Smith normal form

$A$: $n \times n$ matrix over commutative ring $\mathbb{R}$ (with 1)

Suppose there exist $P, Q \in \text{GL}(n, \mathbb{R})$ such that

$$PAQ := B = \text{diag}(d_1, d_1d_2, \ldots d_1d_2 \cdots d_n),$$

where $d_i \in \mathbb{R}$. We then call $B$ a Smith normal form (SNF) of $A$. 
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where $d_i \in R$. We then call $B$ a Smith normal form (SNF) of $A$.

**Note.** (1) Can extend to $m \times n$.

(2) unit $\cdot \det(A) = \det(B) = d_1^n d_2^{n-1} \cdots d_n$.

Thus SNF is a refinement of $\det$. 
Row and column operations

Can put a matrix into SNF by the following operations.

- Add a multiple of a row to another row.
- Add a multiple of a column to another column.
- Multiply a row or column by a unit in $R$. 
Row and column operations

Can put a matrix into SNF by the following operations.

- Add a multiple of a row to another row.
- Add a multiple of a column to another column.
- Multiply a row or column by a **unit** in $R$.

Over a field, SNF is **row reduced echelon form** (with all unit entries equal to 1).

Smith Normal Form and Combinatorics – p. 3
Existence of SNF

If $R$ is a PID, such as $\mathbb{Z}$ or $K[x]$ ($K$ = field), then $A$ has a unique SNF up to units.
Existence of SNF

If $R$ is a PID, such as $\mathbb{Z}$ or $K[x] \ (K = \text{field})$, then $A$ has a unique SNF up to units.

Otherwise $A$ “typically” does not have a SNF but may have one in special cases.
$\mathbb{R}$: a PID

$A$: an $n \times n$ matrix over $\mathbb{R}$ with rows $v_1, \ldots, v_n \in \mathbb{R}^n$

$\text{diag}(e_1, e_2, \ldots, e_n)$: SNF of $A$
Algebraic interpretation of SNF

$R$: a PID

$A$: an $n \times n$ matrix over $R$ with rows $v_1, \ldots, v_n \in R^n$

$\text{diag}(e_1, e_2, \ldots, e_n)$: SNF of $A$

Theorem.

$$R^n/(v_1, \ldots, v_n) \cong (R/e_1R) \oplus \cdots \oplus (R/e_nR).$$
Algebraic interpretation of SNF

\( \mathbf{R} \): a PID

\( \mathbf{A} \): an \( n \times n \) matrix over \( \mathbf{R} \) with rows
\( v_1, \ldots, v_n \in \mathbf{R}^n \)

\( \text{diag}(e_1, e_2, \ldots, e_n) \): SNF of \( \mathbf{A} \)

Theorem.

\[
\mathbf{R}^n / (v_1, \ldots, v_n) \cong (\mathbf{R}/e_1 \mathbf{R}) \oplus \cdots \oplus (\mathbf{R}/e_n \mathbf{R}).
\]

\( \mathbf{R}^n / (v_1, \ldots, v_n) \): (Kastelyn) cokernel of \( \mathbf{A} \)
An explicit formula for SNF

$R$: a PID

$A$: an $n \times n$ matrix over $R$ with $\det(A) \neq 0$

$\text{diag}(e_1, e_2, \ldots, e_n)$: SNF of $A$
An explicit formula for SNF

\( R \): a PID

\( A \): an \( n \times n \) matrix over \( R \) with \( \det(A) \neq 0 \)

\( \text{diag}(e_1, e_2, \ldots, e_n) \): SNF of \( A \)

**Theorem.** \( e_1 e_2 \cdots e_i \) is the gcd of all \( i \times i \) minors of \( A \).

**minor**: determinant of a square submatrix.

**Special case**: \( e_1 \) is the gcd of all entries of \( A \).
An example

Reduced Laplacian matrix of $K_4$:

$$A = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}$$
An example

Reduced Laplacian matrix of $K_4$:

$$A = \begin{bmatrix}
3 & -1 & -1 \\
-1 & 3 & -1 \\
-1 & -1 & 3
\end{bmatrix}$$

Matrix-tree theorem $\implies \det(A) = 16$, the number of spanning trees of $K_4$. 

An example

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Matrix-tree theorem $\implies \det(A) = 16$, the number of spanning trees of $K_4$.

What about SNF?
An example (continued)

\[
\begin{bmatrix}
3 & -1 & -1 \\
-1 & 3 & -1 \\
-1 & -1 & 3
\end{bmatrix}
\rightarrow
\begin{bmatrix}
0 & 0 & -1 \\
-4 & 4 & -1 \\
8 & -4 & 3
\end{bmatrix}
\rightarrow
\begin{bmatrix}
0 & 0 & -1 \\
-4 & 4 & 0 \\
8 & -4 & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
0 & 0 & -1 \\
0 & 4 & 0 \\
4 & -4 & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
0 & 0 & -1 \\
0 & 4 & 0 \\
4 & 0 & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
4 & 0 & 0 \\
0 & 4 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]
Laplacian matrices

$L_0(G)$: reduced Laplacian matrix of the graph $G$

Matrix-tree theorem. $\det L_0(G) = \kappa(G)$, the number of spanning trees of $G$. 
**Laplacian matrices**

$L_0(G)$: reduced Laplacian matrix of the graph $G$

**Matrix-tree theorem.** $\det L_0(G) = \kappa(G)$, the number of spanning trees of $G$.

**Theorem.** $L_0(K_n) \xrightarrow{\text{SNF}} \text{diag}(1, n, n, \ldots, n)$, a refinement of Cayley’s theorem that $\kappa(K_n) = n^{n-2}$. 
**Laplacian matrices**

$L_0(G)$: reduced Laplacian matrix of the graph $G$

**Matrix-tree theorem.** $\det L_0(G) = \kappa(G)$, the number of spanning trees of $G$.

**Theorem.** $L_0(K_n) \xrightarrow{\text{SNF}} \text{diag}(1, n, n, \ldots, n)$, a refinement of Cayley’s theorem that $\kappa(K_n) = n^{n-2}$.

In general, SNF of $L_0(G)$ not understood.
**Abelian sandpile**: a finite collection $\sigma$ of indistinguishable chips distributed among the vertices $V$ of a (finite) connected graph. Equivalently,

$$\sigma : V \rightarrow \{0, 1, 2, \ldots \}.$$
**Abelian sandpile**: a finite collection $\sigma$ of indistinguishable chips distributed among the vertices $V$ of a (finite) connected graph. Equivalently, 

$$\sigma : V \rightarrow \{0, 1, 2, \ldots \}.$$

**Toppling** of a vertex $v$: if $\sigma(v) \geq \deg(v)$, then send a chip to each neighboring vertex.
Choose a vertex to be a **sink**, and ignore chips falling into the sink.

**stable** configuration: no vertex can topple

**Theorem** (easy). After finitely many topples a *stable configuration will be reached, which is independent of the order of topples.*
The monoid of stable configurations

Define a commutative monoid $M$ on the stable configurations by vertex-wise addition followed by stabilization.

**Ideal** of $M$: subset $J \subseteq M$ satisfying $\sigma J \subseteq J$ for all $\sigma \in M$
The monoid of stable configurations

Define a commutative monoid $M$ on the stable configurations by vertex-wise addition followed by stabilization.

**ideal** of $M$: subset $J \subseteq M$ satisfying $\sigma J \subseteq J$ for all $\sigma \in M$

**Exercise.** The (unique) minimal ideal of a finite commutative monoid is a group.
Sandpile group

sandpile group of $G$: the minimal ideal $K(G)$ of the monoid $M$

Fact. $K(G)$ is independent of the choice of sink up to isomorphism.
Sandpile group

sandpile group of $G$: the minimal ideal $K(G)$ of the monoid $M$

Fact. $K(G)$ is independent of the choice of sink up to isomorphism.

Theorem. Let

$$L_0(G) \xrightarrow{\text{SNF}} \text{diag}(e_1, \ldots, e_{n-1}).$$

Then

$$K(G) \cong \mathbb{Z}/e_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/e_{n-1}\mathbb{Z}.$$
Second example

Some matrices connected with Young diagrams
\( \lambda \): a partition \((\lambda_1, \lambda_2, \ldots)\), identified with its Young diagram

\[
\begin{array}{cccc}
\ & \ & \ & \\
\ & \ & \ & \\
\ & \ & \ & \\
\ & \ & \ & \\
\end{array}
\quad (3,1)
\]
Extended Young diagrams

\( \lambda \): a partition \((\lambda_1, \lambda_2, \ldots)\), identified with its Young diagram

\( \lambda^* \): \( \lambda \) extended by a border strip along its entire boundary
\( \lambda \): a partition \((\lambda_1, \lambda_2, \ldots)\), identified with its Young diagram

\( \lambda^* \): \( \lambda \) extended by a border strip along its entire boundary

\[(3,1)^* = (4,4,2)\]
Initialization

Insert 1 into each square of $\lambda^*/\lambda$.

$(3,1)^* = (4,4,2)$
Let \( t \in \lambda \). Let \( M_t \) be the largest square of \( \lambda^* \) with \( t \) as the upper left-hand corner.
Let $t \in \lambda$. Let $M_t$ be the largest square of $\lambda^*$ with $t$ as the upper left-hand corner.
Let $t \in \lambda$. Let $M_t$ be the largest square of $\lambda^*$ with $t$ as the upper left-hand corner.
Suppose all squares to the southeast of $t$ have been filled. Insert into $t$ the number $n_t$ so that $\det M_t = 1$. 
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Determinantal algorithm

Suppose all squares to the southeast of $t$ have been filled. Insert into $t$ the number $n_t$ so that $\det M_t = 1$. 

\[
\begin{array}{ccc}
1 & 1 & 1 \\
 & 1 & 1 \\
2 & 1 & 1 \\
1 & 1 & 1
\end{array}
\]
Suppose all squares to the southeast of $t$ have been filled. Insert into $t$ the number $n_t$ so that $\det M_t = 1$. 

\[
\begin{array}{ccc}
1 & 1 & 2 \\
& 1 & 1 \\
1 & 1 & 1 \\
\end{array}
\]
Suppose all squares to the southeast of $t$ have been filled. Insert into $t$ the number $n_t$ so that $\det M_t = 1$. 
Suppose all squares to the southeast of \( t \) have been filled. Insert into \( t \) the number \( n_t \) so that \( \det M_t = 1 \).
Suppose all squares to the southeast of $t$ have been filled. Insert into $t$ the number $n_t$ so that $\det M_t = 1$. 

\[
\begin{array}{ccc}
9 & 5 & 2 \\
3 & 2 & 1 \\
1 & 1 & 1 \\
\end{array}
\]
Uniqueness

Easy to see: the numbers $n_t$ are well-defined and unique.
Uniqueness

Easy to see: the numbers \( n_t \) are well-defined and unique.

Why? Expand \( \det M_t \) by the first row. The coefficient of \( n_t \) is 1 by induction.
If $t \in \lambda$, let $\lambda(t)$ consist of all squares of $\lambda$ to the southeast of $t$. 
If \( t \in \lambda \), let \( \lambda(t) \) consist of all squares of \( \lambda \) to the southeast of \( t \).
If \( t \in \lambda \), let \( \lambda(t) \) consist of all squares of \( \lambda \) to the southeast of \( t \).
\( u_\lambda = \#\{\mu : \mu \subseteq \lambda\} \)
\[ u_\lambda = \#\{\mu : \mu \subseteq \lambda\} \]

Example. \[ u_{(2,1)} = 5: \]

\[
\begin{array}{cccc}
\boxed{\phantom{1}} & \boxed{\phantom{1}} & \boxed{\phantom{1}} & \boxed{\phantom{1}} & \varnothing \\
\boxed{\phantom{1}} & \boxed{\phantom{1}} & \boxed{\phantom{1}} & \boxed{\phantom{1}} & \boxed{\phantom{1}}
\end{array}
\]
\( u_\lambda = \#\{\mu : \mu \subseteq \lambda\} \)

Example. \( u_{(2,1)} = 5 \):

\[
\begin{array}{cccc}
\square & \square & \square & \square & \phi \\
\square & \square & \square & \square & \square \\
\end{array}
\]

There is a determinantal formula for \( u_\lambda \), due essentially to MacMahon and later Kremeras (not needed here).
Berlekamp (1963) first asked for $n_t \pmod{2}$ in connection with a coding theory problem.

Carlitz-Scoville-Roselle theorem

- Berlekamp (1963) first asked for $n_t \pmod{2}$ in connection with a coding theory problem.

**Theorem.** \( n_t = f(\lambda(t)) \).
Carlitz-Scoville-Roselle theorem

- Berlekamp (1963) first asked for \( n_t \pmod{2} \) in connection with a coding theory problem.

- **Carlitz-Roselle-Scoville** (1971): combinatorial interpretation of \( n_t \) (over \( \mathbb{Z} \)).

**Theorem.** \( n_t = f(\lambda(t)) \).

**Proofs.**
1. Induction (row and column operations).
2. Nonintersecting lattice paths.
An example

\[
\begin{array}{cccc}
7 & 3 & 2 & 1 \\
2 & 1 & 1 & 1 \\
1 & 1 & & \\
& & & \\
\end{array}
\]
An example
Many indeterminates

For each square \((i, j) \in \lambda\), associate an indeterminate \(x_{ij}\) (matrix coordinates).
Many indeterminates

For each square \((i, j) \in \lambda\), associate an indeterminate \(x_{ij}\) (matrix coordinates).

<table>
<thead>
<tr>
<th>(x_{11})</th>
<th>(x_{12})</th>
<th>(x_{13})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x_{21})</td>
<td>(x_{22})</td>
<td></td>
</tr>
</tbody>
</table>
A refinement of $u_\lambda$

$$u_\lambda(x) = \sum_{\mu \subseteq \lambda} \prod_{(i, j) \in \lambda / \mu} x_{ij}$$
A refinement of $u_\lambda$

\[ u_\lambda(x) = \sum_{\mu \subseteq \lambda} \prod_{(i,j) \in \lambda/\mu} x_{ij} \]

\[
\begin{array}{ccc}
  a & b & c \\
  d & e \\
\end{array}
\quad
\begin{array}{c}
\end{array}
\quad
\begin{array}{cc}
  c \\
  d & e \\
\end{array}
\]

\[ \prod_{(i,j) \in \lambda/\mu} x_{ij} = cde \]
An example

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>d</td>
<td></td>
<td></td>
<td>e</td>
</tr>
</tbody>
</table>

\[
\begin{array}{c}
\text{abcde} + \text{bcde} + \text{bce} + \text{cde} \\
+ \text{ce} + \text{de} + \text{c} + \text{e} + 1
\end{array}
\begin{array}{c}
\text{bce} + \text{ce} + \text{c} \\
+ \text{e} + 1
\end{array}
\begin{array}{c}
\text{c} + 1
\end{array}
\begin{array}{c}
1
\end{array}
\begin{array}{c}
\text{de} + \text{e} + 1
\end{array}
\begin{array}{c}
\text{e} + 1
\end{array}
\begin{array}{c}
1
\end{array}
\begin{array}{c}
1
\end{array}
\begin{array}{c}
1
\end{array}
\begin{array}{c}
1
\end{array}
\begin{array}{c}
1
\end{array}
\begin{array}{c}
1
\end{array}
\begin{array}{c}
1
\end{array}
\begin{array}{c}
1
\end{array}
\begin{array}{c}
1
\end{array}
\begin{array}{c}
1
\end{array}
\begin{array}{c}
1
\end{array}\]

Smith Normal Form and Combinatorics – p. 26
\[ A_t = \prod_{(i,j) \in \lambda(t)} x_{i,j} \]
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\[ A_t = \prod_{(i,j) \in \lambda(t)} x_{i,j} \]

\[ A_t = \text{bcdeghiklmo} \]
The main theorem

**Theorem.** Let \( t = (i, j) \). Then \( M_t \) has SNF

\[
\text{diag}(A_{ij}, A_{i-1,j-1}, \ldots, 1).
\]
The main theorem

Theorem. Let \( t = (i, j) \). Then \( M_t \) has SNF

\[ \text{diag}(A_{ij}, A_{i-1,j-1}, \ldots, 1). \]

Proof. 1. Explicit row and column operations putting \( M_t \) into SNF.
2. (C. Bessenrodt) Induction.
An example

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>b</td>
<td>c</td>
</tr>
<tr>
<td>d</td>
<td>e</td>
<td></td>
</tr>
</tbody>
</table>

\[
\begin{align*}
\text{abcde} + \text{bcde} + \text{bce} + \text{cde} + \text{ce} + \text{de} + \text{c} + \text{e} + \text{1} \\
\text{de} + \text{e} + \text{1} \\
\text{1} 
\end{align*}
\]

- \( \text{abcde} + \text{bcde} + \text{bce} + \text{cde} + \text{ce} + \text{de} + \text{c} + \text{e} + \text{1} \)
- \( \text{bce} + \text{ce} + \text{c} + \text{e} + \text{1} \)
- \( \text{c} + \text{1} \)
- \( \text{1} \)
- \( \text{de} + \text{e} + \text{1} \)
- \( \text{e} + \text{1} \)
- \( \text{1} \)
- \( \text{1} \)
- \( \text{1} \)
An example

<table>
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<th>a</th>
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<tbody>
<tr>
<td>d</td>
<td>e</td>
<td></td>
</tr>
</tbody>
</table>

\[
\begin{array}{c|c|c|c}
  abcde + bcde + bce + cde + ce + de + c + e + 1 & bce + ce + c + e + 1 & c + 1 & 1 \\
  de + e + 1 & e + 1 & 1 & 1 \\
  1 & 1 & 1 & 1 \\
\end{array}
\]

\[\text{SNF} = \text{diag}(abcde, e, 1)\]
A special case

Let $\lambda$ be the **staircase** $\delta_n = (n - 1, n - 2, \ldots, 1)$.
Set each $x_{ij} = q$. 
A special case

Let $\lambda$ be the **staircase** $\delta_n = (n - 1, n - 2, \ldots, 1)$. Set each $x_{ij} = q$. 
A special case

Let $\lambda$ be the **staircase** $\delta_n = (n - 1, n - 2, \ldots, 1)$. Set each $x_{ij} = q$.

$u_{\delta_{n-1}}(x) \bigg|_{x_{ij}=q}$ counts Dyck paths of length $2n$ by (scaled) area, and is thus the well-known $q$-analogue $C_n(q)$ of the Catalan number $C_n$. 
A \( q \)-Catalan example

\[ C_3(q) = q^3 + q^2 + 2q + 1 \]
A $q$-Catalan example

$C_3(q) = q^3 + q^2 + 2q + 1$

\[
\begin{array}{ccc}
C_4(q) & C_3(q) & 1 + q \\
C_3(q) & 1 + q & 1 \\
1 + q & 1 & 1 \\
\end{array}
\xrightarrow{\text{SNF}} \sim \text{diag}(q^6, q, 1)
\]
A $q$-Catalan example

$C_3(q) = q^3 + q^2 + 2q + 1$

\[
\begin{vmatrix}
C_4(q) & C_3(q) & 1 + q \\
C_3(q) & 1 + q & 1 \\
1 + q & 1 & 1
\end{vmatrix} \overset{\text{SNF}}{\sim} \text{diag}(q^6, q, 1)
\]

- $q$-Catalan determinant previously known
- SNF is new
Huge literature on random matrices, mostly connected with eigenvalues.

Very little work on SNF of random matrices over a PID.
Is the question interesting?

\[ \text{Mat}_k(n) : \text{all } n \times n \mathbb{Z}-\text{matrices with entries in } [-k, k] \text{ (uniform distribution)} \]

\[ p_k(n, d) : \text{probability that if } M \in \text{Mat}_k(n) \text{ and } \text{SNF}(M) = (e_1, \ldots, e_n), \text{ then } e_1 = d. \]
Is the question interesting?

\( \text{Mat}_k(n) \): all \( n \times n \mathbb{Z} \)-matrices with entries in \([-k, k]\) (uniform distribution)

\( p_k(n, d) \): probability that if \( M \in \text{Mat}_k(n) \) and \( \text{SNF}(M) = (e_1, \ldots, e_n) \), then \( e_1 = d \).

**Recall:** \( e_1 = \gcd \) of \( 1 \times 1 \) minors (entries) of \( M \)
Is the question interesting?

\( \text{Mat}_k(n) \): all \( n \times n \mathbb{Z} \)-matrices with entries in \([-k, k]\) (uniform distribution)

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Recall: \( e_1 = \gcd \) of \( 1 \times 1 \) minors (entries) of \( M \)

**Theorem.** \( \lim_{k \to \infty} p_k(n, d) = \frac{1}{d^{n^2} \zeta(n^2)} \)
Sample result. $\mu_k(n)$: probability that the SNF of a random $A \in \text{Mat}_k(n)$ satisfies $e_1 = 2$, $e_2 = 6$.

$$\mu(n) = \lim_{k \to \infty} \mu_k(n).$$
\[
\mu(n) = 2^{-n^2} \left( 1 - \sum_{i=(n-1)^2}^{n(n-1)} 2^{-i} + \sum_{i=n(n-1)+1}^{n^2-1} 2^{-i} \right) \\
\cdot \frac{3}{2} \cdot 3^{-(n-1)^2} (1 - 3^{(n-1)^2}) (1 - 3^{-n})^2 \\
\cdot \prod_{p>3} \left( 1 - \sum_{i=(n-1)^2}^{n(n-1)} p^{-i} + \sum_{i=n(n-1)+1}^{n^2-1} p^{-i} \right).
\]
A note on the proof

uses a 2014 result of C. Feng, R. W. Nóbrega, F. R. Kschischang, and D. Silva, Communication over finite-chain-ring matrix channels: number of $m \times n$ matrices over $\mathbb{Z}/p^s\mathbb{Z}$ with specified SNF
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**Note.** \( \mathbb{Z}/p^s\mathbb{Z} \) is not a PID, but SNF still exists because its ideals form a finite chain.
\( \kappa(n) \): probability that an \( n \times n \mathbb{Z} \)-matrix has SNF \( \text{diag}(e_1, e_2, \ldots, e_n) \) with \( e_1 = e_2 = \cdots = e_{n-1} = 1 \).
Cyclic cokernel

\( \kappa(n) \): probability that an \( n \times n \ \mathbb{Z} \)-matrix has SNF \( \text{diag}(e_1, e_2, \ldots, e_n) \) with \( e_1 = e_2 = \cdots = e_{n-1} = 1 \).

Theorem. \( \kappa(n) = \prod_p \left( 1 + \frac{1}{p^2} + \frac{1}{p^3} + \cdots + \frac{1}{p^n} \right) \frac{\zeta(2)\zeta(3)\cdots}{p^n} \)
Cyclic cokernel

$\kappa(n)$: probability that an $n \times n \mathbb{Z}$-matrix has SNF $\text{diag}(e_1, e_2, \ldots, e_n)$ with $e_1 = e_2 = \cdots = e_{n-1} = 1$.

Theorem. $\kappa(n) = \prod_{p} \left( 1 + \frac{1}{p^2} + \frac{1}{p^3} + \cdots + \frac{1}{p^n} \right) \zeta(2) \zeta(3) \cdots$

Corollary. $\lim_{n \to \infty} \kappa(n) = \frac{1}{\zeta(6) \prod_{j \geq 4} \zeta(j)} \approx 0.846936 \cdots$. 
Third example

In collaboration with Tommy Wuxing Cai.
Third example

In collaboration with 蔡吴兴.
Third example

In collaboration with 蔡吴兴.

\( \text{Par}(n) \): set of all partitions of \( n \)

E.g., \( \text{Par}(4) = \{4, 31, 22, 211, 1111\} \).
Third example

In collaboration with 蔡吴兴.

**Par**(*n*): set of all partitions of *n*

E.g., Par(4) = {4, 31, 22, 211, 1111}.

**V**_n_: real vector space with basis Par(*n*)
Define \( \mathbf{U} = \mathbf{U}_n : V_n \to V_{n+1} \) by

\[
U(\lambda) = \sum_{\mu} \mu,
\]

where \( \mu \in \text{Par}(n + 1) \) and \( \mu_i \geq \lambda_i \ \forall i \).

Example.

\[
U(42211) = 52211 + 43211 + 42221 + 422111
\]
Dually, define $D = D_n : V_n \to V_{n-1}$ by

$$D(\lambda) = \sum_{\nu} \nu,$$

where $\nu \in \text{Par}(n - 1)$ and $\nu_i \leq \lambda_i \ \forall i$.

**Example.** $D(42211) = 32211 + 42111 + 4221$
**NOTE.** Identify $V_n$ with the space $\Lambda^n_{\mathbb{Q}}$ of all homogeneous symmetric functions of degree $n$ over $\mathbb{Q}$, and identify $\lambda \in V_n$ with the Schur function $s_\lambda$. Then

$$U(f) = p_1 f, \quad D(f) = \frac{\partial}{\partial p_1} f.$$
Basic commutation relation: \[ DU - UD = I \]

Allows computation of eigenvalues of \( DU : V_n \to V_n \).

Or note that the eigenvectors of \( \frac{\partial}{\partial p_1} p_1 \) are the \( p_\lambda \)'s, \( \lambda \vdash n \).
Let \( p(n) = \#\text{Par}(n) = \dim V_n \).

**Theorem.** Let \( 1 \leq i \leq n + 1, i \neq n \). Then \( i \) is an eigenvalue of \( D_{n+1}U_n \) with multiplicity \( p(n + 1 - i) - p(n - i) \). Hence

\[
\det D_{n+1}U_n = \prod_{i=1}^{n+1} i^{p(n+1-i) - p(n-i)}.
\]
Let $p(n) = \# \text{Par}(n) = \dim V_n$.

**Theorem.** Let $1 \leq i \leq n + 1$, $i \neq n$. Then $i$ is an eigenvalue of $D_{n+1}U_n$ with multiplicity $p(n + 1 - i) - p(n - i)$. Hence

$$\det D_{n+1}U_n = \prod_{i=1}^{n+1} i^{p(n+1-i) - p(n-i)}.$$ 

What about SNF of the matrix $[D_{n+1}U_n]$ (with respect to the basis $\text{Par}(n)$)?
Conjecture (first form). Let $e_1, \ldots, e_{p(n)}$ be the eigenvalues of $D_{n+1}U_n$. Then $[D_{n+1}U_n]$ has the same SNF as $\text{diag}(e_1, \ldots, e_{p(n)})$. 
Conjecture (first form). Let $e_1, \ldots, e_{p(n)}$ be the eigenvalues of $D_{n+1} U_n$. Then $[D_{n+1} U_n]$ has the same SNF as $\text{diag}(e_1, \ldots, e_{p(n)})$.

Conjecture (second form). The diagonal entries of the SNF of $[D_{n+1} U_n]$ are:

- $(n+1)(n-1)!$, with multiplicity 1
- $(n-k)!$ with multiplicity $p(k+1) - 2p(k) + p(k-1)$, $3 \leq k \leq n-2$
- $1$, with multiplicity $p(n) - p(n-1) + p(n-2)$. 
Note. \( \{p_{\lambda}\}_{\lambda \vdash n} \) is not an integral basis.
Another form

$m_1(\lambda)$: number of 1’s in $\lambda$

$M_1(n)$: multiset of all numbers $m_1(\lambda) + 1$, $\lambda \in \text{Par}(n)$

Let SNF of $[D_{n+1} U_n]$ be diag$(f_1, f_2, \ldots, f_{p(n)})$.

**Conjecture** (third form). $f_1$ is the product of the distinct entries of $M_1(n)$; $f_2$ is the product of the remaining distinct entries of $M_1(n)$, etc.
An example: $n = 6$

$\text{Par}(6) = \{6, 51, 42, 33, 411, 321, 222, 3111, 2211, 21111, 111111\}$

$\mathcal{M}_1(6) = \{1, 2, 1, 1, 3, 2, 1, 4, 3, 5, 7\}$

$(f_1, \ldots, f_{11}) = (7 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1, 3 \cdot 2 \cdot 1, 1, 1, 1, 1, 1, 1, 1, 1, 1) = (840, 6, 1, 1, 1, 1, 1, 1, 1, 1, 1)$
Conjecture (fourth form). The matrix $[D_{n+1}U_n + xI]$ has an SNF over $\mathbb{Z}[x]$.

Note that $\mathbb{Z}[x]$ is not a PID.
Resolution of conjecture

**Theorem.** The conjecture of Miller is true.
Theorem. The conjecture of Miller is true.

Proof (first step). Rather than use the basis \( \{ s_\lambda \}_{\lambda \in \text{Par}(n)} \) (Schur functions) for \( \Lambda^n_Q \), use the basis \( \{ h_\lambda \}_{\lambda \in \text{Par}(n)} \) (complete symmetric functions). Since the two bases differ by a matrix in \( SL(p(n), \mathbb{Z}) \), the SNF’s stay the same.
Conclusion of proof

(second step) Row and column operations.
(second step) Row and column operations.

Not very insightful.
(second step) Row and column operations.

Not very insightful.
An unsolved conjecture

$m_j(\lambda)$: number of $j$’s in $\lambda$

$\mathcal{M}_j(n)$: multiset of all numbers $j(m_j(\lambda) + 1)$, $\lambda \in \text{Par}(n)$

$p_j$: power sum symmetric function $\sum x_i^j$

Let SNF of the operator $f \mapsto j \frac{\partial}{\partial p_j} p_j f$ with respect to the basis $\{s_\lambda\}$ be $\text{diag}(g_1, g_2, \ldots, g_{p(n)})$. 
An unsolved conjecture

$m_j(\lambda)$: number of $j$’s in $\lambda$

$M_j(n)$: multiset of all numbers $j(m_j(\lambda) + 1)$, $\lambda \in \text{Par}(n)$

$p_j$: power sum symmetric function $\sum x_i^j$

Let SNF of the operator $f \mapsto j \frac{\partial}{\partial p_j} p_j f$ with respect to the basis $\{s_\lambda\}$ be diag$(g_1, g_2, \ldots, g_{p(n)})$.

**Conjecture.** $g_1$ is the product of the **distinct** entries of $M_j(n)$; $g_2$ is the product of the remaining **distinct** entries of $M_j(n)$, etc.
Jacobi-Trudi identity:

\[ s_\lambda = \det[h_{\lambda_i - i + j}], \]

where \( s_\lambda \) is a **Schur function** and \( h_i \) is a **complete symmetric function**.
Jacobi-Trudi specialization

Jacobi-Trudi identity:

\[ s_\lambda = \det[h_{\lambda_i - i + j}], \]

where \( s_\lambda \) is a Schur function and \( h_i \) is a complete symmetric function.

We consider the specialization
\[ x_1 = x_2 = \cdots = x_n = 1, \text{ other } x_i = 0. \]
Then
\[ h_i \rightarrow \binom{n + i - 1}{i}. \]
Specialized Schur function

\[ s_\lambda \to \prod_{u \in \lambda} \frac{n + c(u)}{h(u)}. \]

\( c(u) \): content of the square \( u \)
Diagonal hooks $D_1, \ldots, D_m$

\[
\begin{array}{cccc}
0 & 1 & 2 & 3 & 4 \\
-1 & 0 & 1 & 2 \\
-2 & -1 & 0 & 1 \\
-3 & -2 \\
\end{array}
\]

\[\lambda = (5,4,4,2)\]
Diagonal hooks $D_1, \ldots, D_m$

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$D_1$
Diagonal hooks $D_1, \ldots, D_m$

\[
\begin{array}{cccccc}
0 & 1 & 2 & 3 & 4 \\
-1 & 0 & 1 & 2 \\
-2 & -1 & 0 & 1 \\
-3 & -2 \\
\end{array}
\]

$D_2$
Diagonal hooks $D_1, \ldots, D_m$

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$D_3$
\[ R = \mathbb{Q}[n] \]

Let

\[
\text{SNF} \begin{bmatrix}
(n + \lambda_i - i + j - 1) \\
\lambda_i - i + j
\end{bmatrix} = \text{diag}(e_1, \ldots, e_m).
\]

Then

\[
e_i = \prod_{u \in D_{m-i+1}} \frac{n + c(u)}{h(u)}.
\]
Idea of proof

\[ f_i = \prod_{u \in D_{m-i+1}} \frac{n + c(u)}{h(u)} \]

Then \( f_1 f_2 \cdots f_i \) is the value of the lower-left \( i \times i \) minor. (Special argument for 0 minors.)
Idea of proof

\[ f_i = \prod_{u \in D_{m-i+1}} \frac{n + c(u)}{h(u)} \]

Then \( f_1 f_2 \cdots f_i \) is the value of the lower-left \( i \times i \) minor. (Special argument for 0 minors.)

Every \( i \times i \) minor is a specialized skew Schur function \( s_{\mu/\nu} \). Let \( s_{\alpha} \) correspond to the lower left \( i \times i \) minor.
Let

\[ s_{\mu/\nu} = \sum_{\rho} c_{\nu\rho}^{\mu} s_{\rho}. \]

By Littlewood-Richardson rule,

\[ c_{\nu\rho}^{\mu} \neq 0 \iff \alpha \subseteq \rho. \]
Let

\[ s_{\mu/\nu} = \sum_{\rho} c_{\nu\rho}^{\mu} s_{\rho}. \]

By Littlewood-Richardson rule,

\[ c_{\nu\rho}^{\mu} \neq 0 \iff \alpha \subseteq \rho. \]

Hence

\[ f_i = \gcd(i \times i \text{ minors}) = \frac{e_i}{e_{i-1}}. \]
The last slide 🙈