I. Stern's Diatomic Array and Beyond
II. Spernicity of the Weak Order on S_n

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PART I

Stern's Diatomic Array and Beyond

The arithmetic triangle or Pascal's triangle

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
 & & & & & \vdots & & & & &
\end{array}$$

Properties

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

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$$\sum_{k\geq 0} \binom{n}{k}^2 = \binom{2n}{n}$$

$$\sum_{n\geq 0} \binom{2n}{n} x^n = \frac{1}{\sqrt{1-4x}} \quad \text{(not rational)}$$

Sums of cubes

$$\sum_{k \geq 0} \binom{n}{k}^3 = ??$$

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If $f(n)=\sum_{k\geq 0}\binom{n}{k}^3$ then

$$(n+2)^2 f(n+2) - (7n^2+21n+16) f(n+1) - 8(n+1)^2 f(n) = 0, \;\; n \geq 0$$

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Etc.

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 & & & & 1 & & & & \\
 & & 1 & & & & 1 & & \\
 & 1 & & & & & & 1 & \\
1 & & & & & & & & 1 \\
 & & & & \vdots & & & &
\end{array}
$$

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$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & 1 & & \color{blue}{\mathbf{1}} & & 1 & & \\
 & 1 & & & & & & 1 & \\
1 & & & & & & & & 1 \\
 & & & & \vdots & & & &
\end{array}
$$

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$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & \\
 & & 1 & & & 1 & & & 1 & \\
 & 1 & & & \color{red}{2} & & \color{red}{2} & & & 1 \\
1 & & & & & & & & & & 1 \\
 & & & & & \vdots & & & & &
\end{array}$$

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 & 1 & \mathbf{1} & 2 & \mathbf{1} & 2 & \mathbf{1} & 1 & \\
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 & & 1 & & 1 & & 2 & 1 & 2 & & 1 & & 1 & & \\
1 & & \mathbf{2} & & \mathbf{3} & & \mathbf{3} & & \mathbf{3} & & \mathbf{3} & & \mathbf{2} & & 1 \\
 & & & & & & & \vdots & & & & & & &
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 & 1 & & 1 & & 2 & & 1 & & 2 & & 1 & & 1 & \\
1 & \mathbf{1} & 2 & \mathbf{1} & 3 & \mathbf{2} & 3 & \mathbf{1} & 3 & \mathbf{2} & 3 & \mathbf{1} & 2 & \mathbf{1} & 1 \\
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 & 1 & & 1 & & 2 & & 1 & & 2 & & 1 & & 1 & \\
1 & 1 & 2 & 1 & 3 & 2 & 3 & 1 & 3 & 2 & 3 & 1 & 2 & 1 & 1 \\
 & & & & & & & \vdots & & & & & & &
\end{array}
$$

Stern's triangle

Some properties

- Number of entries in row n (beginning with row 0): $2^{n+1} - 1$ (so not really a triangle)

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Some properties

- Number of entries in row n (beginning with row 0): $2^{n+1} - 1$ (so not really a triangle)
- Sum of entries in row n: 3^n
- Largest entry in row n: F_{n+1} (Fibonacci number)
- Let $\left\langle {n \atop k} \right\rangle$ be the kth entry (beginning with $k = 0$) in row n. Write

$$P_n(x) = \sum_{k \geq 0} \left\langle {n \atop k} \right\rangle x^k.$$

Then $\boxed{P_{n+1}(x) = (1 + x + x^2)P_n(x^2)}$, since $x\,P_n(x^2)$ corresponds to bringing down the previous row, and $(1 + x^2)P_n(x^2)$ to summing two consecutive entries.

Stern's diatomic sequence

- **Corollary.** $$\boxed{P_n(x) = \prod_{i=0}^{n-1} \left(1 + x^{2^i} + x^{2\cdot 2^i}\right)}$$

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$$\begin{aligned} P(x) &= \prod_{i=0}^{\infty}\left(1 + x^{2^i} + x^{2\cdot 2^i}\right) \\ &:= \sum_{n\geq 0} \boldsymbol{b_n} x^n. \end{aligned}$$

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- The sequence $b_0, b_1, b_2, \ldots$ is **Stern's diatomic sequence**:

$$1,\ 1,\ 2,\ 1,\ 3,\ 2,\ 3,\ 1,\ 4,\ 3,\ 5,\ 2,\ 5,\ 3,\ 4,\ 1,\ \ldots$$

(often prefixed with 0)

Partition interpretation

$$\sum_{n\geq 0} b_n x^n = \prod_{i\geq 0}\left(1 + x^{2^i} + x^{2\cdot 2^i}\right)$$

$\Rightarrow$ b_n is the number of partitions of n into powers of 2, where each power of 2 can appear at most twice.

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Note. If each power of 2 can appear at most once, then we obtain the (unique) binary expansion of n:

$$\frac{1}{1-x} = \prod_{i\geq 0} \left(1 + x^{2^i}\right).$$

Historical note

An essentially equivalent array is due to **Moritz Abraham Stern** around 1858 and is known as **Stern's diatomic array**:

$$
\begin{array}{ccccccccccccccccc}
1 & & & & & & & & & & & & & & & & 1 \\
1 & & & & & & & & 2 & & & & & & & & 1 \\
1 & & & & 3 & & & & 2 & & & & 3 & & & & 1 \\
1 & & 4 & & 3 & & 5 & & 2 & & 5 & & 3 & & 4 & & 1 \\
1 & 5 & 4 & 7 & 3 & 8 & 5 & 7 & 2 & 7 & 5 & 8 & 3 & 7 & 4 & 5 & 1 \\
 & & & & & & & & \vdots & & & & & & & &
\end{array}
$$

Amazing property

Theorem (**Stern**, 1858). *Let* $b_0, b_1, \ldots$ *be Stern's diatomic sequence. Then every positive rational number occurs exactly once among the ratios* b_i/b_{i+1}*, and moreover this expression is in lowest terms.*

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Can be proved inductively from

$$b_{2n} = b_n,\ \ b_{2n+1} = b_n + b_{n+1},$$

but better is to use **Calkin-Wilf tree**, though following Stigler's law of eponymy was earlier introduced by **Jean Berstel** and **Aldo de Luca** as the **Raney tree**. Closely related tree by Stern, called the **Stern-Brocot tree**, and a much earlier similar tree by **Kepler** (1619).

Stigler's law of eponymy

Stephen M. Stigler (1980): No scientific discovery is named after its original discoverer.

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Note. Stigler's law of eponymy implies that Stigler's law of eponymy was not originally discovered by Stigler.

Sums of squares

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & 1 & & & & 1 & & & & 1 & & & \\
 & 1 & & 1 & & 2 & & 1 & & 2 & & 1 & & 1 & \\
1 & 1 & 2 & 1 & 3 & 2 & 3 & 1 & 3 & 2 & 3 & 1 & 2 & 1 & 1 \\
 & & & & & & & \vdots & & & & & & &
\end{array}
$$

$$
\boldsymbol{u_2(n)} := \sum_k \left\langle {n \atop k} \right\rangle^2 = 1,\ 3,\ 13,\ 59,\ 269,\ 1227,\ \ldots
$$

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\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & 1 & & & & 1 & & & & 1 & & & \\
 & 1 & & 1 & & 2 & & 1 & & 2 & & 1 & & 1 & \\
1 & 1 & 2 & 1 & 3 & 2 & 3 & 1 & 3 & 2 & 3 & 1 & 2 & 1 & 1 \\
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\end{array}
$$

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$$
u_2(n+1) = 5u_2(n) - 2u_2(n-1), \quad n \geq 1
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$$\boldsymbol{u_2(n)} := \sum_k \left\langle {n \atop k} \right\rangle^2 = 1,\ 3,\ 13,\ 59,\ 269,\ 1227,\ \ldots$$

$$u_2(n+1) = 5u_2(n) - 2u_2(n-1), \quad n \geq 1$$

$$\sum_{n \geq 0} u_2(n)x^n = \frac{1-2x}{1-5x+2x^2}$$

Sums of cubes

$$u_3(n) := \sum_k \left\langle {n \atop k} \right\rangle^3 = 1,\ 3,\ 21,\ 147,\ 1029,\ 7203,\ \ldots$$

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$$u_3(n) = 3 \cdot 7^{n-1}, \quad n \geq 1$$

Proof for $u_2(n)$

$$
\begin{aligned}
u_2(n+1) &= \cdots + \left\langle {n \atop k} \right\rangle^2 + \left(\left\langle {n \atop k} \right\rangle + \left\langle {n \atop k+1} \right\rangle \right)^2 + \left\langle {n \atop k+1} \right\rangle^2 + \cdots \\
&= 3u_2(n) + 2\sum_k \left\langle {n \atop k} \right\rangle \left\langle {n \atop k+1} \right\rangle
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Thus define $\boldsymbol{u_{1,1}(n)} := \sum_k \left\langle {n \atop k} \right\rangle \left\langle {n \atop k+1} \right\rangle$, so

$$u_2(n+1) = 3u_2(n) + 2u_{1,1}(n).$$

What about $u_{1,1}(n)$?

$$
\begin{aligned}
u_{1,1}(n+1) &= \cdots + \left(\left\langle {n \atop k-1} \right\rangle + \left\langle {n \atop k} \right\rangle \right) \left\langle {n \atop k} \right\rangle \\
&\quad + \left\langle {n \atop k} \right\rangle \left(\left\langle {n \atop k} \right\rangle + \left\langle {n \atop k+1} \right\rangle \right) \\
&\quad + \left(\left\langle {n \atop k} \right\rangle + \left\langle {n \atop k+1} \right\rangle \right) \left\langle {n \atop k+1} \right\rangle + \cdots \\
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$$

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$$
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&\quad + \left\langle {n \atop k} \right\rangle \left(\left\langle {n \atop k} \right\rangle + \left\langle {n \atop k+1} \right\rangle \right) \\
&\quad + \left(\left\langle {n \atop k} \right\rangle + \left\langle {n \atop k+1} \right\rangle \right) \left\langle {n \atop k+1} \right\rangle + \cdots \\
&= 2u_2(n) + 2u_{1,1}(n)
\end{aligned}
$$

Recall also $u_2(n+1) = 3u_2(n) + 2u_{1,1}(n)$.

Two recurrences in two unknowns

Let

$$\color{red}{\boldsymbol{A}} := \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}.$$

Then

$$A \begin{bmatrix} u_2(n) \\ u_{1,1}(n) \end{bmatrix} = \begin{bmatrix} u_2(n+1) \\ u_{1,1}(n+1) \end{bmatrix}.$$

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minimum (or characteristic) polynomial of A: $x^2 - 5x + 2$

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$$\begin{array}{rrcl} \Rightarrow & A^{n-1}(A^2 - 5A + 2) & = & 0 \\ \Rightarrow & u_2(n+1) & = & 5u_2(n) - 2u_2(n-1) \end{array}$$

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Also $u_{1,1}(n+1) = 5u_{1,1}(n) - 2u_{1,1}(n-1)$.

What about $u_3(n)$?

Now we need

$$\begin{aligned} u_{2,1}(n) &:= \sum_k \left\langle {n \atop k} \right\rangle^2 \left\langle {n \atop k+1} \right\rangle \\ u_{1,2}(n) &:= \sum_k \left\langle {n \atop k} \right\rangle \left\langle {n \atop k+1} \right\rangle^2 . \end{aligned}$$

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However, by symmetry about a vertical axis,

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However, by symmetry about a vertical axis,

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We get

$$\begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} u_3(n) \\ u_{2,1}(n) \end{bmatrix} = \begin{bmatrix} u_3(n+1) \\ u_{2,1}(n+1) \end{bmatrix}.$$

Unexpected eigenvalue

Characteristic polynomial of $\begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}$: $x(x-7)$

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Unexpected eigenvalue

Characteristic polynomial of $\begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}$: $x(x-7)$

Thus $u_3(n+1) = 7u_3(n)$ and $u_{2,1}(n+1) = 7u_{2,1}(n)$ $(n \geq 1)$.

In fact, for $n \geq 1$ we have

$$\begin{aligned} u_3(n) &= 3 \cdot 7^{n-1} \\ u_{2,1}(n) &= 2 \cdot 7^{n-1}. \end{aligned}$$

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Get a matrix of size $\lceil (r+1)/2 \rceil$, so expect a recurrence of this order.

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Conjecture. The least order of a homogenous linear recurrence with constant coeffcients satisfied by $u_r(n)$ is $\frac{1}{3}r + O(1)$.

A more accurate conjecture

Write $[a_0, \ldots, a_{m-1}]_m$ for the periodic function $f : \mathbb{N} \to \mathbb{R}$ satisfying $f(n) = a_i$ if $n \equiv i \pmod{m}$.

$$\boldsymbol{A_r} : \text{ matrix arising from } u_r(n)$$

$$\boldsymbol{e_i(r)} : \ \# \text{ eigenvalues of } A_r \text{ equal to } i$$

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$$A_r : \text{ matrix arising from } u_r(n)$$

$$e_i(r) : \ \# \text{ eigenvalues of } A_r \text{ equal to } i$$

Conjecture. We have

$$e_0(2k-1) = \frac{1}{3}k + \left[0, -\frac{1}{3}, \frac{1}{3}\right]_3,$$

and all 0 eigenvalues are semisimple. There are no other multiple eigenvalues.

Even d

Conjecture. We have

$$\begin{aligned} e_1(2k) &= \frac{1}{6}k + \left[-1, -\frac{1}{6}, -\frac{1}{3}, -\frac{1}{2}, -\frac{2}{3}, \frac{1}{6}\right]_6 \\ e_{-1}(2k) &= e_1(2k+6). \end{aligned}$$

The eigenvalues 1 and -1 are semisimple, and there are no other multiple eigenvalues.

Minimum order of recurrence

$\mathbf{mo}(r)$: minimum order of recurrence satisfied by $u_r(n)$

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Conjecture. We have $\mathrm{mo}(2) = 2$, $\mathrm{mo}(6) = 4$, and otherwise

$$\begin{aligned}
\mathrm{mo}(2s) &= 2\left\lfloor \frac{s}{3} \right\rfloor + 3 \quad (s \neq 1, 3) \\
\mathrm{mo}(6s+1) &= 2s+1, \quad s \geq 0 \\
\mathrm{mo}(6s+3) &= 2s+1, \quad s \geq 0 \\
\mathrm{mo}(6s+5) &= 2s+2, \quad s \geq 0.
\end{aligned}$$

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\mathrm{mo}(2s) &= 2\left\lfloor \frac{s}{3} \right\rfloor + 3 \quad (s \neq 1,3) \\
\mathrm{mo}(6s+1) &= 2s+1, \quad s \geq 0 \\
\mathrm{mo}(6s+3) &= 2s+1, \quad s \geq 0 \\
\mathrm{mo}(6s+5) &= 2s+2, \quad s \geq 0.
\end{aligned}$$

True for $r \leq 125$.

Minimum order of recurrence

$\mathrm{mo}(r)$: minimum order of recurrence satisfied by $u_r(n)$

Conjecture. We have $\mathrm{mo}(2) = 2$, $\mathrm{mo}(6) = 4$, and otherwise

$$\begin{aligned}
\mathrm{mo}(2s) &= 2\left\lfloor \frac{s}{3} \right\rfloor + 3 \quad (s \neq 1, 3) \\
\mathrm{mo}(6s+1) &= 2s+1, \quad s \geq 0 \\
\mathrm{mo}(6s+3) &= 2s+1, \quad s \geq 0 \\
\mathrm{mo}(6s+5) &= 2s+2, \quad s \geq 0.
\end{aligned}$$

True for $r \leq 125$.

$$\sum_{r \geq 0} \mathrm{mo}(r) x^r = \frac{\text{irred. deg } 13}{(1-x)(1-x^6)}$$

Work of David Speyer (November 12, 2018)

Theorem. *The matrix A_r is realized by the operator $\phi\colon V_r \to V_r$ defined by*

$$\phi(f)(x,y) = f(x+y,y) + f(x,x+y),$$

where $\boldsymbol{V_r}$ is the space of homogeneous polynomials (over $\mathbb{Z}$) of degree r in the variables x, y, modulo the subspace generated by all $f(x,y) - f(y,x)$.

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Corollary. *The minimum order $\mathrm{mo}(r)$ of a recurrence satisfied by u_r is no larger than the conjectured value.*

General α

$$
\begin{aligned}
\alpha &= (\alpha_0, \ldots, \alpha_{m-1}) \\
u_\alpha(n) &:= \sum_k \left\langle {n \atop k} \right\rangle^{\alpha_0} \left\langle {n \atop k+1} \right\rangle^{\alpha_1} \cdots \left\langle {n \atop k+m-1} \right\rangle^{\alpha_{m-1}}
\end{aligned}
$$

A closer look at $\alpha = (1,1,1,1)$

$$u_{1,1,1,1}(n) = \sum_k \left\langle {n \atop k} \right\rangle \left\langle {n \atop k+1} \right\rangle \left\langle {n \atop k+2} \right\rangle \left\langle {n \atop k+3} \right\rangle$$

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$$A_{(1,1,1,1)} = \begin{array}{cc} \begin{bmatrix} 3 & 8 & 6 & 0 & 0 & 0 \\ 2 & 5 & 3 & 0 & 0 & 0 \\ 2 & 4 & 2 & 0 & 0 & 0 \\ 1 & 4 & 2 & 1 & 0 & 0 \\ 1 & 3 & 1 & 2 & 1 & 0 \\ 0 & 2 & 2 & 2 & 2 & 0 \end{bmatrix} & \begin{array}{c} \mathbf{4} \\ \mathbf{3,1} \\ \mathbf{2,2} \\ \mathbf{1,2,1} \\ \mathbf{2,1,1} \\ \mathbf{1,1,1,1} \end{array} \end{array}$$

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Reduction to $\alpha = (r)$

$$\text{min. polynomial for } \alpha = (4): \quad (x+1)(2x^2 - 11x + 1)$$

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$\mathrm{mp}(\alpha)$: minimum polynomial of A_α

Theorem. *Let $\alpha \in \mathbb{N}^m$ and $\sum \alpha_i = r$. Then $\mathrm{mp}(\alpha)$ has the form $x^{w_\alpha}(x-1)^{z_\alpha}\mathrm{mp}(r)$ for some $w_\alpha, z_\alpha \in \mathbb{N}$.*

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No conjecture for value of w_α, z_α.

A generalization

Let $\boldsymbol{p(x)}, \boldsymbol{q(x)} \in \mathbb{C}[x]$, $\boldsymbol{\alpha} = (\alpha_0, \ldots, \alpha_{m-1}) \in \mathbb{N}^m$, and $b \geq 2$. Set

$$q(x) \prod_{i=0}^{n-1} p(x^{b^i}) = \sum_k \left\langle {\boldsymbol{n} \atop \boldsymbol{k}} \right\rangle_{\boldsymbol{p,q,\alpha,b}} x^k = \sum_k \left\langle {\boldsymbol{n} \atop \boldsymbol{k}} \right\rangle x^k$$

and

$$u_{p,q,\alpha,b}(n) = \sum_k \left\langle {n \atop k} \right\rangle^{\alpha_0} \left\langle {n \atop k+1} \right\rangle^{\alpha_1} \cdots \left\langle {n \atop k+m-1} \right\rangle^{\alpha_{m-1}}.$$

Main theorem

Theorem. *For fixed p, q, α, b, the function $u_{p,q,\alpha,b}(n)$ satisfies a linear recurrence with constant coefficients ($n \gg 0$). Equivalently, $\sum_n u_{p,q,\alpha,b}(n)x^n$ is a rational function of x.*

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Note. $\exists$ multivariate generalization.

PART II

A Weak Order Conjecture

Graded posets

$$\begin{array}{rl} \color{red}{P}: & \text{finite poset} \\ \color{blue}{\textit{chain}}: & u_1 < u_2 < \cdots < u_k \end{array} \quad .$$

Graded posets

$$\begin{array}{rl} \boldsymbol{P}: & \text{finite poset} \\ \textbf{\textit{chain}}: & u_1 < u_2 < \cdots < u_k \end{array} .$$

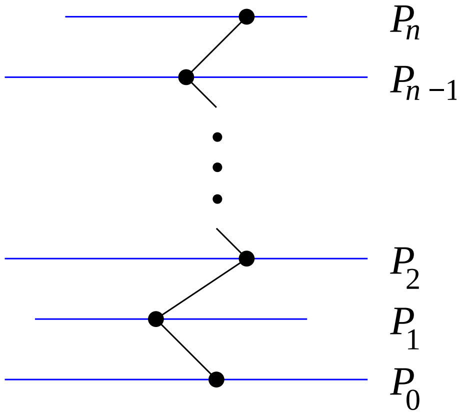
Assume P is **finite**. P is **graded of rank *n*** if

$$P = P_0 \cup P_1 \cup \cdots \cup P_n,$$

such that every maximal chain has the form

$$t_0 < t_1 < \cdots < t_n, \quad t_i \in P_i.$$

Diagram of a graded poset



Rank-symmetry and unimodality

Let $p_i = \#P_i$.

Rank-generating function: $F_P(q) = \sum_{i=0}^{n} p_i q^i$

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rank-unimodal and rank-symmetric $\Rightarrow j = \lfloor n/2 \rfloor$

The Sperner property

antichain $A \subseteq P$:

$$s, t \in A, \quad s \leq t \Rightarrow s = t$$

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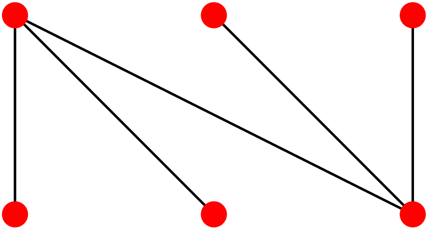
• • • •

Note. P_i is an antichain

P is **Sperner** (or has the **Sperner property**) if

$$\max_A \#A = \max_i p_i$$

An example



rank-symmetric, rank-unimodal, $F_P(q) = 3 + 3q$

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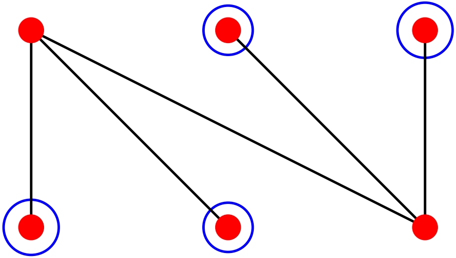

rank-symmetric, rank-unimodal, $F_P(q) = 3 + 3q$ not Sperner

The boolean algebra

B_n: subsets of $\{1, 2, \ldots, n\}$, ordered by inclusion

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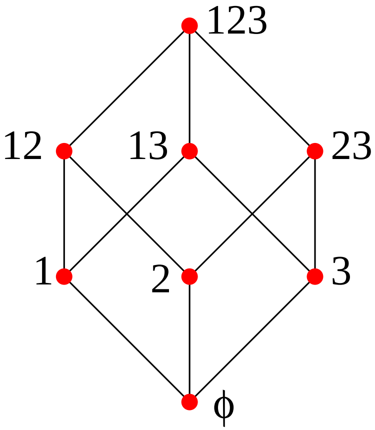
$$p_i = \binom{n}{i}, \quad F_{B_n}(q) = (1+q)^n$$

rank-symmetric, rank-unimodal

Diagram of B_3



Sperner's theorem, 1927

Theorem. *B_n is Sperner.*

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Emanuel Sperner
9 December 1905 – 31 January 1980

Linear algebra to the rescue!

$$\boldsymbol{P} = P_0 \cup \cdots \cup P_m : \quad \text{graded poset}$$

$$\mathbb{Q}\boldsymbol{P_i} : \quad \text{vector space with basis } P_i$$

$U \colon \mathbb{Q}P_i \to \mathbb{Q}P_{i+1}$ is **order-raising** if

$$U(s) \in \mathrm{span}_{\mathbb{Q}}\{t \in P_{i+1} \ : \ s < t\}$$

Order-matchings

Order matching: $\mu\colon P_i \to P_{i+1}$: injective and $\mu(t) > t$

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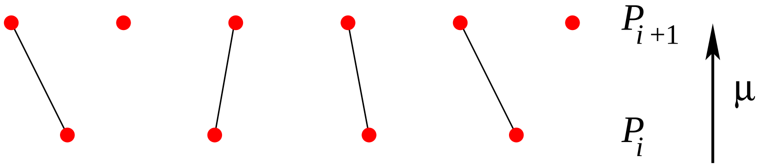

Order-raising and order-matchings

Key Lemma. *If* $U\colon \mathbb{Q}P_i \to \mathbb{Q}P_{i+1}$ *is injective and order-raising, then there exists an order-matching* $\mu\colon P_i \to P_{i+1}$.

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Proof. Consider the matrix of U with respect to the bases P_i and P_{i+1}.

Key lemma proof

$$\begin{array}{cc} & \overbrace{\begin{array}{ccccc} t_1 & \cdots & t_m & \cdots & t_n \end{array}}^{P_{i+1}} \\ P_i\left\{\begin{array}{c} s_1 \\ \vdots \\ s_m \end{array}\right. & \left[\begin{array}{ccc|c} \neq 0 & & & * \\ & \ddots & & * \\ & & \neq 0 & * \end{array}\right] \end{array}$$

$$\mathbf{det \neq 0}$$

Key lemma proof

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\begin{bmatrix} \neq 0 & & & \vert & * \\ & \ddots & & \vert & * \\ & & \neq 0 & \vert & * \end{bmatrix}
\end{array}
$$

$$\mathbf{det \neq 0}$$

$$\Rightarrow s_1 < t_1, \ldots, s_m < t_m \qquad \square$$

Minor variant

Similarly if there exists **surjective** order-raising $U\colon \mathbb{Q}P_i \to \mathbb{Q}P_{i+1}$, then there exists an order-matching $\mu\colon P_{i+1} \to P_i$.

A criterion for Spernicity

$$\boldsymbol{P} = P_0 \cup \cdots \cup P_n : \text{ finite graded poset}$$

Proposition. *If for some j there exist order-raising operators*

$$\mathbb{Q}P_0 \overset{\text{inj.}}{\to} \mathbb{Q}P_1 \overset{\text{inj.}}{\to} \cdots \overset{\text{inj.}}{\to} \mathbb{Q}P_j \overset{\text{surj.}}{\to} \mathbb{Q}P_{j+1} \overset{\text{surj.}}{\to} \cdots \overset{\text{surj.}}{\to} \mathbb{Q}P_n,$$

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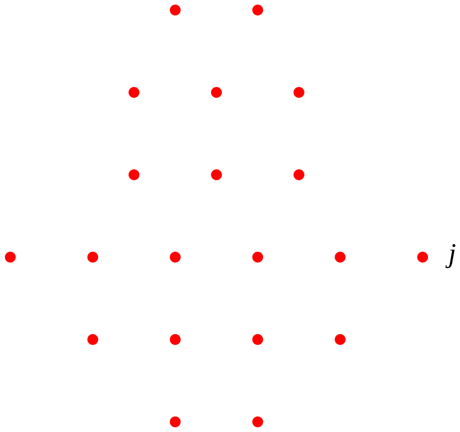
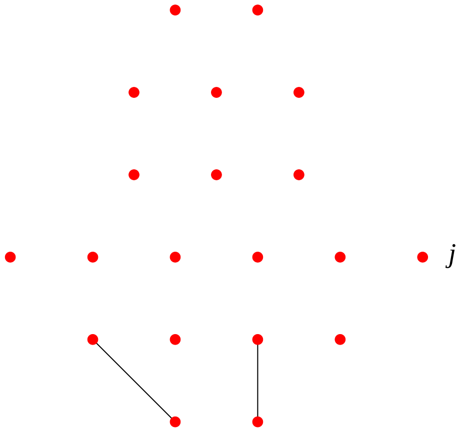
then P is rank-unimodal and Sperner.

Proof. "Glue together" the order-matchings.

Gluing example



Gluing example



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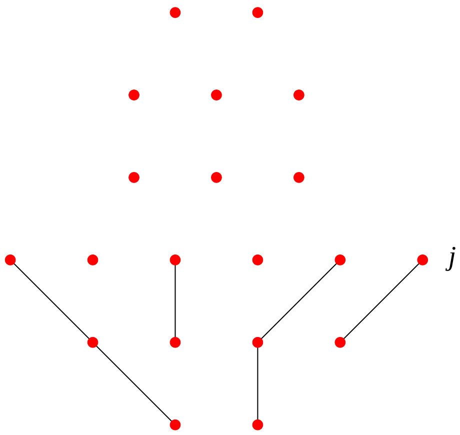

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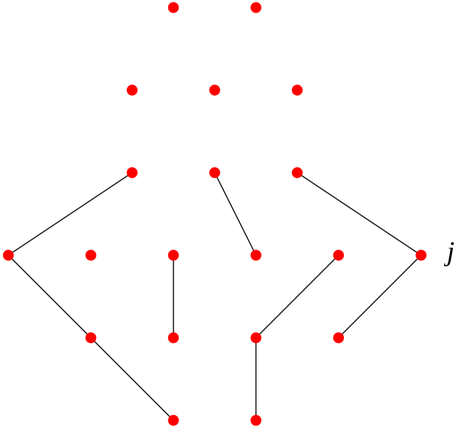

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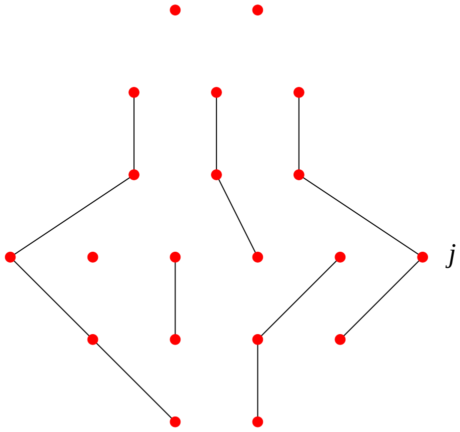

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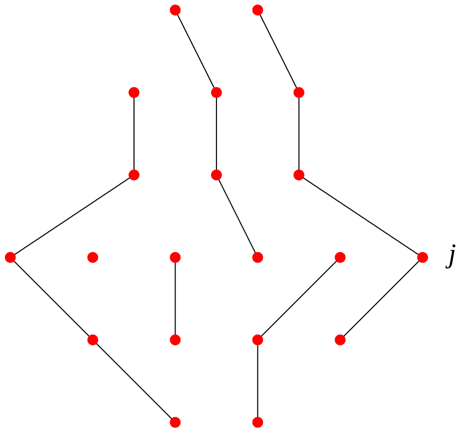

A chain decomposition

$$P = C_1 \cup \cdots \cup C_{p_j} \quad \text{(chains)}$$

$$A = \text{ antichain}, C = \text{ chain} \Rightarrow \#(A \cap C) \leq 1$$

$$\Rightarrow \#A \leq p_j. \qquad \square$$

The weak order $W(S_n)$ on S_n

$$s_i = (i, i+1), \quad 1 \le i \le n-1$$

$$w \in S_n, \ \ell(w) = \#\{1 \le i < j \le n \,:\, w(i) > w(j)\}$$

For $u, v \in S_n$ define $u \le v$ if $v = us_{i_1} \cdots s_{i_k}$, where $\ell(v) = k + \ell(u)$.

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$W(S_n)$ is graded of rank $\binom{n}{2}$, rank-symmetric, and rank-unimodal, with

$$\begin{aligned} F_{W(S_n)}(q) \quad &:= \quad \sum_{k=0}^{\binom{n}{2}} \#W(S_n)_k \, q^k \\ &= \quad (1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1}). \end{aligned}$$

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How to define $U_k : \mathbb{Q}W(S_n)_k \to \mathbb{Q}W(S_n)_{k+1}$?

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Theorem (Macdonald 1991, Fomin-S. 1994). $\mathfrak{S}_w$: Schubert polynomial indexed by $w \in S_n$. Let $\boldsymbol{k} = \ell(w)$.

$$k!\,\mathfrak{S}_w(1,1,\ldots,1) = \sum_{(a_1,\ldots,a_k)\in R(w)} a_1 \cdots a_k,$$

where $\boldsymbol{R(w)}$ is the set of reduced decompositions of w, i.e.,

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Example. $321 = s_1 s_2 s_1 = s_2 s_1 s_2$, and

$$1 \cdot 2 \cdot 1 + 2 \cdot 1 \cdot 2 = 6 = \ell(321)!.$$

An equivalent formulation

Define for $\ell(w) = k$ (or $w \in W(S_n)_k$),

$$U(w) = U_k(w) = \sum_{i\,:\,s_i w > w} i \cdot s_i w.$$

If $u < v$ and $\ell(v) - \ell(u) = r$, then

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Thus U is a "natural" order-raising operator for $W(S_n)$.

A matrix

$\mathcal{U}(n,k)$: matrix of

$$U^{\binom{n}{2}-2k} : \mathbb{Q}W(S_n)_k \to \mathbb{Q}W(S_n)_{\binom{n}{2}-k}$$

with respect to the bases $W(S_n)_k$ and $W(S_n)_{\binom{n}{2}-k}$.

A matrix

$\mathcal{U}(n,k)$: matrix of

$$U^{\binom{n}{2}-2k} : \mathbb{Q}W(S_n)_k \to \mathbb{Q}W(S_n)_{\binom{n}{2}-k}$$

with respect to the bases $W(S_n)_k$ and $W(S_n)_{\binom{n}{2}-k}$.

If $u \in W(S_n)_k$ and $v \in W(S_n)_{\binom{n}{2}-k}$, then

$$\mathcal{U}(n,k)_{uv} = \begin{cases} (\cdots)\mathfrak{S}_{u^{-1}v}(1,\ldots,1), & u \leq v \\ 0, & u \not\leq v. \end{cases}$$

A determinant

To show: $\det \mathcal{U}(n,k) \neq 0$ (implies $W(S_n)$ is Sperner).

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Corollary. *$W(S_n)$ is Sperner.*

Another approach

Theorem (**C. Gaetz** and **Y. Gao**, November 13, 2018). *There exists a "down" operator*

$$D\colon \mathbb{C}W(S_n)_k \to \mathbb{C}W(S_n)_{k-1}$$

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Corollary. *For $k < \frac{1}{2}\binom{n}{2}$ the linear transformation*

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What is D?

Lehmer code of $w = a_1 \cdots a_n \in S_n$: $\boldsymbol{L(w)} = (c_1, \ldots, c_n)$, where

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If $w \in (W_n)_k$, then $Dw := \sum_{v \in (W_n)_{k-1}} \gamma_{vw} v$, where

$$\gamma_{vw} = \begin{cases} \| \, L(w) - L(v) \, \|_1, & \text{if } v < w \text{ (\textbf{strong} order)} \\ 0, & \text{otherwise.} \end{cases}$$

Example of D

$$v = 231654, \quad w = 251634$$

$$\ell(v) = 5, \quad \ell(w) = 6, \quad v < w \text{ (strong order)}$$

$$L(v) = (1,1,0,2,1,0), \quad L(w) = (1,3,0,2,0,0)$$

$$L(w) - L(v) = (0,2,0,0,-1,0)$$

$$\gamma_{vw} = 2 + 1 = 3$$

Combining the two proofs

Gaetz and Gao combined their ideas with those of Hamaker, Pechenik, Speyer, and Weigandt to find the Smith normal form of $\mathcal{U}(n,k)$ (stronger result than $\det \mathcal{U}(n,k)$).

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  Is there a nice q-analogue?
- Other types, i.e., the weak order of other Coxeter groups?

The final slide

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