrence due to Bergeron, Favreau, and Krob.

Baik-Deift-Johansson:

Define $u(x)$ by

$$\frac{d^2}{dx^2}u(x) = 2u(x)^3 + xu(x) \quad (*),$$

with certain initial conditions.

$(*)$ is the **Painlevé II** equation (roughly, the branch points and essential singularities are independent of the initial conditions).

Paul Painlevé

1863: born in Paris.

1890: Grand Prix des Sciences Mathématiques

1908: first passenger of Wilbur Wright;
set flight duration record of one hour, 10
minutes.

1917, 1925: Prime Minister of France.

1933: died in Paris.

Tracy-Widom distribution:

$$F(t)$$

$$= \exp \left(- \int_t^\infty (x - t) u(x)^2 dx \right)$$

Theorem (Baik-Deift-Johansson) *For random (uniform) $w \in \mathfrak{S}_n$ and all $t \in \mathbb{R}$ we have*

$$\lim_{n \rightarrow \infty} \text{Prob} \left(\frac{\text{is}_n(w) - 2\sqrt{n}}{n^{1/6}} \leq t \right) = F(t).$$

Corollary.

$$\begin{aligned} \text{is}_n(w) &= 2\sqrt{n} + \left(\int t dF(t) \right) n^{1/6} + o(n^{1/6}) \\ &= 2\sqrt{n} - (1.7711 \dots) n^{1/6} + o(n^{1/6}) \end{aligned}$$

Gessel's theorem reduces the problem to “just” analysis, viz., the **Riemann-Hilbert problem** in the theory of integrable systems, and the **method of steepest descent** to analyze the asymptotic behavior of integrable systems.

Where did the Tracy-Widom distribution $F(t)$ come from?

$$F(t) = \exp \left(- \int_t^\infty (x - t)u(x)^2 dx \right)$$

$$\frac{d^2}{dx^2}u(x) = 2u(x)^3 + xu(x) \quad (*),$$

Gaussian Unitary Ensemble (GUE):

Consider an $n \times n$ hermitian matrix $\mathbf{M} = (M_{ij})$ with probability distribution

$$Z_n^{-1} e^{-\text{tr}(M^2)} dM,$$

$$dM = \prod_i dM_{ii} \cdot \prod_{i < j} d(\text{Re}(M_{ij})) d(\text{Im}(M_{ij})),$$

where Z_n is a normalization constant.

Tracy-Widom (1994): let α_1 denote the largest eigenvalue of M . Then

$$\lim_{n \rightarrow \infty} \text{Prob} \left(\left(\alpha_1 - \sqrt{2n} \right) \sqrt{2n}^{1/6} \leq t \right) = F(t).$$

Is the connection between $\text{is}(w)$ and GUE a coincidence?

Okounkov provides a connection, via the theory of **random topologies on surfaces**. Very briefly, a surface can be described in two ways:

- Gluing polygons along their edges, connected to random matrices via quantum gravity.
- Ramified covering of a sphere, which can be formulated in terms of permutations.

Symmetry.

$$\mathfrak{I}_n = \{w \in \mathfrak{S}_n : w^2 = 1\}$$

$$\mathfrak{I}_{2n}^* = \{w \in \mathfrak{I}_n : w(i) \neq i \forall i\}.$$

$$\#\mathfrak{I}_{2n}^* = (2n - 1)!! = 1 \cdot 3 \cdot 5 \cdots (2n - 1)$$

Theorem (Baik-Rains). (a) *We have for random (uniform) $w \in \mathfrak{I}_n$ and all $t \in \mathbb{R}$ that*

$$\begin{aligned} \lim_{n \rightarrow \infty} \text{Prob} \left(\frac{\text{is}_n(w) - 2\sqrt{n}}{n^{1/6}} \leq t \right) \\ = F(t)^{1/2} \exp \left(\frac{1}{2} \int_t^\infty u(s) ds \right), \end{aligned}$$

where $F(t)$ denotes the Tracy-Widom distribution and $u(s)$ the Painlevé II function. (By symmetry we can replace $\text{is}(w)$ with $\text{ds}(w)$.)

(b) *We have for random (uniform) $w \in \mathfrak{J}_{2n}^*$ and all $t \in \mathbb{R}$ that*

$$\begin{aligned} & \lim_{n \rightarrow \infty} \text{Prob} \left(\frac{\text{ds}_{2n}(w) - 2\sqrt{2n}}{(2n)^{1/6}} \leq t \right) \\ &= F(t)^{1/2} \exp \left(\frac{1}{2} \int_t^\infty u(s) ds \right) \end{aligned}$$

(same up to scaling as largest eigenvalue of a real symmetric matrix from GOE model).

(c) *We have for random (uniform) $w \in \mathfrak{J}_{2n}^*$ and all $t \in \mathbb{R}$ that*

$$\begin{aligned} & \lim_{n \rightarrow \infty} \text{Prob} \left(\frac{\text{is}_{2n}(w) - 2\sqrt{2n}}{(2n)^{1/6}} \leq t \right) \\ &= F(t)^{1/2} \cosh \left(\frac{1}{2} \int_t^\infty u(s) ds \right) \end{aligned}$$

(same up to scaling as largest eigenvalue of a real skew-symmetric matrix from GSE model).

Pattern Avoidance

$$\begin{aligned} \mathbf{v} &= b_1 \cdots b_k \in \mathfrak{S}_k \\ \mathbf{w} &= a_1 \cdots a_n \in \mathfrak{S}_n \end{aligned}$$

w **avoids** v if no subsequence $a_{i_1} \cdots a_{i_k}$ of w is in the same relative order as v .

3 **5 2** 9 6 **8** 1 **4** 7 does **not** avoid 3142.

w has no increasing (decreasing) subsequence of length $k \Leftrightarrow w$ avoids $12 \cdots k$ ($k \cdots 21$).

Let $v \in \mathfrak{S}_k$. Define

$$\mathfrak{S}_n(v) = \{w \in \mathfrak{S}_n : w \text{ avoids } v\}$$

$$s_n(v) = \#\mathfrak{S}_n(v).$$

Hammersley-Knuth-Rotem:

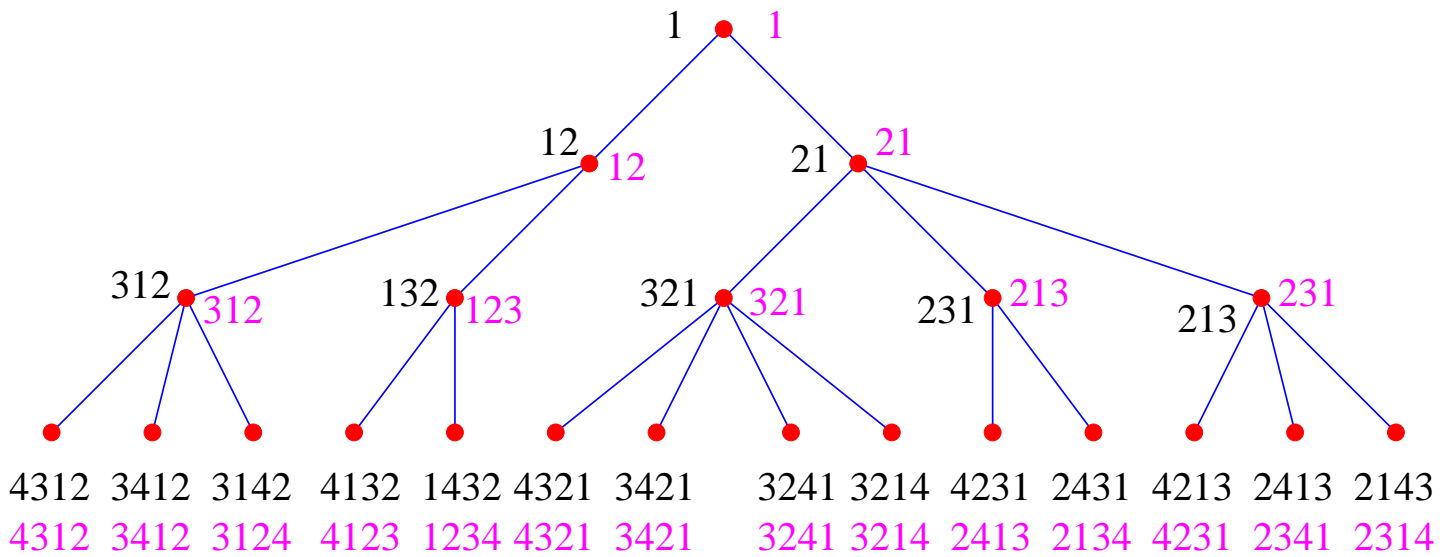
$$s_n(123) = s_n(321) = C_n.$$

Knuth:

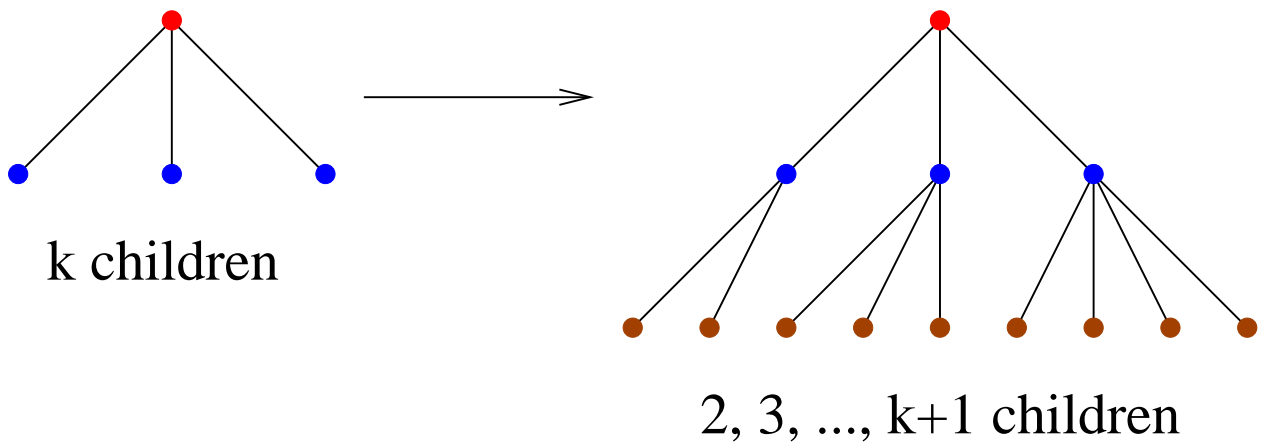
$$s_n(132) = s_n(213) = s_n(231) = s_n(312) = C_n.$$

Method of **generating trees** (Chung-Graham-Hoggatt-Kleiman, West): define $u \leq v$ if u is a **subsequence** of v .

$$3142 \leq 8\mathbf{3}5\mathbf{1}96\mathbf{4}\mathbf{2}7$$



black: 123-avoiding
 magenta: 132-avoiding



Define $u \rightsquigarrow v$ if $s_n(u) = s_n(v)$ for all n .

One equivalence class for $k = 3$.

Three equivalence classes for $k = 4$.

Gessel: $s_n(1234) =$

$$\frac{1}{(n+1)^2(n+2)} \sum_{j=0}^n \binom{2j}{j} \binom{n+1}{j+1} \binom{n+2}{j+2}$$

Bóna:

$$\sum_{n \geq 0} s_n(1342)x^n = \frac{32x}{1 + 20x - 8x^2 - (1 - 8x)^{3/2}},$$

Open: $s_n(1324)$

Typical application (**Ryan, Lakshmibai-Sandhya, Haiman**): Let $w \in \mathfrak{S}_n$. The Schubert variety Ω_w in the complete flag variety $\mathrm{GL}(n, \mathbb{C})$ is smooth if and only if w avoids 4231 and 3412.

$$\sum_{n \geq 0} f(n)x^n = \frac{1}{1 - x - \frac{x^2}{1-x} \left(\frac{2x}{1+x-(1-x)C(x)} - 1 \right)},$$

where

$$C(x) = \sum_{n \geq 0} C_n x^n = \frac{1 - \sqrt{1 - 4x}}{2x}.$$

Joint with:

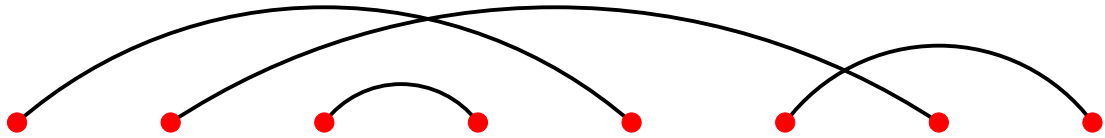
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(complete) matching:



crossing:



nesting:



total number of matchings on $[2n] := \{1, 2, \dots, 2n\}$ is

$$(2n - 1)!! := 1 \cdot 3 \cdot 5 \cdots (2n - 1).$$

Theorem. *The number of matchings on $[2n]$ with no crossings (or with no nestings) is*

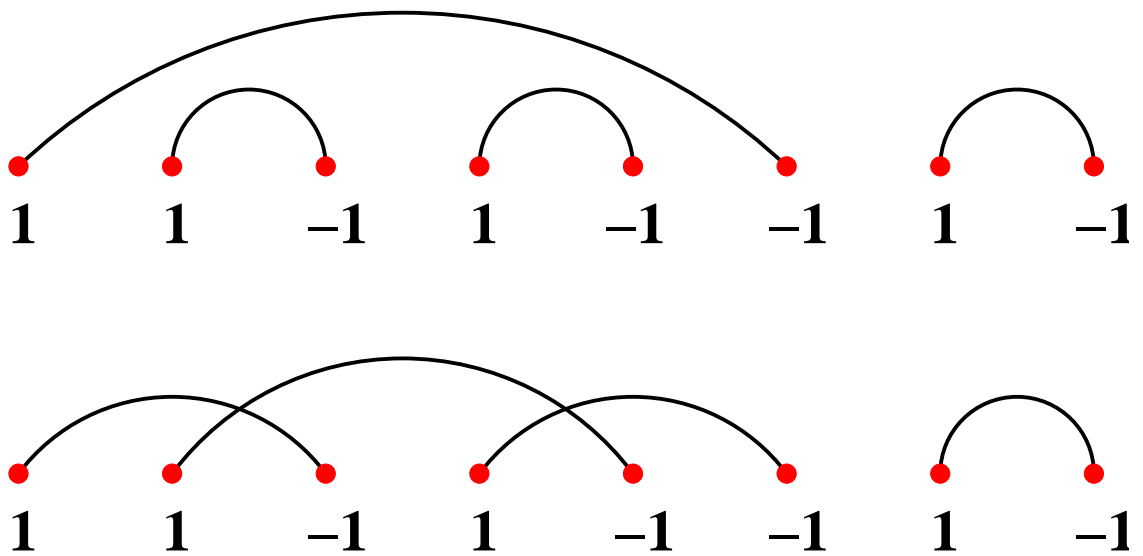
$$C_n := \frac{1}{n + 1} \binom{2n}{n}.$$

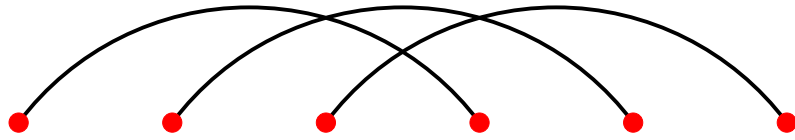
Well-known:

$$C_n = \#\{a_1 \cdots a_{2n} : a_i = \pm 1,$$

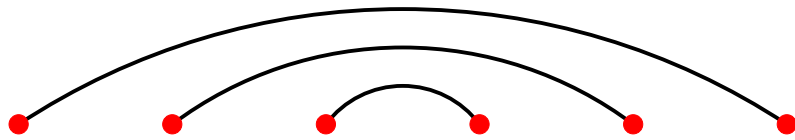
$$a_1 + \cdots + a_i \geq 0, \sum a_i = 0\}$$

(**ballot sequence**).





3-crossing



3-nesting

M = matching

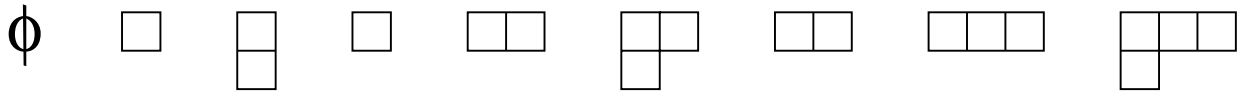
$\mathbf{cr}(M) = \max\{k : \exists k\text{-crossing}\}$

$\mathbf{ne}(M) = \max\{k : \exists k\text{-nesting}\}.$

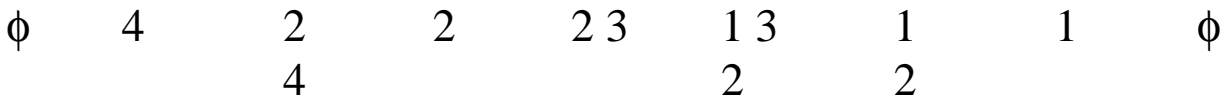
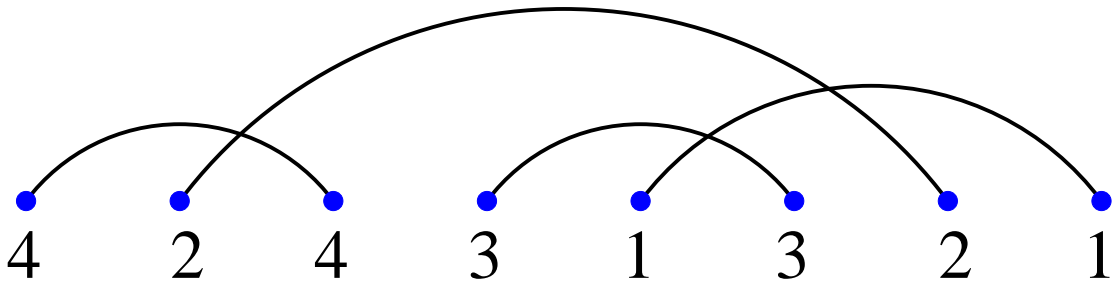
Theorem. Let $f_n(i, j) = \#$ matchings M on $[2n]$ with $\mathbf{cr}(M) = i$ and $\mathbf{ne}(M) = j$. Then $\mathbf{f}_n(i, j) = \mathbf{f}_n(j, i)$.

Corollary. $\#$ matchings M on $[2n]$ with $\mathbf{cr}(M) = k$ equals $\#$ matchings M on $[2n]$ with $\mathbf{ne}(M) = k$.

Main tool: oscillating tableaux.



shape $(3, 1)$, length 8



$\Phi(M) = (\phi \square \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array} \square \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array} \square \phi)$

Φ is a bijection from matchings on $1, 2, \dots, 2n$ to oscillating tableaux of length $2n$, shape \emptyset .

$$\tilde{f}_n^\lambda := \#\{\text{osc. tab. of shape } \lambda, \text{ length } n\}$$

Corollary.

$$\sum_{\lambda} \left(\tilde{f}_n^\lambda\right)^2 = (2n - 1)!!$$

Proof. Number of oscillating tableaux

$$(\emptyset = \lambda^0, \lambda^1, \dots, \lambda^{2n} = \emptyset)$$

of length $2n$, shape \emptyset , and with $\lambda^n = \lambda$ is $\left(\tilde{f}_n^\lambda\right)^2$. Sum on all λ to get the total number of matchings on $[2n]$, viz., $(2n - 1)!!$. \square

Brauer algebra \mathcal{B}_n : a complex semisimple algebra (depending on a parameter x) of dimension $(2n - 1)!!$.

Dimensions of irreducible representations of \mathcal{B}_n : \tilde{f}_n^λ , confirming

$$\sum_{\lambda} \left(\tilde{f}_n^\lambda \right)^2 = (2n - 1)!!$$

Compare

$$\sum_{\lambda \vdash n} \left(f^\lambda \right)^2 = n!$$

Schensted's theorem for matchings. *Let*

$$\Phi(M) = (\emptyset = \lambda^0, \lambda^1, \dots, \lambda^{2n} = \emptyset).$$

Then

$$\begin{aligned} \text{cr}(M) &= \max\{(\lambda^i)'_1 : 0 \leq i \leq n\} \\ \text{ne}(M) &= \max\{\lambda^i_1 : 0 \leq i \leq n\}. \end{aligned}$$

Proof. Reduce to ordinary RSK.

Now let $\text{cr}(M) = i$, $\text{ne}(M) = j$, and
 $\Phi(M) = (\emptyset = \lambda^0, \lambda^1, \dots, \lambda^{2n} = \emptyset)$.

Define M' by

$$\Phi(M') = (\emptyset = (\lambda^0)', (\lambda^1)', \dots, (\lambda^{2n})' = \emptyset).$$

By Schensted's theorem for matchings,

$$\text{cr}(M') = j, \quad \text{ne}(M') = i.$$

Thus $M \mapsto M'$ is an involution on matchings of $[2n]$ interchanging cr and ne .

\Rightarrow **Theorem.** *Let $f_n(i, j) = \#$ matchings M on $[2n]$ with $\text{cr}(M) = i$ and $\text{ne}(M) = j$. Then $f_n(i, j) = f_n(j, i)$.*

Open: simple description of $M \mapsto M'$, the analogue of

$$a_1 a_2 \cdots a_n \mapsto a_n \cdots a_2 a_1,$$

which interchanges is and ds .

Enumeration of k -noncrossing matchings (or nestings).

Recall: The number of matchings M on $[2n]$ with no crossings, i.e., $\text{cr}(M) = 1$, (or with no nestings) is $C_n = \frac{1}{n+1} \binom{2n}{n}$.

What about the number with $\text{cr}(M) \leq k$?

Assume $\text{cr}(M) \leq k$. Let

$$\Phi(M) = (\emptyset = \lambda^0, \lambda^1, \dots, \lambda^{2n} = \emptyset).$$

Regard each $\lambda^i = (\lambda_1^i, \dots, \lambda_k^i) \in \mathbb{N}^k$.

Corollary. *The number $f_k(\mathbf{n})$ of matchings M on $[2n]$ with $\text{cr}(M) \leq k$ is the number of lattice paths of length $2n$ from $\mathbf{0}$ to $\mathbf{0}$ in the region*

$$\mathcal{C}_n := \{(a_1, \dots, a_k) \in \mathbb{N}^k : a_1 \leq \dots \leq a_k\}$$

with steps $\pm e_i$ ($e_i = i$ th unit coordinate vector).

$\mathcal{C}_n \otimes \mathbb{R}_{\geq 0}$ is a fundamental chamber for the Weyl group of type B_k .

Grabiner-Magyar: applied **Gessel-Zeilberger reflection principle** to solve this lattice path problem (not knowing connection with matchings).

Theorem. *Define*

$$\mathbf{H}_k(\mathbf{x}) = \sum_n f_k(n) \frac{x^{2n}}{(2n)!}.$$

Then

$$H_k(x) = \det \left[I_{|i-j|}(2x) - I_{i+j}(2x) \right]_{i,j=1}^k$$

where

$$I_m(2x) = \sum_{j \geq 0} \frac{x^{m+2j}}{j!(m+j)!}$$

as before.

Example. $k = 1$ (noncrossing matchings):

$$\begin{aligned} H_1(x) &= I_0(2x) - I_2(2x) \\ &= \sum_{j \geq 0} C_j \frac{x^{2j}}{(2j)!}. \end{aligned}$$

Compare:

$u_k(n) := \#\{w \in \mathfrak{S}_n : \text{longest increasing subsequence of length } \leq k\}.$

$$\sum_{n \geq 0} u_k(n) \frac{x^{2n}}{n!^2} = \det [I_{i-j}(2x)]_{i,j=1}^k.$$

Baik-Rains (implicitly):

$$\lim_{n \rightarrow \infty} \text{Prob} \left(\frac{\text{cr}_n(M) - \sqrt{2n}}{(2n)^{1/6}} \leq \frac{t}{2} \right) = F_1(t),$$

where

$$F_1(t) = \sqrt{F(t)} \exp \left(\frac{1}{2} \int_t^\infty u(x) dx \right),$$

where $F(t)$ is the Tracy-Widom distribution and $u(x)$ the Painlevé II function.

$$F(t) = \exp \left(- \int_t^\infty (x - t) u(x)^2 dx \right)$$

$$\frac{d^2}{dx^2} u(x) = 2u(x)^3 + xu(x)$$

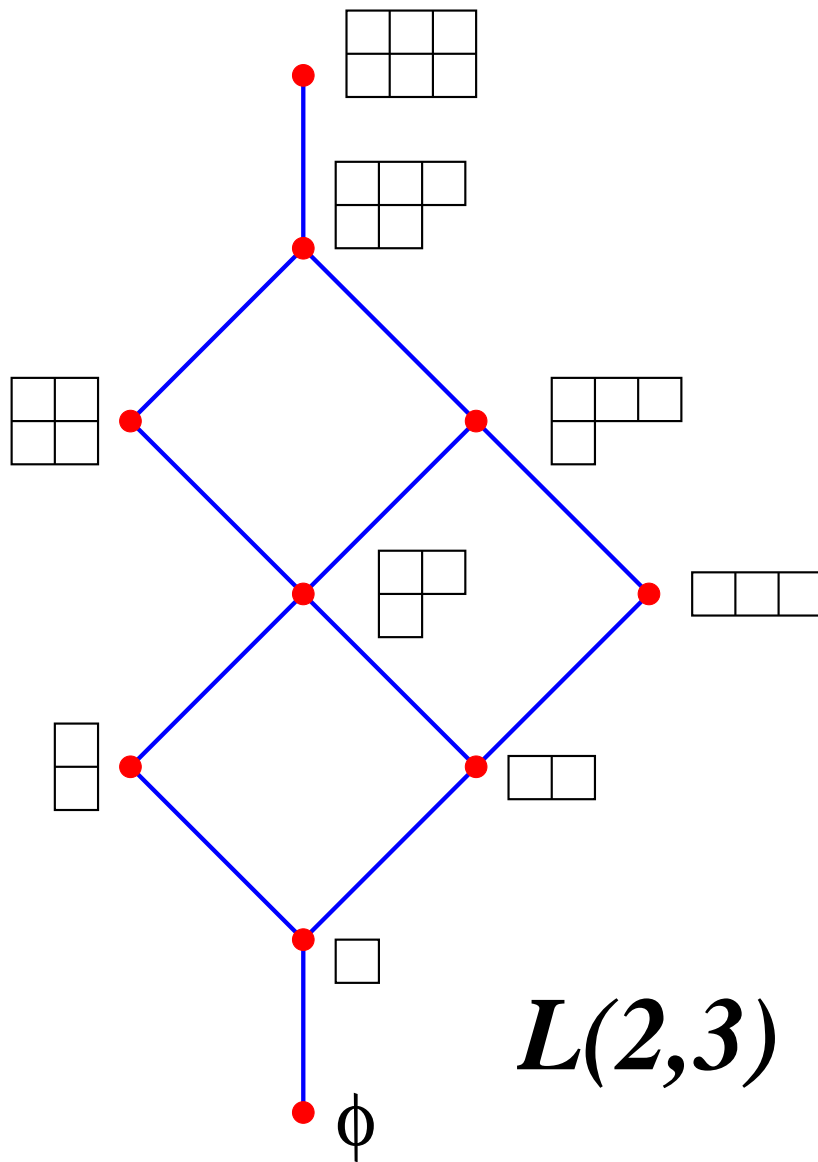
$$g_{j,k}(n) := \#\{\text{matchings } M \text{ on } [2n], \\ \text{cr}(M) \leq j, \text{ne}(M) \leq k\}$$

Now

$g_{j,k}(n) = \#\{(\emptyset = \lambda^0, \lambda^1, \dots, \lambda^{2n} = \emptyset) : \\ \lambda^{i+1} = \lambda^i \pm \square, \lambda^i \subseteq j \times k \text{ rectangle}\},$
a walk on the Hasse diagram $\mathcal{H}(j, k)$
of

$$L(j, k) := \{\lambda \subseteq j \times k \text{ rectangle}\},$$

ordered by inclusion.



\mathbf{A} = adjacency matrix of $\mathcal{H}(j, k)$
 \mathbf{A}_0 = adjacency matrix of $\mathcal{H}(j, k) - \{\emptyset\}$.

Transfer-matrix method \Rightarrow

$$\sum_{n \geq 0} g_{j,k}(n) x^{2n} = \frac{\det(I - xA_0)}{\det(I - xA)}.$$

Theorem (Grabiner, implicitly) Every zero of $\det(I - xA)$ has the form

$$2(\cos(\pi r_1/m) + \cdots + \cos(\pi r_j/m)),$$

where each $r_i \in \mathbb{Z}$ and $m = j + k + 1$.

Corollary. *Every factor of $\det(I - xA)$ over \mathbb{Q} has degree dividing*

$$\frac{1}{2}\phi(2(j + k + 1)),$$

where ϕ is the Euler phi-function.

Example.

$$j = 2, k = 5, \frac{1}{2}\phi(16) = 4:$$

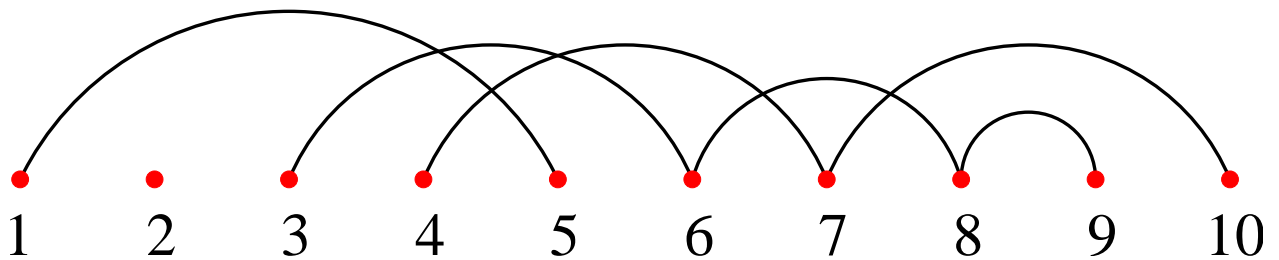
$$\begin{aligned} \det(I - xA) &= (1 - 2x^2)(1 - 4x^2 + 2x^4) \\ &\quad (1 - 8x^2 + 8x^4)(1 - 8x^2 + 8x^3 - 2x^4) \\ &\quad (1 - 8x^2 - 8x^3 - 2x^4) \end{aligned}$$

$$j = k = 3, \frac{1}{2}\phi(14) = 3:$$

$$\begin{aligned} \det(I - xA) &= (1 - x)(1 + x)(1 + x - 9x^2 - x^3) \\ &\quad (1 - x - 9x^2 + x^3)(1 - x - 2x^2 + x^3)^2 \\ &\quad (1 + x - 2x^2 - x^3)^2 \end{aligned}$$

Partition of the set $[n]$:

$\{1, 5\}, \{2\}, \{3, 6, 8, 9\}, \{4, 7, 10\}$



Generalize oscillating tableaux to
vacillating tableaux (related to the
partition algebra).

Alternating Subsequences

A sequence $b_1 b_2 \cdots b_k$ is **alternating** if

$$b_1 > b_2 < b_3 > b_4 < \cdots b_k.$$

E_n : number of alternating $w \in \mathfrak{S}_n$
(**Euler number**)

$$E_4 = 5: 2143, 3142, 3241, 4132, 4231$$

Desiré André (1879):

$$\sum_{n \geq 0} E_n \frac{x^n}{n!} = \sec x + \tan x.$$

$as_n(w)$: length of longest alternating subsequence of $w \in \mathfrak{S}_n$

$$as_9(3 \mathbf{86} 4 1 \mathbf{925} 7) = 5$$

$$b_k(n) = \#\{w \in \mathfrak{S}_n : as_n(w) \leq k\}$$

$$b_1(n) = 1 \quad (w = 12 \cdots n)$$

$$b_n(n) = n!$$

$$b_n(n) - b_{n-1}(n) = E_n$$

Define

$$\mathbf{B}(x, t) = \sum_{k, n \geq 0} b_k(n) t^k \frac{x^n}{n!},$$

and set $\boldsymbol{\rho} = \sqrt{1 - t^2}$.

Theorem. *We have*

$$B(x, t) = \frac{1 + \rho + 2te^{\rho x} + (1 - \rho)e^{2\rho x}}{1 + \rho - t^2 + (1 - \rho - t^2)e^{2\rho x}}.$$

Corollary (with I. Gessel).

$$b_k(n) = \frac{1}{2^{k-1}} \sum_{\substack{i+2j \leq k \\ i \equiv k \pmod{2}}} (-2)^j \binom{k-j}{(k+i)/2} \binom{n}{j} i^n.$$

E.g.,

$$b_2(n) = 2^{n-1}$$

$$b_3(n) = \frac{1}{4}(3^n - 2n + 3)$$

$$b_4(n) = \frac{1}{8}(4^n - 2(n-2)2^n).$$

Corollary.

$$\frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \text{as}_n(w) = \frac{4n+1}{6}, \quad n \geq 2$$

$$\text{Var}(\text{as}_n) = \frac{32n-3}{180}, \quad n \geq 4$$

Corollary. $\#\{w \in \mathfrak{S}_n : \text{as}(w) \text{ even}\}$

$$= \#\{w \in \mathfrak{S}_n : \text{as}(w) \text{ odd}\}, \quad n > 1$$

Simple proof due to **Bóna** and **Pylyavskyy** (independently).

Pemantle: limiting distribution of as_n . Let

$$G(t) = \lim_{n \rightarrow \infty} \text{Prob} \left(\frac{as_n(w) - 2n/3}{\sqrt{n}} \leq t \right).$$

Then $G(t)$ is Gaussian.

Key lemma: *Some longest alternating subsequence of $w \in \mathfrak{S}_n$ contains n .*

Leads to recurrence for

$$\begin{aligned} c_k(n) &= b_k(n) - b_{k-1}(n) \\ &= \#\{w \in \mathfrak{S}_n : as_n(w) = k\}, \end{aligned}$$

namely,

$$\begin{aligned} c_k(n) &= \sum_{j=1}^n \binom{n-1}{j-1} \\ &\quad \sum_{2r+s=k-1} (c_{2r}(j-1) + c_{2r+1}(j-1)) c_s(n-j). \end{aligned}$$