Ranks of My Students 1977–2004

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41									

Some Problems I Couldn't Solve

- serious effort
- still open
- may be tractable
- easily explained
- inspired by G.-C. Rota, Ten mathematics problems I will never solve, Oaxaca, 1997

Prehistory: **Circulant Hadamard matrices**

A **circulant Hadamard matrix** of order n is an $n \times n$ matrix of ± 1's such that any two distinct rows are orthogonal, and each row is a cyclic shift one unit right of the previous row (circulant matrix).

$$\begin{bmatrix} 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \end{bmatrix}$$

Conjecture (Ryser). If n is the order of a circulant Hadamard matrix, then $n = 4$.

If A is a circulant Hadamard matrix with first row $(a_0, \ldots, a_{n-1})$, compute $\det(A)$ in two ways to get:

$$n^{n/2} = \prod_{k=0}^{n-1} (a_0 + a_1\zeta^k + \cdots + a_{n-1}\zeta^{(n-1)k}),$$

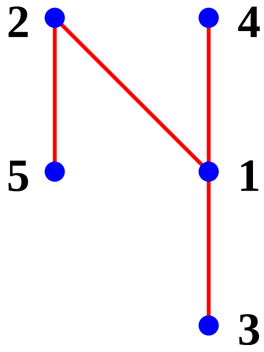
where $\zeta = e^{2\pi i/n}$.

Theorem (Turyn, 1965). *There does not exist a circulant Hadamard matrix of order* $8m$ *(and various* $4(2k+1)$*).*

RS (∼1968): There does not exist a circulant Hadamard matrix of order $2^j > 4$.

The Poset Conjecture (or Neggers-Stanley conjecture)

$\boldsymbol{P}$ = labelled poset



$\boldsymbol{\mathcal{L}(P)}$: set of linear extensions of P

w	des(w)
35124	1
35142	2
31524	2
31452	2
53124	2
31542	3
53142	3

If $\pi = a_1 a_2 \ldots a_n \in \mathfrak{S}_n$, then define

$$\mathbf{des}(\pi) = \#\{i \,:\, a_i > a_{i+1}\},$$

the number of **descents** of π.

$$\boldsymbol{A}_P(x) = \sum_{\pi \in \mathcal{L}(P)} x^{\mathrm{des}(\pi)},$$

the ***P*-Eulerian polynomial**.

For above example,

$$A_P(x) = x + 4x^2 + 2x^3.$$

Conjecture. $A_P(x)$ has only real zeros.

Neggers (1978) for naturally labelled posets; RS (1986) for any labeling.

Sample result (Wagner). If conjecture is true for P and Q, then also true for $P+Q$ (compatibly labelled).

Let $\boldsymbol{L}$ be a finite distributive lattice, so $L = J(P)$ for some finite poset P. Let

$$\boldsymbol{c_i} = \#\ i\text{-element chains of } L$$

Proposition. *If P is naturally labelled, then $A_P(x)$ has only real zeros if and only if the* ***chain polynomial*** $\sum_i c_i x^i$ *has only real zeros.*

Conjecture. The chain polynomial of a modular lattice has only real zeros.

Possible hint: M. Chudnovsky and P. Seymour, The roots of the stable set polynomial of a clawfree graph.

Gorenstein Hilbert functions

Let $R = R_0 \oplus R_1 \oplus \cdots \oplus R_s$ be an artinian graded Gorenstein algebra over the field $K = R_0$, generated by R_1, with $R_s \neq 0$. Define

$$\boldsymbol{h_i} = \dim_K R_i,$$

the **Hilbert function** of R. Thus $h_0 = 1$.

Well-known (Macaulay): $h_i = h_{s-i}$.

What more can be said about $(h_0, h_1, \ldots, h_s)$ (**Gorenstein sequence**)?

Is a complete characterization possible?

If $s = 4$ and $h_1 = n$, how small can h_2 be?

$$(1, n, h_2, n, 1)$$

Denote this minimum by $f(n)$.

Fact:

$$\frac{1}{2}6^{2/3} \le \liminf_{n\to\infty} f(n)n^{-2/3} \le \limsup_{n\to\infty} f(n)n^{-2/3} \le 6^{2/3}$$

Linear algebra reformulation:

Fix $s \geq 0$. Let

$\boldsymbol{M_i} = \{\text{monomials of degree } i \text{ in } x_1, \ldots, x_m\}$.

Fix a nonzero $\boldsymbol{\sigma} : M_s \to K$. For $0 \leq j \leq s$, let $\boldsymbol{A^{(j)}}$ be the matrix with rows indexed by M_j and columns by M_{s-j}, defined by

$$\boldsymbol{A^{(j)}_{uv}} = \sigma(uv).$$

Let $\boldsymbol{h_j} = \operatorname{rank} A^{(j)}$.

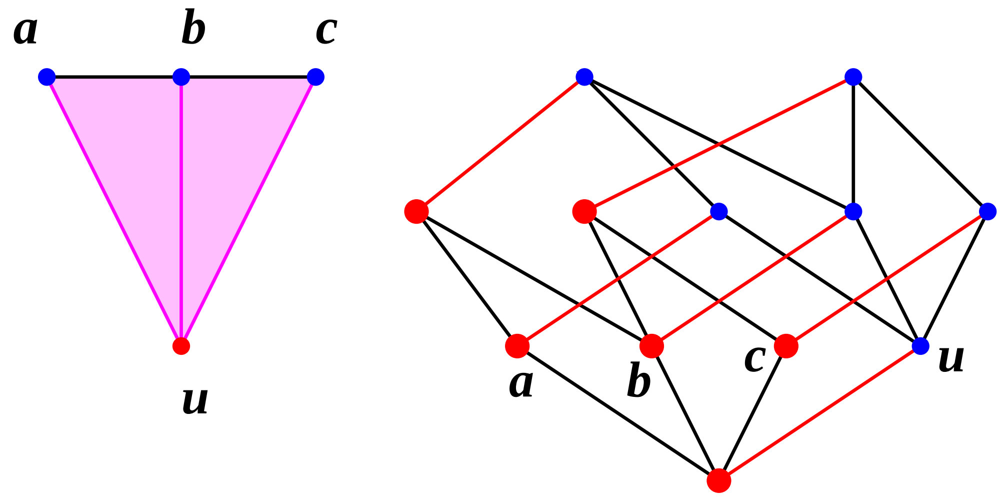
Fact:

Gorenstein sequence same as $(h_0, h_1, \ldots, h_s)$.

Partitions of simplicial complexes

Let $\boldsymbol{\Gamma}$ be a finite (abstract) simplicial complex, i.e., an order ideal of a boolean algebra B_n.

***c*Γ**: **cone** over Γ

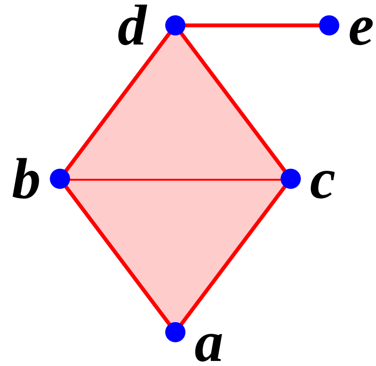


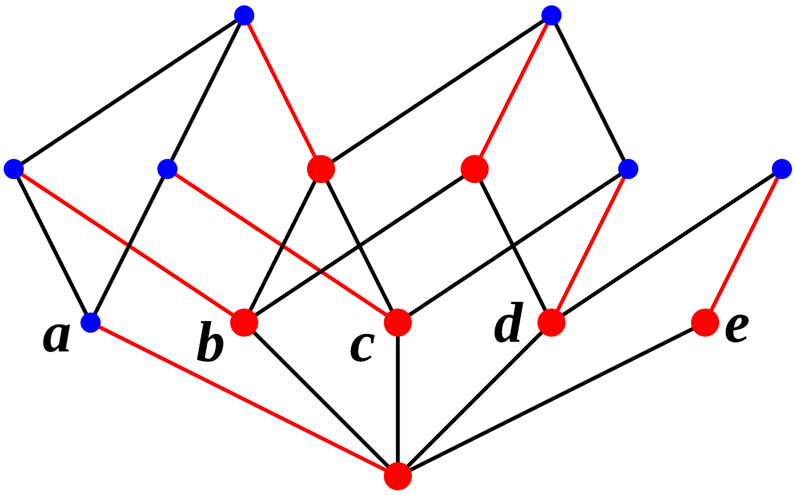
$\exists$ partition $\boldsymbol{\pi}$ of $c\Gamma$ into 2-element intervals $[F, F']$ such that

$$\{F \ : \ [F, F'] \in c\Gamma\}$$

is a subcomplex (namely, Γ) of $c\Gamma$.

Suppose Δ is only **acyclic** (vanishing reduced homology).





Theorem (1993). *If Δ is acyclic, then there is a partition π of Δ into 2-element intervals $[F, F']$ such that*

$$\{F : [F, F'] \in \Delta\}$$

is a subcomplex of Γ.

Proof uses Marriage Theorem and exterior algebra.

f_i: number of i-dimensional faces of Δ

$(f_0, f_1, \ldots)$: **f-vector** of Δ

Corollary (Kalai). *f-vectors of acyclic Δ coincide with f-vectors of cones.*

Nice generalization by Duval for **any** Δ (partition into 1-element and 2-element intervals).

Many open questions remain.

Sample:

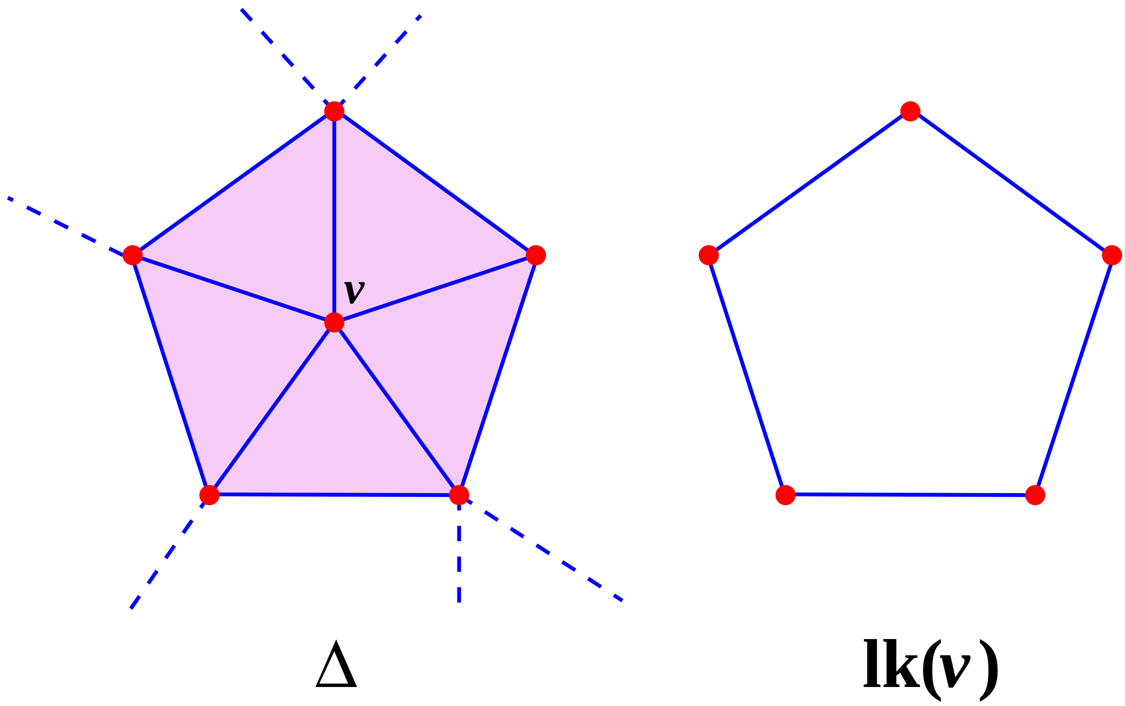
Obvious fact: For any Γ, there is a partition π of $cc\Gamma$ into intervals $[F, F'] \cong B_2$ such that

$$\{F : [F, F'] \in \pi\}$$

is a subcomplex (namely, Γ) of $cc\Gamma$.

If v is a vertex of Δ, define the **link** of v by

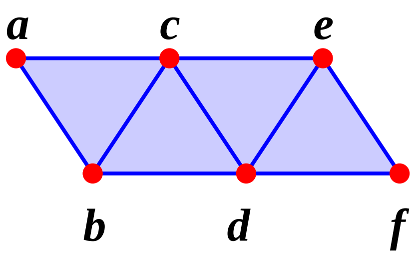
$$\mathrm{lk}(v) = \{F \in \Delta : v \notin F,\ F \cup v \in \Delta\}.$$

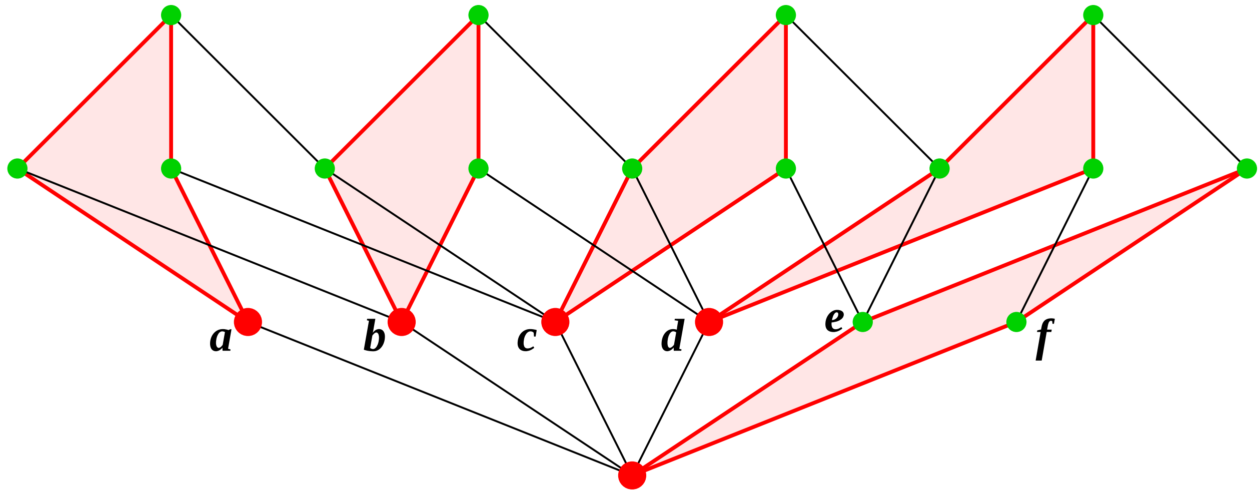


Conjecture. If Δ and $\mathrm{lk}(v)$ are acyclic for every vertex v of Δ (i.e., Δ is **doubly acyclic**), then there is a partition π of Δ into intervals $[F, F'] \cong B_2$ such that

$$\{F : [F, F'] \in \pi\}$$

is a subcomplex of Δ.





Perhaps a "generalized Marriage Theorem" is involved.

Above conjecture $\Rightarrow$ f-vectors of doubly acyclic Δ coincide with f-vectors of double cones $cc\Gamma$ (proved by Kalai using algebraic shifting).